SATURATION AND SIMPLE EXTENSIONS
OF MODELS OF PEANO ARITHMETIC

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Recursive saturation has come to play an important part in the model theory of Peano Arithmetic (PA). One of the neater results is the theorem of Smoryński and Stavi [16] that recursive saturation is preserved by cofinal extensions. (We give a quite simple proof of this in Corollary 1.3(i).) It is natural to inquire about a converse to this theorem, but the answer comes too quickly: A model of PA has a recursively saturated, cofinal extension iff it is tall.

The Smoryński-Stavi theorem loses nothing if the cofinal extensions are required to be simple. Besides, simple cofinal extensions arise quite naturally here because of Kotlarski's observation (Corollary 1.3(iii)) that the Smoryński-Stavi theorem implies that $\omega_1$-saturation is preserved by simple, cofinal extensions. Notice also that a simple elementary extension must be cofinal in order to be recursively saturated (Proposition 1.6). It is thus we are led to the following two questions:

(1) Does some tall model of PA that is not recursively saturated have a recursively saturated simple extension?

(2) Does every tall model of PA have a recursively saturated simple extension?

We will show in this paper that (1) has a positive answer whereas (2) is answered negatively. Moreover, we characterize among countable models of PA those which do have recursively saturated simple extensions: they are precisely the lofty models. The notion of a lofty model of PA is the central new concept introduced here, and it is the main purpose of this paper to study it and its relation to recursive saturation. This will be done in Sections 1–4.

More precisely, we shall be concerned with a refinement of the notion of loftiness: for a model $\mathcal{N}$ of PA and $e \in N$, we define when $\mathcal{N}$ is $e$-lofty. In Theorem 3.1 we give the following characterization. If $\mathcal{N}$ is countable and $e \in N$, then $\mathcal{N}$ is

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e-lofty iff $\mathcal{N}$ has a recursively saturated simple, cofinal extension $\mathcal{N}(b)$ with $b < e$. If we let $L(\mathcal{N})$ be the set \{e $\in N$: $\mathcal{N}$ is not e-lofty\}, then Theorem 2.8 asserts that $L(\mathcal{N})$ is an initial segment which is closed under multiplication. We show in Theorem 4.11 that essentially there is no more that can be said about $L(\mathcal{N})$.

In Section 5 we consider simple cofinal extensions of $\kappa^+$-saturated models, answering a question of Kotlarski concerning uncountable $\kappa$.

The countable, recursively saturated models of PA form a countably PC$^*$ class (Definition 6.1), as can be seen by using satisfaction classes. By some general results on countably PC$^*$ classes, obtained in Section 6, we show that the countable lofty models do not form a countably PC$^*$ class.

The paper concludes with Section 7 which contains some open questions.

1. Preliminaries

This paper deals with models of Peano Arithmetic (PA) and their extensions. We assume throughout that all models are models of PA, except as otherwise noted.

Let us begin by reviewing some fundamental definitions and properties. All extensions that we consider are elementary extensions. Suppose $\mathcal{M} < \mathcal{N}$. Then $\mathcal{N}$ is a cofinal extension of $\mathcal{M}$ if

$$(\forall b \in N)(\exists a \in M)(b < a).$$

In contrast, $\mathcal{N}$ is an end extension of $\mathcal{M}$ if

$$(\forall a \in M)(\forall b \in N - M)(a < b).$$

$\mathcal{N}$ is a simple extension of $\mathcal{M}$ if for some $b \in N$, every element of $N$ is definable in $\mathcal{N}$ with parameters from $M \cup \{b\}$; in this case we may write $\mathcal{N} = \mathcal{M}(b)$.

A model $\mathcal{M}$ is short if for some $a \in M$, $\{x \in M$: $x$ is definable from $a$ in $\mathcal{M}\}$ is cofinal in $\mathcal{M}$; otherwise $\mathcal{M}$ is tall. $\mathcal{M}$ is recursively saturated if $\mathcal{M}$ realizes every recursive type over $\mathcal{M}$, i.e. for every set $\Sigma(v, y)$ with a recursive set of G"odel numbers and every finite $\bar{a}$ from $\mathcal{M}$, if $\Sigma(v, \bar{a})$ is finitely satisfiable in $\mathcal{M}$, then $\mathcal{M} \models \Sigma(b, \bar{a})$ for some $b \in M$. $\mathcal{M}$ is boundedly recursively saturated if $\mathcal{M}$ realizes every bounded recursive type over $\mathcal{M}$, where $\Sigma(v, y)$ is bounded if it contains a formula $v < z$. (Smoryński has called such models 'short recursively saturated'.) We will abbreviate 'boundedly' by 'bdd'. It follows easily that short models are not recursively saturated, yet every non-standard model has a bdd recursively saturated cofinal extension. The reader may also check the following useful criterion: $\mathcal{M}$ is tall iff every type $\Sigma(x, a)$ over $\mathcal{M}$ is contained in a bounded type.

We use freely the observation that PA is essentially just set theory with the negation of the axiom of infinity. So we feel free to write, for example, $x \in s$ instead of more cumbersome phrases like "the $x$th prime divides $s$". Also it is useful to code sequences, i.e. there is (in PA) a definable function $(x, y) \mapsto (x)_y$.
such that for all $\phi$,

$$\text{PA} \vdash \forall u \exists v \phi(u, v) \rightarrow \forall z \exists x \forall y < z \phi(y, (x)),$$

Since $\text{PA}$ has definable Skolem functions it is possible (without loss of generality) and useful to write expressions like $\tau(a)$, where $\tau$ is a definable term. For example, if $\mathcal{M} = \mathcal{N}(b)$, then every element of $\mathcal{N}$ is of the form $\tau(a, b)$ for some $a \in \mathcal{M}$ and some definable term $\tau$.

A subset $I$ of a model $\mathcal{M}$ is a cut of $\mathcal{M}$ if $I$ is a non-empty initial segment that is closed under successor. The standard cut, which we always assume to be $\omega$, is just the set of standard elements of $\mathcal{M}$.

Suppose $I$ is a cut in $\mathcal{M}$ and $\mathcal{M} < \mathcal{N}$. Then $\mathcal{M}$ fills $I$ with $b$ if $b \in \mathcal{M}$ and $i < b < j$ for all $i \in I$ and $j \in \mathcal{M} - I$. Let

$$I^\mathcal{N} = \{ x \in \mathcal{N} : (\exists y \in I)(x < y) \}.$$

We write $\mathcal{M} <^I \mathcal{N}$ if $I^\mathcal{N} = I$.

A type $\Sigma(v)$ (not necessarily complete) defines the cut $I$ of $\mathcal{M}$ if for all $\mathcal{N} > \mathcal{M}$ and $b \in \mathcal{N}$, $b$ realizes $\Sigma$ if $\mathcal{M}$ fills $I$ with $b$. In this case we say that $I$ is definable; if $\Sigma$ is recursive, we say that $I$ is recursively definable. For example, it is shown in [4] that there are nonstandard minimal models $\mathcal{M}$ and $\mathcal{N}$ such that the standard cut is recursively definable in $\mathcal{M}$ but not in $\mathcal{N}$; and in fact, every minimal model has a cofinal simple extension in which the standard cut is recursively definable.

A sequence $(a_n : n < \omega)$ of elements of $\mathcal{M}$ is coded in $\mathcal{M}$ if for some $c \in \mathcal{M}$, $(c)_n = a_n$ for all $n < \omega$. In this case we say that $c$ codes $(a_n : n < \omega)$. A weaker notion is that the sequence $(a_n : n < \omega)$ is definable in $\mathcal{M}$; this means that for some $c \in \mathcal{M}$ and a sequence $\langle \tau_n : n < \omega \rangle$ of terms, $a_n = \tau_n(c)$ for all $n < \omega$. If in addition the sequence $\langle \tau_n : n < \omega \rangle$ is recursive, then $(a_n : n < \omega)$ is recursively definable.

1.1. Proposition. (i) $\mathcal{M}$ is tall iff every definable sequence in $\mathcal{M}$ is bounded, iff every recursively definable sequence in $\mathcal{M}$ is bounded.

(ii) $\mathcal{M}$ is bdd recursively saturated iff every recursively definable bounded sequence in $\mathcal{M}$ is coded.

(iii) $\mathcal{M}$ is $\omega$-saturated iff every definable sequence in $\mathcal{M}$ is coded.

(iv) $\mathcal{M}$ is $\omega_1$-saturated iff every $\omega$-sequence of elements of $\mathcal{M}$ is coded.

Proof. (i) If $\mathcal{M}$ is tall and $a \in \mathcal{M}$, then there exists $b \in \mathcal{M}$ which is greater than every element of $\mathcal{M}$ that is definable from $a$. Then $b$ is an upper bound for every sequence of the form $\langle \tau_n(a) : n < \omega \rangle$ with each $\tau_n$ a (definable) term. Inversely, if $\mathcal{M}$ is short, then there is $a \in \mathcal{M}$ such that $\{ x \in \mathcal{M} : x$ is definable in $\mathcal{M} from $a$ is cofinal in $\mathcal{M}$. So setting $\tau_n(x) = y$ iff $y$ is least such that $\phi_n(x, y)$ (for some recursive enumeration $\phi_0, \phi_1, \ldots$ of the formulas with two free variables), then $\langle \tau_n(a) : n < \omega \rangle$ is unbounded in $\mathcal{M}$.

(ii) If $\mathcal{M}$ is recursively saturated, $a \in \mathcal{M}$, and $\langle \tau_n : n < \omega \rangle$ is recursive, then the type $\Sigma(v, a) = \{ (v)_n = \tau_n(a) : n < \omega \}$ is recursive and finitely satisfiable in $\mathcal{M}$. So
$M \models \Sigma(c, a)$ for some $c \in M$, i.e. $c$ codes $\langle \tau_n(a) : n < \omega \rangle$. Conversely, suppose $M$ codes every recursively definable sequence in $M$, and suppose $\Sigma(v, a)$ is finitely satisfiable in $M$ and $\Sigma$ is recursive. By (i) we may choose $b \in M$ such that $\Sigma(v, a) \cup \{ v < b \}$ is finitely satisfiable in $M$. Choose (for all $n < \omega$) $s_n \in M$ such that $M \models \langle s_n = \{ x < b : \bigwedge_{i < n} \sigma_i(x, a) \} \rangle$, where $\{ \sigma_i : i < \omega \}$ is a recursive enumeration of $\Sigma$. Then clearly $\langle s_n : n < \omega \rangle$ is recursively definable in $M$. So by hypothesis, it is coded by $c$ for some $c \in M$. Now $\langle \langle c \rangle : i < n \rangle$ is a decreasing sequence of non-empty sets, for each standard $n$; so this holds for some non-standard $n$. Then if $M \models \langle d \in (c)_n \rangle$, it follows that $d$ realizes $\Sigma(v, a)$ in $M$. (The bounded case is similar.)

The proof of (iii) and (iv) are similar to the proof of (ii). Also, a proof of (iv) can be found in Pabion [11, Proposition 1].

We now formulate a tool that enables us to construct recursively saturated simple extensions in Section 3. In fact, we will also use this tool to give easy proofs of two known results.

1.2. Lemma. Suppose $%$ is a cofinal extension of $M$ which realizes all bounded recursive types with parameters in $%$. Then $%$ is bdd recursively saturated. (Hence, if $%$ is tall, then $%$ is recursively saturated.)

Proof. By Proposition 1.1(ii) it suffices to show that every definable bounded sequence in $%$ is coded in $%$. Let $\langle \tau_n(v) : n < \omega \rangle$ be recursive, and let $b \in N$; we show that $\langle \tau_n(b) : n < \omega \rangle$ is coded in $%$. Choose $a \in M$ with $a > b$. In $%$ define $c_n = \langle \tau_n(d) : d < a \rangle$. Then by hypothesis there exists $e \in N$ which codes $\langle c_n : n < \omega \rangle$. Then $\langle (e)_n \rangle_b = (c_n)_b = \tau_n(b)$ for all $n < \omega$. Choose $f \in N$ which codes $\langle ((e)_n) : n < \omega \rangle$, i.e. $f$ codes $\langle \tau_n(b) : n < \omega \rangle$.

1.3. Corollary (Smoryński–Stavi [16]).

(i) If $%$ is a cofinal extension of $M$ and $%$ is (bdd) recursively saturated, then so is $%$.

(ii) If $%$ is a cofinal extension of $M$ and $%$ is $\omega$-saturated, then so is $%$.

(iii) (Kotlarski [8]). If $%$ is a simple cofinal extension of $M$ and $%$ is $\omega_1$-saturated, then so is $%$.

Proof. (i) is immediate from Lemma 1.2. Then (ii) holds because, under the hypothesis, $%$ is recursively saturated (by (i)) and hence $\omega$-homogeneous, so since $%$ realizes all pure types, $%$ is $\omega$-saturated. (Alternate argument: a recursively saturated model of PA represents all reals iff it is $\omega$-saturated.) For (iii), suppose $% = % (b)$ and suppose $\langle \tau_n(a_n, b) : n < \omega \rangle$ is given, with each $a_n \in M$. By hypothesis we may choose $c \in M$ with $\langle c \rangle_n = a_n$ for all $n < \omega$. Let $\tau_n(u, v) = \tau_n((u)_n, v)$. Then $\langle \tau_n(a_n, b) : n < \omega \rangle = \langle \tau_n(c, b) : n < \omega \rangle$, so the result follows from (ii) and Proposition 1.1.

As remarked in the Introduction, this paper involves a study of notions lying
between recursive saturation and tallness (and their bounded versions). These notions are presented in the following definition.

1.4. Definitions. Let $\mathcal{M}$ be a model and $I$ a cut of $\mathcal{M}$.

(i) Given $e \in M$, $\Sigma(v, y)$ and $a \in M$, where $\Sigma(v, a)$ is a type over $\mathcal{M}$ (i.e. $\Sigma(v, a)$ is finitely satisfiable in $\mathcal{M}$), we say that $\Sigma(v, a)$ is $e$-lofty (in $\mathcal{M}$) if for some $s \in M$, $\Sigma(v, a) \cup \{v = s\}$ is finitely satisfiable in $\mathcal{M}$ and $\mathcal{M} \vdash |s| = e$. $\mathcal{M}$ is $e$-lofty if every consistent recursive $\Sigma(v, a)$ ($a \in M$) is $e$-lofty in $\mathcal{M}$. $\mathcal{M}$ is lofty if $\mathcal{M}$ is $e$-lofty for some $e \in M$.

(ii) $\mathcal{M}$ is $I$-lofty if $\mathcal{M}$ is $e$-lofty for all $e \in M - I$. In particular, $\mathcal{M}$ is $\omega$-lofty if $\mathcal{M}$ is $e$-lofty for all non-standard $e \in M$.

(iii) $\mathcal{M}$ is uniformly $I$-lofty if for every recursive $\Sigma(v, a)$ which is finitely satisfiable in $\mathcal{M}$, there exists $c \in M$ such that for all $e \in M - I$, $\Sigma(v, a) \cup \{(\exists i < e)(v = (c)_i)\}$ is finitely satisfiable in $\mathcal{M}$. In particular, $\mathcal{M}$ is uniformly $\omega$-lofty if for every recursive $\Sigma(v, a)$ one can find such $c$ which works for all non-standard $e \in M$.

(iv) $\mathcal{M}$ is strongly $I$-lofty if for every recursive consistent $\Sigma(v, a)$ ($a \in M$), $\Sigma(v, a)$ is $e$-lofty in $\mathcal{M}$ for some $e \in I$.

[We will occasionally refer to the notions defined above for $\mathcal{M}$ as 'notions of loftiness'.]

(v) $\mathcal{M}$ is bdd $e$-lofty, bdd lofty, and so on, if the respective notions above hold when restricted to bounded types $\Sigma(v, y)$.

The following proposition gives some immediate consequences of these definitions. The implications in (iii) and (iv) below are all proper, as we show in Section 4.

1.5. Proposition. Let $\mathcal{M}$ be a model.

(i) $\mathcal{M}$ is recursively saturated iff $\mathcal{M}$ is 1-lofty iff $\mathcal{M}$ is strongly $\omega$-lofty.

(ii) $\mathcal{M}$ is tall iff $\mathcal{M}$ is strongly $M$-lofty.

(iii) Let $I$ be a cut. Then: $\mathcal{M}$ is strongly $I$-lofty $\Rightarrow$ $\mathcal{M}$ is uniformly $I$-lofty $\Rightarrow$ $\mathcal{M}$ is $I$-lofty.

(iv) Let $J \subsetneq I$ be cuts. If $\mathcal{M}$ is $J$-lofty, then $\mathcal{M}$ is strongly $I$-lofty.

Proof. The only part of the Proposition which possibly is not trivial is (ii). Suppose $\mathcal{M}$ is tall and let $\Sigma(v, a)$ be a recursive type. By tallness, there is $b \in M$ such that $b$ is greater than every element of $M$ definable from $a$. Then, the set $\{x \in M : x < b\}$ demonstrates that $\Sigma(v, a)$ is $b$-lofty. Conversely, suppose $\mathcal{M}$ is strongly $M$-lofty. Let $a \in M$ and consider the recursive type $\Sigma(v, a)$ which asserts: “$v$ is greater than every element definable from $a$”. If $s \in M$ is such that $\mathcal{M} \vdash |s| = e$, and $s$ demonstrates that $\Sigma(v, a)$ is $e$-lofty, then max $s$ is greater than every element definable from $a$. Hence, $\mathcal{M}$ is tall. □
Each of the statements in Proposition 1.5 remains true when the notions are replaced by their bounded versions. Notice that any notion of loftiness is equivalent to its bounded version plus tallness.

In Section 3 we will relate the various notions of loftiness to the existence of recursively saturated simple extensions. In fact, such extensions are cofinal, as we now check. We also get the existence of models which are not bdd lofty.

1.6. Proposition. Suppose $\mathcal{R} = \mathcal{M}(b)$. Then $\mathcal{R}$ is a cofinal extension of $\mathcal{M}$ iff $b < a$ for some $a \in M$. If $\mathcal{R}$ is not a cofinal extension of $\mathcal{M}$, then $\mathcal{R}$ is short and not bdd lofty.

Proof. The proof of the proposition, except the failure of bdd loftiness, is immediate from the following observation: For each formula $\phi(v, z_1, z_2, \ldots, z_n)$ there is a term $\tau(x)$ such that the sentence

$$\left( \forall z_1 \leq x \right) \left( \forall z_2 \leq x \right) \cdots \left( \forall z_n \leq x \right) [\exists \bar{v} \phi(v, \bar{z}) \rightarrow \exists \bar{v} \leq \tau(x) \phi(v, \bar{z})]$$

is a consequence of PA.

To show the failure of bdd $e$-loftiness for arbitrary $e \in N$, choose $h, c \in N$ such that $h$ is nonstandard and $b \cdot e \cdot h < c$. Now consider the recursive, bounded type $\Sigma(v, b, c, h)$ consisting of $v < 2^c$ (so that $v$ codes a subset of $[0, c)$) together with $|v| \leq bh$ and all formulas of the form $\forall y < b [\tau(y, b) < c \rightarrow \tau(y, b) \in v]$, where $\tau(y, b)$ ranges over all terms without parameters or free variables other than $y$ and $b$. If $\Sigma \cup \{v \in s\}$ were consistent for some $s \in N$ of internal cardinality $e$, then $S = \{v \in s : |v| \leq bh\}$ would contain every predecessor of $c$, since $\mathcal{R} = \mathcal{M}(b)$. This is a contradiction, since (in $\mathcal{R}$) $|S| \leq e \cdot bh < c$. \qed

This argument also shows that no nonstandard minimal model is bdd lofty.

2. Reducing to types that define cuts

In this section we show how to reduce the notions of loftiness to types that define cuts. The following definition is the key to this reduction. We also apply the method to give a short proof of a theorem of Pabion and Richard in [11].

Recall that we identify PA with finite set theory. (In fact, the results below hold for ZFC as well, if 'k-onto' is defined only for $k \in \omega$.)

2.1. Definition (within PA). For $f$ any function from a closed interval $[a, b]$ to a (finite) set $A$, the notion "$f$ is $k$-onto $A$" is defined by induction on $k$. $f$ is 0-onto if $f$ is onto $A$. $f$ is $(n+1)$-onto if for all $B \subseteq A$ there exists $[c, d] \subseteq [a, b]$ such that $f \upharpoonright [c, d]$ is $n$-onto $B$.

2.2. Lemma (within PA). For all $k$ and every (finite) set $A$ there exists an interval $[a, b]$ and a function $f : [a, b] \rightarrow A$ which is $k$-onto.
Proof (in PA). The proof proceeds by induction on $k$. The case $k = 0$ is obvious. Assume the result for $k = n$. Given $A$, we may choose for each $B \subseteq A$ an interval $[a_B, b_B]$ and a function $f_B : [a_B, b_B] \to B$ which is $n$-onto. Of course, we may require $[a_B, b_B] \cap [a_C, b_C] = \emptyset$ for distinct $B, C \subseteq A$. Using the finite axiom of choice, this may all be done canonically within PA. Let $a$ be the minimum of all the $a_B$, let $b$ be the maximum of all the $b_B$, and let $g : [a, b] \to A$ be arbitrary extending $\bigcup \{f_B : B \subseteq A\}$. Then $g$ is $(n+1)$-onto $A$. □

The following definition gives a special class of types, namely those that define an intersection of intervals.

2.3. Definition. A set $\Gamma(v, \bar{y})$ of formulas is an interval set if every formula in $\Gamma$ is equivalent to one of the form $\tau_1 \equiv v \equiv \tau_2$, where $\tau_1$ and $\tau_2$ are terms in variables from $\bar{y}$; and moreover, if $\gamma, \phi \in \Gamma$ then $[\gamma \rightarrow \phi]$ or $[\phi \rightarrow \gamma]$ is valid.

2.4. Lemma. For every set of formulas $\Sigma(v, \bar{y}) \cup \{v < y_0\}$ there is an interval set $\Gamma(v, \bar{y}, z)$, a term $\tau(x, y_0, z)$ and a map $g$ from $\Sigma$ onto $\Gamma$, with $g$ and $\Gamma$ recursive in $\Sigma$, which satisfy the following conditions.

(i) For any model $M$ and $\bar{a}$ from $M$, any non-standard $d \in M$, and any $\sigma \in \Sigma$, if $M \models \exists v \sigma(v, \bar{a})$, then $M \models \exists v [g(\sigma)](v, \bar{a}, d)$.

(ii) For any model $M$ and $\bar{a}$, $b$ and $d$ from $M$, and any $\sigma \in \Sigma$, if $M \models g(\sigma)(b, \bar{a}, d)$, then $M \models \sigma(\tau(b, a_0, d), \bar{a})$.

Proof. Let $\{\sigma_n : n < \omega\}$ enumerate $\Sigma$. We define terms $s_n$, $t_n$ and $\tau$ and formulas $\gamma_n$ for $n < \omega$ as follows; then $\Gamma = \{\gamma_n : n < \omega\}$ will be the required set, where $g(\sigma_n) = \gamma_n$. Choose $s_0$, $t_0$ and $\tau$ so that

$$PA \vdash \\tau(\cdot, \gamma_n, z) : [s_n(\bar{y}, z), t_n(\bar{y}, z)] \to [0, y_0] \text{ is } z\text{-onto}$$.

In general, given $s_n$ and $t_n$ choose $s_{n+1}$, $t_{n+1}$ such that

$$PA \vdash z > n \to [s_n(\bar{y}, z) \leq s_{n+1}(\bar{y}, z) \leq t_{n+1}(\bar{y}, z) \leq t_n(\bar{y}, z)$$

$$^*\tau(\cdot, y_0, z) \upharpoonright [s_{n+1}(\bar{y}, z), t_{n+1}(\bar{y}, z)] \equiv (z - n - 1)\text{-onto } \{x \leq y_0 : \sigma_n(x, \bar{y})\}$$.

Let $\gamma_n$ be $z > n \land s_{n+1}(\bar{y}, z) \leq v \leq t_{n+1}(\bar{y}, z)$. The reader may easily verify that if the terms $s_i$ and $t_i$ are chosen canonically, then this construction works. □

Lemma 2.4 allows us to consider just interval sets, rather than arbitrary types, in verifying loftiness. For this, the notion of the cofinality of a cut, defined by Kirby [7], is useful.

2.5. Definition. Suppose $J$ is a cut of a model $M$. Then $\text{cf}^M(J)$, the cofinality of $J$ in $M$, is the set of $e \in M$ such that whenever $s$ is an internal set of internal cardinality $\leq e$, i.e. $M \models |s| \leq e$, then $s \cap J$ is not cofinal in $J$.  

2.6. **Lemma.** Suppose that for every proper recursively definable cut \( I \) of \( \mathcal{M} \), \( e \in \text{cf}^{\omega}(I) \). Then \( \mathcal{M} \) is bdd \( e \)-lofty.

**Proof.** Suppose \( \Sigma(v, a_0, a_1) \cup \{ v < a_0 \} \) is recursive and finitely satisfiable in \( \mathcal{M} \), where \( a_0 \) is nonstandard. Choose \( g, \tau \) and \( I \) as given by Lemma 2.4. Then property 2.4(i) guarantees that each element of \( \Gamma(v, \alpha, a_0) \) has a witness, so that \( \Gamma(v, \alpha, a_0) \) is finitely satisfiable (by definition of interval set). Since each formula in \( \Gamma(v, \alpha, a_0) \) defines a closed interval, there are two possibilities only: either \( \Gamma \) recursively defines a proper cut \( I \), or \( \Gamma \) is realized. By hypothesis, in the former case there is an internal set \( s \) of internal power \( e \) with \( s \cap I \) cofinal in \( I \). It follows that for all \( \gamma \in \Gamma(v, \alpha, a_0) \) there is \( b \in s \) with \( \mathcal{M} \models \gamma(b, \alpha, a_0) \). Let \( s' \) be the range of \( \tau(x, a_0, a_0) \) on \( s \). It follows from 2.4(ii) that \( \Sigma(v, \alpha) \cup \{ v \in s' \} \) is finitely satisfiable in \( \mathcal{M} \), and this completes the proof since \( \mathcal{M} \models |s'| \leq |s| = e \).

The following theorem can be proved in much the same manner as was Lemma 2.6.

2.7. **Theorem.** (i) There are no definable (resp., recursively definable) cuts in \( \mathcal{M} \) iff \( \mathcal{M} \) is \( \omega \)-saturated (resp., recursively saturated).

(ii) ([11]). For any model \( \mathcal{M} \) of \( \text{PA} \) and any cardinal \( \kappa > \omega \), if \( (M, <) \) is \( \kappa \)-saturated, then \( \mathcal{M} \) is \( \kappa \)-saturated.

**Proof.** We leave the proof of (i) to the reader who followed the proof of lemma 2.6. To prove (ii) one uses an induction on \( \kappa \). The proof of the base step \( \kappa = \omega_1 \) is similar to (i) and is also left to the reader. The case when \( \kappa \) is a limit cardinal is trivial. So we indicate the proof for the successor step \( \kappa = \lambda^+ \). Let \( \Sigma = \{ \sigma_\alpha(v) : \alpha < \lambda \} \) be a consistent set of formulas, closed under finite conjunction, where \( \sigma_\alpha(v) = v < c \) for some \( c \in M \). By an induction on \( \alpha \) we will construct a decreasing sequence of nonempty internal sets \( s_\alpha(\alpha < \lambda) \) with \( s_\alpha \subseteq \{ x : \mathcal{M} \models \sigma_\alpha(x) \} \). Moreover, we require that \( s_\alpha \cap \{ x : \mathcal{M} \models \sigma_\gamma(x) \} \neq \emptyset \) for all \( \gamma < \lambda \). Let \( s_0 = \{ x : \mathcal{M} \models \sigma_0(x) \} \) and \( s_{\alpha+1} = s_\alpha \cap \{ x : \mathcal{M} \models \sigma_{\alpha+1}(x) \} \). For \( \alpha \) a limit ordinal, consider the following consistent set of formulas in the variables \( v \) and \( v_\beta \) (\( \beta < \alpha \)):

\[
\{(v)_\alpha = s_\beta : \beta < \alpha \} \cup \{v_\beta < v_\beta : \beta < \beta' < \alpha \} \cup \{ \forall x \forall y (x < y \rightarrow (v)_x \equiv (v)_y) \}.
\]

There exist \( a, a_0, a_1, \ldots, a_\beta, \ldots \) satisfying this set in \( \mathcal{M} \), since \( \mathcal{M} \) is \( \lambda \)-saturated by the inductive hypothesis. Consider the cut \( I = \{ x \in M : x < a_\beta \) for some \( \beta < \alpha \} \). It follows by overspill that for all \( \gamma < \lambda \) there exists \( b \in M - I \) such that

\[
\mathcal{M} \models \exists z \in (a)_b \land \sigma_\gamma(z).
\]

By the \( \lambda^+ \)-saturation of \( (M, <) \) \( I \) has downward cofinality at least \( \lambda^+ \), so there exists \( b \in M - I \) which satisfies (*) for all \( \gamma < \lambda \). I.e. \( s_\alpha = (a)_b \).

Finally, the argument of Lemma 2.4, together with the \( \lambda \)-saturation of \( \mathcal{M} \), transfers the type \( \{ v \in s_\alpha : \alpha < \lambda \} \) to a type \( \Gamma \) whose formulas define intervals. The
$\lambda^+$-saturation of $(M, <)$ is used to guarantee that $\Gamma$ is realized in $M$; thus, $M$ also realizes $\Sigma$. □

**Remark.** The proof sketched above shows the analogous result for ZFC: if the ordinals of a model of ZFC are $\kappa$-saturated ($\kappa > \omega$), then so is the model.

Our main use of Lemma 2.6 is to prove the following useful theorem. (For a converse and variations, see Section 4.)

**2.8. Theorem.** Let $I = \{ e \in M : \mathcal{M} \text{ is not } e\text{-lofty} \}$. Then $I$ is an initial segment closed under multiplication.

**Proof.** Clearly $I$ is an initial segment. Suppose $e^2 \notin I$, i.e. $M$ is $e^2$-lofty; we show $e \notin I$, i.e. $\mathcal{M}$ is $e$-lofty. By Lemma 2.6 it suffices to show that for every recursively definable cut $J$, $e \notin \text{cf}_{\mathcal{M}}(J)$. Since $\mathcal{M}$ is $e^2$-lofty, $e^2 \notin \text{cf}_{\mathcal{M}}(J)$; so it suffices to prove the following lemma.

**2.9. Lemma.** For any cut $J$ of a model $\mathcal{M}$, $\text{cf}_{\mathcal{M}}(J)$ is closed under multiplication.

**Proof.** Suppose $e^2 \notin \text{cf}_{\mathcal{M}}(J)$; say $(s_i : i < e^2)$ is an increasing (internal) enumeration of a set $s$ such that $s \cap J$ is cofinal in $J$. Let $s' = \{ s_{e^i} : i < e \}$. If $s' \cap J$ is cofinal in $J$, then $e \notin \text{cf}_{\mathcal{M}}(J)$ and we are done. Otherwise there is $j \in J$ such that for no $s_{e^i} > j$ is $s_{e^i} \in J$. Let $k = \max \{ i : s_{e^i} \in J \}$. Then $(s_{e^{i+1}} : i < e) \cap J$ is cofinal in $J$, and again $e \notin \text{cf}_{\mathcal{M}}(J)$.

The following corollary shows that the restriction to recursive types in the definition of $e$-loftiness is inessential, except for standard $e$.

**2.10. Corollary.** Suppose $\mathcal{M}$ is $e$-lofty, where $e$ is nonstandard, and let $\Sigma(v, a)$ be a type over $\mathcal{M}$, not necessarily recursive. Then $\Sigma(v, a)$ is $e$-lofty in $\mathcal{M}$.

**Proof.** Let $f = \max \{ x : x^2 \leq e \}$. By Theorem 2.8, $\mathcal{M}$ is $f$-lofty. Now it suffices to show that for some $s \in M$, $\Sigma(v, a) \cup \{ v \in s \}$ is consistent and $\mathcal{M} \models \Sigma \equiv e$. Let $\Phi(v, a) = \{ \exists x \phi_n(x, a) \rightarrow \phi_n((v)_n, a) : n < \omega \}$, where $\phi_n : n < \omega$ is a recursive enumeration of all formulas $\phi(x, y)$. Since $\mathcal{M}$ is $f$-lofty, there exists $r$ with $\mathcal{M} \models |r| = f$, such that $\Phi \cup \{ v \in r \}$ is consistent. Then since $f$ is non-standard, the set $s = \{ (x)_i : x \in r, i < f \}$ is the desired set of internal cardinality $\leq |e|$. □

3. Loftiness and simple extensions

In this section we show that a countable model is lofty if and only if it has a recursively saturated simple extension. In fact the following sharper result holds.
3.1. Theorem. Let $\mathcal{M}$ be countable and $e \in M$. Then $\mathcal{M}$ is (bdd) $e$-lofty iff $\mathcal{M}$ has a (bdd) recursively saturated simple extension $\mathcal{N} = \mathcal{M}(b)$ with $b < e$. (The right-to-left direction does not require countability.)

Proof. ($\Leftarrow$) Suppose $\Sigma = \Sigma(v, a)$ is (bounded and) finitely satisfiable in $\mathcal{M}$, where $\Sigma$ is recursive. Then $\mathcal{N}$ realizes $\Sigma$, say by $\tau(b, c)$ with $e \in M$. Let $s = \{r(i, c): i \leq 2^e\}$; then $s$ is an internal subset of $\mathcal{M}$ of internal power at most $e$, and it is easy to see that $\Sigma(v, a) \cup \{v \in s\}$ is finitely satisfiable in $\mathcal{N}$ and hence in $\mathcal{M}$.

($\Rightarrow$) By Lemma 1.2, it suffices to find $\mathcal{N}(b)$, with $b < e$, which realizes every (bounded) recursive type over $\mathcal{M}$. By Theorem 2.8, we may choose $\langle d, n < \omega \rangle$ such that $d_n^{2^{n-1}} = e$ and $\mathcal{M}$ is $d_n$-lofty. For the moment, suppose $\langle \Sigma_n(v): n < \omega \rangle$ enumerates some (bounded) recursive types over $\mathcal{M}$. We may use a coding so that sequences $\langle a_i: i < n \rangle$ with $a_i < b_n$ are coded by numbers less than $\prod_{i < n} b_i$. In that case, notice that if $b$ codes $\langle a_i: i < n \rangle$, then

$$b^{2^n} \leq \prod_{i < n} d_i^{2^n} = \prod_{n < i} (d_i^{2^{n-1}})^{2^{n-1}} \leq \prod_{i < n} e^{2^{n-1-i}} < e^{2^n},$$

hence $b < e$. It follows by an easy compactness argument that there exists $\mathcal{N}(b)$ with $b < e$ such that $\mathcal{M}(b) \models \forall \Sigma_n((b)_n)$, for all $n < \omega$ such that $\Sigma_n(v) \cup \{v < d_n\}$ is consistent.

It suffices then to choose the types $\Sigma_n$ so that $\Sigma_n(v) \cup \{v < d_n\}$ is consistent for all $n < \omega$, and so that by realizing all $\Sigma_n$ in $\mathcal{N}(b)$, it should follow that every (bounded) recursive type over $\mathcal{M}$ is realized in $\mathcal{N}(b)$. Let $\langle \Gamma_n(v): n < \omega \rangle$ enumerate all (bounded) recursive types over $\mathcal{M}$. For each $n$ there exists $s_n$ of internal cardinality $d_n$ such that $\Gamma_n(v) \cup \{v \in s_n\}$ is consistent over $\mathcal{M}$. Choose $f_n \in \mathcal{M}$ so that $\mathcal{M} \models "f_n$ maps $[0, d_n)$ onto $s_n"$, and let $\Sigma_n(v) = \Gamma_n(f_n(v))$, all $n$. Then for all $n$ we have $\Sigma_n(v) \cup \{v < d_n\}$ is finitely satisfiable in $\mathcal{M}$, and the construction above guarantees that $\mathcal{N}(b)$ realizes each $\Sigma_n$ and hence each $\Gamma_n$. \hfill \Box

Next we prove similar characterizations of uniform and strong loftiness.

3.2. Theorem. Let $I$ be a proper cut of a countable model $\mathcal{M}$. Then $\mathcal{M}$ is (bdd) uniformly $I$-lofty iff $\mathcal{M}$ has a (bdd) recursively saturated simple extension $\mathcal{N}(b)$ such that $b < a$ for all $a \in M - I$.

Proof. The reverse direction has a similar proof to the corresponding direction of Theorem 3.1, so we omit it. Suppose $\mathcal{M}$ is (bdd) uniformly $I$-lofty. Let $\langle \Gamma_n(v): n < \omega \rangle$ enumerate all (bounded) recursive types over $\mathcal{M}$. For each $n$ choose $a_n \in M$ such that for all $e \in M - I$, $\Gamma_n(v) \cup \{\forall \theta < e(v = (a_n)_n)\}$ is finitely satisfiable in $\mathcal{M}$. We may assume that $I$ is closed under multiplication; otherwise $\mathcal{M}$ is $e$-lofty for some $e \in I$ (by Theorem 2.8), and we are done by Theorem 3.1. As in the proof of Theorem 3.1, the desired model is obtained by applying the compactness theorem, in this case, to obtain a model of the following theory:

$$\text{Th}(\mathcal{M}) \cup \{\Gamma_n((a)_n): n < \omega\} \cup \{b < e : e \in M - I\},$$
where $b$ is a new constant symbol whose interpretation will be the desired generator.

### 3.3. Theorem

Let $I$ be a cut in a countable model $\mathfrak{M}$. Then $\mathfrak{M}$ is $(\text{bdd})$ strongly $I$-lofty iff $\mathfrak{M}$ has a $(\text{bdd})$ recursively saturated extension $\mathfrak{N} = \mathfrak{M}(b_0, b_1, \ldots)$ such that for all $i < \omega$, $b_i \in I^\mathfrak{N}$.

**Sketch of proof.** The reverse direction follows as in 3.1. So does the forward direction, which is however somewhat easier in this case since there is no need to code the realizations of the appropriate types into a single generator.

### 3.4. Corollary

Suppose $\mathfrak{N}$ is a cofinal extension of $\mathfrak{M}$ and $\mathfrak{N}$ is $e$-lofty, uniformly $I$-lofty, or strongly $I$-lofty respectively. Then $\mathfrak{N}$ is $e$-lofty, uniformly $I^\mathfrak{N}$-lofty, or strongly $I^\mathfrak{N}$-lofty, respectively. (The corresponding bounded notions are similarly preserved.)

**Proof.** By a routine Löwenheim–Skolem argument, it suffices to consider only the case that $\mathfrak{M}$ and $\mathfrak{N}$ are countable. Suppose $\mathfrak{M}$ is $e$-lofty; the other cases follow similarly. By Theorem 3.1, $\mathfrak{M}$ has a simple extension $\mathfrak{M}(b)$ which is recursively saturated, where $b < e$.

By compactness there exists $\mathfrak{N}(b) > \mathfrak{M}(b)$. But then $\mathfrak{N}(b)$ is recursively saturated by the Smoryński–Stavi theorem (Corollary 1.3(i)). Hence $\mathfrak{N}$ is $e$-lofty, by the other direction of Theorem 3.1.

In Section 7 we raise some questions pertaining to uncountable models. Here is a preliminary result in that direction. (See also Section 5.)

### 3.5. Theorem

The following are equivalent. (We use the notation $\text{card}_{\mathfrak{M}}(e) = |\{x \in M : x < ^{\mathfrak{M}} e\}|$.)

(i) $\mathfrak{M}$ has an $\omega_1$-saturated simple extension.

(ii) Conditions (a) and (b) hold for some $e \in M$.

(a) For all countable $X \subseteq M$ there exists $a \in M$ such that $\{(a)_i : i < e\} \supseteq X$.

(b) $(\exists d)[(\text{card}_{\mathfrak{M}}(e))^\mathfrak{N}_d \leq \text{card}_{\mathfrak{M}}(d)]$.

(iii) Conditions (a') and (b') hold.

(a') For some $e$, every countable type over $\mathfrak{M}$ is consistent with $v \in s$ for some $s$ of internal cardinality $e$.

(b') $\forall c \exists a [(\text{card}_{\mathfrak{M}}(c))^\mathfrak{N}_d \leq \text{card}_{\mathfrak{M}}(a)]$.

**Proof.** (iii)$\Rightarrow$(ii). Assume (iii); say $e$ witnesses (a'). Let $d$ be any non-standard element; we show that (ii) holds for $d \cdot e$. Since (b) follows from (b'), we focus on (a). Given countable $X \subseteq M$, say $X = \{x_i : i < \omega\}$, let $\Sigma(v) = \{(v)_i = x_i : i < \omega\}$. By (a') we may choose $s$ of internal cardinality $e$ such that $\Sigma(v) \cup \{v \in s\}$ is consistent; say $s = \{(b)_i : i < e\}$. For $i < e$ and $j < d$ let $(c)_{d+i} = ((b)_i)_j$; then $\{(c)_k : k < d \cdot e\} \supseteq X$. 


(i) \implies (iii). Assume \( M(b) \) is \( \omega_1 \)-saturated. Then \( M(b) \) is a cofinal extension of \( M \) (by Proposition 1.6), so we may choose \( e \in M \) such that \( b < e \). Then (a') follows as in the proof of Theorem 3.1. For (b'), fix \( c \in M \). Since \( M(b) \) is \( \omega_1 \)-saturated, it codes every \( \omega \)-sequence from \( M(b) \) (hence from \( M \)), by Proposition 1.1. So for any non-standard \( k \in M \), if \( a_0 = e^k \), then \( (\text{card}_{M}(c))^\kappa_0 \leq \text{card}_{M(b)}(a_0) \). So it suffices to find \( a \in M \) such that \( \text{card}_{M}(a) \geq \text{card}_{M(b)}(a_0) \). To do this, first we apply the collection schema in \( M \) so that for each term \( (u,v) \) (without parameters) there is \( a_0 \) such that for all \( d \), there is \( d' < a_0 \) for which \( \min(\tau(d,i),a_0) = \min(\tau(d',i),a_0) \) for all \( i < e \). Since \( M(b) \) has uncountable cofinality, so does \( M \) so there exists non-standard \( a \in M \) with \( a > a_0 \) for all \( \tau(u,v) \). Notice that if \( z \in M(b) \), then \( z = \tau(d',b) \) for some \( d \in M \) and some \( \tau(u,v) \); so if \( z < a_0 \), then \( z = \tau(d',b) \) for some \( d' < a_0 \), by choice of \( d \). Hence

\[
\text{card}_{M}(a) \geq \left| \bigcup \{ \tau(d,b) : d < a, \tau \text{ a term} \} \right| \geq \text{card}_{M(b)}(a_0),
\]
as desired.

(ii) \implies (i). Fix \( e \) and \( d \) witnessing (ii). Let \( \{s^i : i < (\text{card}_{M}(e))^{\kappa_0}\} \) enumerate all \( \omega \)-sequences \( s^i = (s^i_n : n < \omega) \) of elements of \( M \) less than \( e \), and let \( \{a_i : i < (\text{card}_{M}(e))^{\kappa_0}\} \) be a one-one enumeration of a set of predecessors in \( M \) of \( d \). It follows from an easy use of compactness that there exists \( M(b) = (b)_{(a_n,n)} = s^i_n \) (for all \( i < (\text{card}_{M}(e))^{\kappa_0} \) and \( n < \omega \), with \( b < e^{d^2} \) (which is possible as long as we use a reasonable coding), so that \( M(b) \) is a cofinal extension of \( M \). It suffices to show that \( M(b) \) codes all \( \omega \)-sequences of elements of \( M \), by Lemma 1.2. Suppose \( \{d_n : n < \omega\} \subseteq M \). By hypothesis (a) we may choose \( a \in A \) such that \( \{(a)_n : i < e\} \supseteq \{d_n : n < \omega\} \). For each \( n < \omega \) choose \( i_n < e \) such that \( (a)_{i_n} = d_n \); then choose \( i \in M(b) \) such that \( (i)_n = i_n \) for all \( n < \omega \), by choice of \( M(b) \). Then \( (d_n : n < \omega) = ((a)_{(i)_n} : n < \omega) \), which is of course coded in \( M(b) \). \( \square \)

We will see in Corollary 4.20 that the model \( M \) need not be \( \omega_1 \)-saturated (or even recursively saturated) in order that \( M(b) \) be \( \omega_1 \)-saturated, at least if one assumes the continuum hypothesis.

4. Models with prescribed loftiness

Our purpose in this section is to construct models which show that the various notions of loftiness do not collapse.

4.1. Definition. The loftiness of \( M \) is \( L(M) = \{e \in M : \text{M is not e-lofty}\} \).

Recall that \( L(M) \) is an initial segment of \( M \) that is closed under multipication (Theorem 2.8). Our main theorems of this section, Theorem 4.11 and 4.16, are converses to this.
We have seen in Proposition 1.6 that there are models which are not bdd lofty. We begin this section with Lemma 4.4, which produces models that are tall but not lofty (see Corollary 4.6). First we recall the following work of MacDowell and Specker [10].

4.2. Definition. $\mathcal{R}$ is a conservative extension of $\mathcal{M}$ if $\mathcal{M} \preceq \mathcal{R}$ and for all $X \subseteq N$ which are definable with parameters in $\mathcal{R}$, $X \cap M$ is definable with parameters in $\mathcal{M}$. (Note: It follows that $\mathcal{R}$ is an end extension of $\mathcal{M}$.)

4.3. Lemma ([10]). Every model of PA has a conservative (end) extension. □

4.4. Lemma. Suppose $\mathcal{R}$ is a conservative extension of $\mathcal{M}$. Then $\mathcal{R}$ is not bdd uniformly $M$-lofty.

Proof. Suppose for a contradiction that $\mathcal{R}$ is bdd uniformly $M$-lofty. Fix a recursive enumeration $\{\tau_n(x): n < \omega\}$ of all terms, and fix $b \in N - M$. Consider the recursive type $\{\sigma_n(b, v): n < \omega\}$, where

$$\sigma_n(b, v) \equiv \forall i < b \left[ ((v)_i)_n = \tau_n(i) \right].$$

This type is consistent with $v < b^b$ so there is $s \in N$ such that for each $n < \omega$ there is $k_n \in M$ such that $\mathcal{M} \models \sigma_n(b, (s)_k)$. In $\mathcal{R}$ define

$$g(u) = \mu v \{v \notin ((s)_k)_u : k, n < u\}.$$

Clearly, if $u \in M$, then $g(u) \in M$. Therefore, there are $c \in M$ and term $\tau(c, u)$ such that $\tau(c, u) = g(u)$ for all $u \in M$. In fact, we can arrange so that there are arbitrarily large $d \in M$ such that $\tau(d, u) = \tau(c, u)$. Let $\tau_n(x) = \tau(x, x)$. Pick $d > k_n$ so that $\tau(d, u) = \tau(c, u)$ for all $u \in M$. Then $g(d) = \tau_n(d)$, but $\tau_n(d) = ((s)_k)_d$ and $k_n, n < d$. Contradiction. □

In particular we have reproved the following observation of Kotlarski. Notice that Theorem 3.3 and Corollary 4.5 together imply Lemma 4.4 with the word ‘uniformly’ replaced by ‘strongly’.

4.5. Corollary. No proper conservative extension of a model of PA is recursively saturated. □

Notice that by using Lemma 4.3 to iterate conservative extensions of a given model $\omega$ times (or even $\alpha$ times, for any limit ordinal $\alpha$), one obtains a tall model which, by Lemma 4.4, is not lofty. This proves:

4.6. Corollary. Every model has a tall conservative extension, of any cofinality, which is not lofty. □
To obtain models with prescribed loftiness, we will use the following combinatorial notion and two pertinent lemmas.

4.7. **Definition.** Let $e$, $i$, and $j$ be elements of the model $M$ and let $C$ be a parametrically definable subset of $M$. (We think of $C$ as consisting of codes of sequences of length $e$.) $C$ is $(i, j)$-rich (for $M$ and $e$) if $j \geq 1$ and for every $d \in M$, there is $c \in C$ such that $(d)_k = (c)_{i+kj}$ whenever $i + kj < e$.

4.8. **Lemma.** Suppose that $C$ is $(i, j)$-rich for $M$ and $e$, and that $f : C \rightarrow [0, a)$ is parametrically definable in $M$. Then for some $b < a$, $f^{-1}(b)$ is $(i + bj, aj)$-rich (for $M$ and $e$).

**Proof.** Suppose not; then for all $b < a$, $f^{-1}(b)$ is not $(i + bj, aj)$-rich, i.e. there is $d_b$ for which there is no $c \in f^{-1}(b)$ such that $(d_b)_k = (c)_{i+bj+ka}$ whenever $i + bj + ka < e$. Choose $d \in M$ so that $(d)_{b+ka} = (d_b)_k$ whenever $i + bj + ka < e$ and $b < a$. Since $C$ is $(i, j)$-rich, there is $c \in C$ such that $(d)_{b+ka} = (c)_{i+(b+ka)}$ whenever $i + (b + ka) < e$. So whenever $b < a$ and $i + bj + ka < e$, $(d_b)_k = (c)_{i+bj+ka}$. If we set $b = f(c)$, then this contradicts the choice of $d_b$. □

In 4.9 through 4.13 we construct models $M$ with prescribed loftiness $I$ such that $M$ is uniformly $I$-lofty. A parallel treatment of the nonuniform case appears in 4.14 through 4.17.

4.9. **Lemma.** Suppose $C$ is $(i, j)$-rich for $M$ and $d$, and suppose $J$ is a proper cut of $M$. Let $\tau(x)$ be any term with parameters in $M$, and let $e \in \text{cf}^M(J)$. Then for some $k \leq e$, there is an $(i + kj, (e + 1)j)$-rich subclass $C'$ of $C$ such that for some $p \in J$ and $q \in M - J$, $\{(\tau(c))_m : m < e\} \cap [p, q] = \emptyset$ for all $c \in C'$.

**Proof.** A sequence $\langle a_k : k \leq e + 1 \rangle$ may be defined internally as follows. Set $a_0 = 0$. Given $a_k$, choose $a_{k+1}$ to be the least such that $\{c \in C : \{(\tau(c))_m : m < e\} \cap [a_k, a_{k+1}) = \emptyset\}$ is not $(i + kj, (e + 1)j)$-rich. (If no such $a_{k+1}$ exists, then set $a_{k+1} = \infty$, and in fact set $a_i = \infty$ whenever $k < l \leq e + 1$.)

For the moment suppose $a_l \notin J$ for some $l \leq e + 1$. Since $e \in \text{cf}^M(J)$, there is $k < l$ such that $a_k \in J$ and $a_{k+1} \notin J$. By the minimality of $a_{k+1}$, we see that if $p = a_k$ and $a_{k+1} > q + 1 \notin J$, then the class $\{c \in C : \{(\tau(c))_m : m < e\} \cap [p, q] = \emptyset\}$ is $(i + kj, (e + 1)j)$-rich, and the conclusion of the lemma follows.

Hence we may assume that $a_l \in J$ for all $l \leq e + 1$. Partition $C$ into $e + 1$ pieces as follows: $f(c)$ is the least $k \leq e$ such that $\{(\tau(c))_m : m < e\} \cap [a_k, a_{k+1}) = \emptyset$. Notice that $f(c) \leq e$ by the pigeon-hole principle. By Lemma 4.8 there exists $k \leq e$ such that $f^{-1}(k)$ is $(i + kj, (e + 1)j)$-rich. Hence, $\{c \in C : \{(\tau(c))_m : m < e\} \cap [a_k, a_{k+1}) = \emptyset\}$ is $(i + kj, (e + 1)j)$-rich. This contradicts the definition of $a_{k+1}$. □

4.10. **Lemma.** Let $I \subseteq J$ be proper cuts of a countable model $M$ such that $\text{cf}^M(J) \geq I$
and $I$ is closed under multiplication. Then there exists countable $N > 1$ $\mathcal{M}$ such that $\mathcal{M}$ fills neither $I$ nor $J$, $\text{cf}^\mathcal{M}(J^\mathcal{M}) \supseteq I^\mathcal{M}$, and for some $b \in N$, $\{(b)_k : k \in I\} \supseteq M$.

**Proof.** We will define a descending sequence of parametrically definable classes $C_n (n < \omega)$ such that $\mathcal{M}$ can be formed as $\mathcal{M}(b)$, where $b$ is the type determined by the classes $C_n$; that is, $\varphi(b)$ will hold iff for some $n$, $\varphi(x)$ holds for all $x \in C_n$. Let $\{(a_n, \varphi_n, \tau_n) : n < \omega\}$ enumerate all triples $(a, \varphi, \tau)$ with $a \in M$, $\varphi(x)$ a formula with parameters from $M$, and $\tau(x)$ a term with parameters from $M$. Fix $d \in M - I$. Set $C_0 = M$, which is $(0, 1)$-rich for $\mathcal{M}$ and $d$. We define $C_{n+1}$ from $C_n$ under the inductive hypothesis that $C_n$ is $(i, j)$-rich for $\mathcal{M}$ and $d$, for some $i, j \in I$ depending on $n$.

First, to guarantee the last conclusion, let $C^0_n = \{c \in C_n : (i, j) = a_n\}$; then $C^0_n$ is $(i_0, j_0)$-rich, where $i_0 = i + j \in I$ and $j_0 = j \in I$.

Second, apply Lemma 4.8 to obtain $i_1, j_1 \in I$ and $C^1_n \subseteq C^0_n$ such that $C^1_i$ is $(i_1, j_1)$-rich and either $\mathcal{M} \vdash \varphi(x)$ for all $x \in C^1_i$ or $\mathcal{M} \vdash \lnot \varphi(x)$ for all $x \in C^1_i$. This can be accomplished by setting $j_1 = 2j_0$ and either $i_1 = i_0$ or $i_1 = i_0 + j_0$. This step will guarantee that the type we are building is complete.

Third, we guarantee that $\tau_n(x)$ does not define a new element below $a_n$ (unless $a_n \notin I$ in which case set $i_2 = i_1, j_2 = j_1$, and $C^2_n = C^1_n$). Partition $C^1_n$ into $a_n + 1$ pieces as follows: $f(x) = \min(\tau_n(x), a_n)$. By Lemma 4.8 we may choose $i_2, j_2$, and $C^2_n \subseteq C^1_n$ such that $i_2 \leq i_1 + a_n j_1 \in I$, $j_2 = (a_n + 1) j_1 \in I$, and $C^2_n = f^{-1}\{b\}$ for some $b \leq a_n$, where $C^2_n$ is $(i_2, j_2)$-rich. Notice that if $b < a_n$, then $\tau_n(x) = b$ for all $x \in C^2_n$, while if $b = a_n$, then $\tau_n(x) > a_n$ for all $x \in C^2_n$.

Our fourth step is to ensure that $I$ is not filled in the extension. (The same argument allows us to guarantee that any given proper cut of $\mathcal{M}$ is not filled in the extension.) Let $h$ be the least such that $\{x \in C^2_n : \tau_n(x) \leq h\} = (i_2, 2j_2)$-rich. (If no such $h$ exists, choose $h \in M - I$ arbitrarily.) If $h \in I$, set $C^3_n = \{x \in C^2_n : \tau_n(x) \leq h\}$. If $h \notin I$, then set $C^3_n = \{x \in C^2_n : \tau_n(x) \geq h\}$ so that $C^3_n$ is $(i_2 + j_2, 2j_2)$-rich by Lemma 4.8. Set $(i_3, j_3) = (i_2, 2j_2)$ in the former case, and $(i_3, j_3) = (i_2 + j_2, 2j_2)$ in the latter case.

Finally, we want to guarantee that $\text{cf}^\mathcal{M}(J^\mathcal{M}) \supseteq I$, and that $J$ is not filled. Suppose $a_n \in I$ (otherwise, set $C_{n+1} = C_n$). An application of Lemma 4.9 yields $p \in J$, $q \in M - J$ along with $k \leq a_n$ and a class $C_{n+1} \subseteq C^3_n$ such that $C_{n+1}$ is $(i_3 + k j_3, (a_n + 1) j_3)$-rich, and for all $c \in C_{n+1}$ and $m < a_n$, $(\tau_n(c))_m \notin [p, q]$. Notice that $i_3 + k j_3, (a_n + 1) j_3 \in I$.

This completes the construction of $\langle C_n : n < \omega\rangle$.

Let $\Sigma(v) = (\varphi_n(v), \varphi_n(c))$ for all $c \in C_{n+1}$, and $n < \omega$. Since each $C_n$ is nonempty, $\Sigma(v)$ is complete over $\mathcal{M}$ and 'contains' all the decisions from the five steps in the construction above. For example, for all $n$ there exist $p \in J$ and $q \in M - J$ such that a formula expressing that $\{(\tau_n(v))_m : m < a_n\} \cap [p, q] = \emptyset$ is in $\Sigma(v)$. This guarantees that if $\mathcal{M}(b) > \mathcal{M}$, where $b$ realizes $\Sigma(v)$ in $\mathcal{M}(b)$, then $\mathcal{M}(b)$ does not fill $J$ and $\{(\tau_n(b))_m : m < a_n\} \cap J^\mathcal{M}(b)$ is not cofinal in $J^\mathcal{M}(b)$. The rest of the details showing that $\mathcal{M}(b)$ is the desired model are left to the reader. $\square$
2.6. Lemma. Suppose that for every proper recursively definable cut \( I \) of \( \mathbb{M} \), \( e \notin \text{cf}^\mathbb{M}(I) \). Then \( \mathbb{M} \) is bdd e-lofty.

Proof. Suppose \( \Sigma(v, a_0, a_1) \cup \{v < a_0\} \) is recursive and finitely satisfiable in \( \mathbb{M} \), where \( a_0 \) is nonstandard. Choose \( g, \tau \) and \( I \) as given by Lemma 2.4. Then property 2.4(i) guarantees that each element of \( \Gamma(v, \bar{a}, a_0) \) has a witness, so that \( \Gamma(v, \bar{a}, a_0) \) is finitely satisfiable (by definition of interval set). Since each formula in \( \Gamma(v, \bar{a}, a_0) \) defines a closed interval, there are two possibilities only: either \( \Gamma \) recursively defines a proper cut \( I \), or \( \Gamma \) is realized. By hypothesis, in the former case there is an internal set \( s \) of internal power \( e \) with \( s \cap I \) cofinal in \( I \). It follows that for all \( \gamma \in \Gamma(v, \bar{a}, a_0) \) there is \( b \in s \) with \( \mathbb{M} \models \gamma(b, \bar{a}, a_0) \). Let \( s' \) be the range of \( \tau(x, a_0, a_0) \) on \( s \). It follows from 2.4(ii) that \( \Sigma(v, \bar{a}) \cup \{v \in s'\} \) is finitely satisfiable in \( \mathbb{M} \), and this completes the proof since \( \mathbb{M} \models |s'| \leq |s| = e \). \( \square \)

The following theorem can be proved in much the same manner as was Lemma 2.6.

2.7. Theorem. (i) There are no definable (resp., recursively definable) cuts in \( \mathbb{M} \) iff \( \mathbb{M} \) is \( \omega \)-saturated (resp., recursively saturated).

(ii) ([11]). For any model \( \mathbb{M} \) of \( \text{PA} \) and any cardinal \( \kappa > \omega \), if \( (M, \lt) \) is \( \kappa \)-saturated, then \( \mathbb{M} \) is \( \kappa \)-saturated.

Proof. We leave the proof of (i) to the reader who followed the proof of lemma 2.6. To prove (ii) one uses an induction on \( \kappa \). The proof of the base step \( \kappa = \omega_1 \) is similar to (i) and is also left to the reader. The case when \( \kappa \) is a limit cardinal is trivial. So we indicate the proof for the successor step \( \kappa = \lambda^+ \). Let \( \Sigma = \{\sigma_\alpha(v) : \alpha < \lambda\} \) be a consistent set of formulas, closed under finite conjunction, where \( \sigma_0(v) \) is \( v < c \) for some \( c \in M \). By an induction on \( \alpha \) we will construct a decreasing sequence of nonempty internal sets \( s_\alpha(\alpha < \lambda) \) with \( s_\alpha \subseteq \{x : \mathbb{M} \models \sigma_\alpha(x)\} \). Moreover, we require that \( s_\alpha \cap \{x : \mathbb{M} \models \sigma_\alpha(x)\} \neq \emptyset \) for all \( \gamma < \lambda \). Let \( s_0 = \{x : \mathbb{M} \models \sigma_0(x)\} \) and \( s_{\alpha+1} = s_\alpha \cap \{x : \mathbb{M} \models \sigma_{\alpha+1}(x)\} \). For \( \alpha \) a limit ordinal, consider the following consistent set of formulas in the variables \( v \) and \( v_{\bar{a}} (\beta < \alpha) \):

\[
\{(v)_{v_{\bar{a}}} = s_{\beta} : \beta < \alpha\} \cup \{(v_{\bar{a}} < v_{\bar{a}'} : \beta < \alpha', \forall x \forall y (x < y \rightarrow (v)_{v_{\bar{a}}'}) \}
\]

There exist \( a, a_0, a_1, \ldots, a_{\bar{a}}, \ldots \) satisfying this set in \( \mathbb{M} \), since \( \mathbb{M} \) is \( \lambda \)-saturated by the inductive hypothesis. Consider the cut \( I = \{x \in M : x < a_{\bar{a}} \} \) for some \( \beta < \alpha \). It follows by overspill that for all \( \gamma < \lambda \) there exists \( b \in M - I \) such that

\[
\mathbb{M} \models \exists z \{z \in (a_{\bar{a}})_b \}.
\]

By the \( \lambda^+ \)-saturation of \( (M, \lt) \) \( I \) has downward cofinality at least \( \lambda^+ \), so there exists \( b \in M - I \) which satisfies (*) for all \( \gamma < \lambda \). Let \( s_\alpha = (a)_{\bar{a}} \).

Finally, the argument of Lemma 2.4, together with the \( \lambda \)-saturation of \( \mathbb{M} \), transfers the type \( \{v \in s_\alpha : \alpha < \lambda\} \) to a type \( \Gamma \) whose formulas define intervals. The
Remark. It is not difficult to construct lofty models of arbitrary cardinality that are not recursively saturated. (But see Question 7.1.) Take any countable lofty model $\mathcal{M}$ that is not recursively saturated, by Theorem 4.11. In fact there exists a recursively definable cut $J$ of $\mathcal{M}$, by Theorem 2.7(i). The result now follows from Theorem 6 of Paris–Mills [13], together with Corollary 3.4. In fact, these lofty models have cofinality $\omega$. Hence these models have recursively saturated simple extensions. This can be seen easily by applying Theorem 3.1 to a countable cofinal submodel, and then invoking the Smoryński–Stavi theorem (Corollary 1.3(i)).

The following corollary has the same proof as Theorem 4.11 but with an $\omega_1$-sequence of models, taking unions at limit ordinals.

4.13. Corollary. Let $\mathcal{M}$ be countable and $I$ a cut of $\mathcal{M}$ which is closed under multiplication. Then there is $\mathcal{N}>^I\mathcal{M}$ such that $\mathcal{N}$ has cofinality $\omega_1$, $\mathcal{N}$ does not fill $I$, $L(\mathcal{N})=I$, and $\mathcal{N}$ is uniformly $I$-lofty.

We now turn our attention to the nonuniform case. We will use the following ad hoc definition. If $\mathcal{M}$ is a model and $I$ and $J$ are cuts of $\mathcal{M}$, then we say that $\text{cf}^\mathcal{M}(J) \supseteq I$ nonuniformly provided $J$ is proper and there is no $b \in M$ such that $\{(b)_i : i \in I\} \cap J$ is cofinal in $J$.

4.14. Lemma. Suppose $C$ is $(i, j)$-rich for $\mathcal{M}$ and $d$, and suppose that $I$ and $J$ are proper cuts such that $\text{cf}^\mathcal{M}(J) \supseteq I$ nonuniformly and $I$ is closed under multiplication. Let $\tau(x)$ be any term with parameters in $M$, and let $r \in M-I$. Then there are $k \leq e \leq r$, with $e \in M-I$, and an $(i+kj, (e+1))$-rich subclass $C'$ of $C$ such that for some $p \in J$ and $q \in M-J$, $\{(\tau(c))_m : m < e\} \cap [p, q] = \emptyset$ for all $c \in C'$.

Proof. For each $b \leq r$ a sequence $\langle a_{b,k} : k \leq b+1 \rangle$ may be defined internally as follows. Set $a_{b,0} = 0$. Given $a_{b,k}$, choose $a_{b,k+1}$ to be the least such that $\{c \in C : \{(\tau(c))_m : m < b\} \cap \{a_{b,k}, a_{b,k+1}\} = \emptyset\}$ is not $(i+kj, (b+1))$-rich. (If there is no such $a_{b,k+1}$, then set $a_{b,1} = \infty$ whenever $k+1 \leq l \leq b+1$.) Consider the doubly-indexed sequence $\langle a_{b,k} : k \leq b+1 \in I \rangle$. For each $b \in I$, the set $\{a_{b,k} : k \leq b+1\}$ has internal cardinality in $I$, because $I$ is closed under multiplication by hypothesis. Since $\text{cf}^\mathcal{M}(J) \supseteq I$ nonuniformly, there are $p \in J$ and $q \in M-J$ such that $\{a_{b,k} : k \leq b+1 \in I\} \cap [p, q] = \emptyset$. By overspill, there exists $e \in M-I$, $e \leq r$, such that $\{a_{e,k} : k \leq e+1\} \cap [p, q] = \emptyset$.

For the moment, suppose $a_{e,l} \notin J$ for some $l \leq e+1$. Then there is $k < l$ such that $a_{e,k} < p$ and $q < a_{e,k+1}$. By the minimality of $a_{e,k+1}$, we see that the class $\{c \in C : \{(\tau(c))_m : m < e\} \cap [p, q] = \emptyset\}$ is $(i+kj, (e+1))$-rich, and the conclusion of the lemma follows.

Hence, we may assume that $a_{e,l} \in J$ for all $l \leq e+1$. But then a contradiction follows the same way it did in the proof of Lemma 4.9.
4.15. **Lemma.** Let $I$, $J$, $K$ be proper cuts of a countable model $\mathcal{M}$ such that $\text{cf}^\mathcal{M}(J) \supseteq I$ nonuniformly and $I$ is closed under multiplication. Suppose that whenever $k \in K$ then there is $i \in K - I$ such that $ik \in K$. Then there exists countable $\mathcal{N} >^1 \mathcal{M}$ such that $\mathcal{N}$ fills neither $I$ nor $J$, $\text{cf}^\mathcal{N}(J^\mathcal{N}) \supseteq I$ nonuniformly, and for some $b \in \mathbb{N}$, \{\{(b)_k : k \in K\} \supseteq M\}.

**Proof.** Proceed in the manner of the proof of Lemma 4.10, defining a decreasing sequence $\langle C_n : n < \omega \rangle$ of parametrically definable classes which will determine the type of $b$, and then set $\mathcal{N} = \mathcal{M}(b)$. Fix $d \in M - K$. Set $C_0 = M$, and then define $C_{n+1}$ from $C_n$ under the inductive hypothesis that $C_n$ is $(i, j)$-rich for $\mathcal{M}$ and $d$, for some $i, j \in K$ depending on $n$. The first, second, third and fourth steps are the same as in the proof of Lemma 4.10 except that $i, i_0, \ldots, j_2, j_3$ are now in $K$. The fifth step is replaced by the following step, which is intended to guarantee that $\text{cf}^\mathcal{N}(J^\mathcal{N}) \supseteq I$ nonuniformly.

By the given condition on $I$ and $K$, there is $r \in K - I$ such that $j_3 r \in K$. An application of Lemma 4.14 yields $p \in J$ and $q \in M - J$ along with $k < e < r$ and a parametrically definable $C_{n+1} \subseteq C_n$ such that $e \in K - I$, $C_{n+1}$ is $(i_3 + k j_3, (e + 1) j_3)$-rich, and for all $c \in C_{n+1}$ and $m < e$, $(\tau_n(c))_m \notin [p, q]$. Notice that $i_3 + k j_3, (e + 1) j_3 \in K$.

This completes the construction of $\langle C_n : n < \omega \rangle$. The proof that $\mathcal{N}(b)$ is the desired model is left to the reader. □

4.16. **Theorem.** Let $\mathcal{M}$ be countable and $I$ a cut of $\mathcal{M}$ which is closed under multiplication. Then there is a countable $\mathcal{N} >^1 \mathcal{M}$ such that $\mathcal{N}$ does not fill $I$, $L(\mathcal{N}) = I$, and $\mathcal{N}$ is not uniformly $I$-lofty.

**Proof.** First, we find a decreasing sequence $\langle K_n : n < \omega \rangle$ of cuts of $\mathcal{M}$ such that $\bigcap \{K_n : n < \omega\} = I$ with the property that whenever $k \in K_n$ then there is $i \in K_n - I$ such that $ik \in K_n$. To obtain such a sequence, let $\langle d_n : n < \omega \rangle$ be a decreasing sequence of elements of $M - I$ converging to $I$ such that $d_{n+1} < d_n$. (Notice that there is such a sequence since $I$ is closed under multiplication.) Then let $K_n = \{x \in M : x^{m+1} < d_m^m \text{ for some } m < \omega\}$. Next, as in the proof of Theorem 4.11, there is a countable end extension $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M}_0$ has a recursively definable cut $J_0$ for which $\text{cf}^\mathcal{M}_0(J_0) \supseteq I$. (In fact it suffices that $\text{cf}^\mathcal{M}(J_0) \supseteq I$ nonuniformly.) Construct an elementary chain $\mathcal{M}_0 < \mathcal{M}_1 < \mathcal{M}_2 < \cdots$ of models, where $\mathcal{M}_{n+1}$ is obtained from $\mathcal{M}_n$ by applying Lemma 4.15 to $\mathcal{M}_n$ with $J = J_0^\mathcal{M}_n$, and $K = K_\mathcal{M}$. The reader will easily verify that $\mathcal{N} = \bigcup \{\mathcal{M}_n : n < \omega\}$ is as desired. □

4.17. **Theorem.** Let $\mathcal{M}$ be a countable model and $I$ a cut of $\mathcal{M}$ which is closed under multiplication. Then there is $\mathcal{N} >^1 \mathcal{M}$ such that $\mathcal{N}$ has cofinality $\omega_1$, $\mathcal{N}$ does not fill $I$, $L(\mathcal{N}) = I$, and $\mathcal{N}$ is not uniformly $I$-lofty.

**Proof** (Sketch). This follows from a slight strengthening of Theorem 4.16. Roughly, iterate Theorem 4.16 $\omega_1$ times, obtaining models $\mathcal{M}_\alpha$ ($\alpha < \omega_1$). Added
care is taken to ensure that for a fixed recursively definable cut $J_0$ of $\mathcal{M}_0$ (as in the proof of 4.16), $\text{cf}^{\mathcal{M}_n}(J_0^{\mathcal{M}_n}) \geq I$ nonuniformly for all $\alpha < \omega_1$. □

What ever became of strong loftiness?

4.18. Theorem. Let $\mathcal{M}$ be countable and $I$ a cut that is the union of proper initial segments of $I$ which are closed under multiplication. Then there is $\mathcal{N} > \mathcal{M}$ such that $\mathcal{N}$ does not fill $I$, $L(\mathcal{N}) = I^\mathcal{M}$, and $\mathcal{N}$ is strongly $I^\mathcal{N}$-lofty.

Before proving the theorem we make two remarks about it. First, the requirement that $I$ is the union of proper initial segments of $I$ which are closed under multiplication is necessary. For, suppose that $L(\mathcal{M}) = I$, $\mathcal{M}$ is strongly $I$-lofty, and $e \in I$. Since $\mathcal{M}$ is not $e$-lofty, there is a recursively definable cut $J$ of $\mathcal{M}$ such that $e \in \text{cf}^\mathcal{M}(J)$, by Theorem 2.6. According to Lemma 2.9, $\text{cf}^\mathcal{M}(J)$ is closed under multiplication. Since $\mathcal{M}$ is strongly $I$-lofty, $\text{cf}^\mathcal{M}(J) \not\subseteq I$.

The second remark is that the conclusion of Theorem 4.18 cannot be strengthened so that $\mathcal{N} > I^\mathcal{M}$. For, let $\mathcal{M}$ be an end extension of the minimal model $\mathcal{M}_0$ (which is not the standard model) of its theory, and let $I = M_0$. Suppose $\mathcal{N} > I^\mathcal{M}$ and $\mathcal{N}$ is strongly $I$-lofty. We will obtain a contradiction. Let $\Sigma(v) = \{(v)_n = \tau_n : n < \omega\}$, i.e. $(v)_n$ is the $n$th definable element in some effective enumeration of the terms. Choose $e \in M_0$ such that $\Sigma(v) \cup \{v \in e\}$ is finitely satisfiable in $\mathcal{N}$ for some $s$ of internal cardinality $e$. Then the set $\{(x)_s : x \in s, i < e\}$ has internal cardinality at most $e^2 \in M_0$; yet, this set includes $M_0$, so, in particular, has more than $e^2$ elements.

Proof of Theorem 4.18. We will use a slight strengthening of Theorem 4.11. Suppose we add to the hypothesis of that theorem that $I_0, I_1, \ldots, I$ are cuts of $\mathcal{M}$ such that $\text{cf}^\mathcal{M}(J_i) \subseteq I$ for each $j < s$. Then we can add to the conclusion that $\mathcal{N}$ does not fill $I_j$ for each $i < r$, and that $\text{cf}^{\mathcal{M}}(J_i^{\mathcal{N}}) = \text{cf}^\mathcal{M}(J_i)$ for each $j < s$. We can obtain the first additional conclusion because Lemma 4.10 can be similarly strengthened as was hinted at by the parenthetical comment in the proof of that lemma. The second conclusion can be obtained by applying Lemma 4.9 a few more times.

Let $I = \bigcup \{I_n : n < \omega\}$, where $\langle I_n : n < \omega \rangle$ is a strictly increasing sequence of cuts of $\mathcal{M}$, each closed under multiplication. Using the strengthened form of Theorem 4.11, construct an elementary chain $\mathcal{M} = \mathcal{M}_0 < \mathcal{M}_1 < \mathcal{M}_2 < \cdots$ and (using the remark following Theorem 4.11) $J_n$ of $\mathcal{M}_n$, such that the following hold for each $n < \omega$:

1. $\mathcal{M}_n < I_n^{\mathcal{M}_n}$;
2. $\mathcal{M}_{n+1}$ does not fill $I_n^{\mathcal{M}_n}$;
3. $L(\mathcal{M}_{n+1}) = I_n^{\mathcal{M}_n}$;
4. $\text{cf}^{\mathcal{M}_{n+1}}(J_n^{\mathcal{M}_{n+1}}) = I_n^{\mathcal{M}_n}$;
5. for $m < n$, $\text{cf}^{\mathcal{M}_{n+1}}(J_{m+1}^{\mathcal{M}_{n+1}}) = \text{cf}^{\mathcal{M}_{n}}(J_{m}^{\mathcal{M}_{n}})$;
6. for $m < n$, $\mathcal{M}_{n+1}$ does fill $J_{m+1}^{\mathcal{M}_{n+1}}$.

Clearly, the model $\mathcal{N} = \bigcup \{\mathcal{M}_n : n < \omega\}$ is as desired. □
In light of Section 3, we can interpret some of the results of this section in terms of the existence of recursively saturated simple extensions.

4.19. **Corollary.** There are countable models $\mathcal{M}$ of any completion of PA with any one of the following properties:

(i) $\mathcal{M}$ has a cofinal, recursively saturated extension, but has none which is a simple extension.

(ii) For some nonstandard $e \in M$, $\mathcal{M}$ has a recursively saturated, simple extension $\mathcal{M}(b)$, but has none with $b < e$.

(iii) For all nonstandard $e \in M$, $\mathcal{M}$ has a recursively saturated, simple extension $\mathcal{M}(b)$ with $b < e$, but has none in which $b$ fills the standard cut.

(iv) $\mathcal{M}$ has a recursively saturated simple extension $\mathcal{M}(b)$ in which $b$ fills the standard cut, yet $\mathcal{M}$ is not recursively saturated.

**Proof.** Any tall model which is not recursively saturated satisfies (i), while (ii) follows from Theorems 3.1 and 4.11. Similarly, part (iii) follows from Theorems 3.2 and 4.16. Finally (iv) follows from Theorem 3.2 and 4.11. \(\square\)

In fact, by using the models constructed in the proof of Corollary 4.13, we can strengthen (iv), using Theorem 3.5.

4.20. **Corollary (CH).** Every complete extension of PA has a model which is not recursively saturated yet has an $\omega_1$-saturated simple extension. \(\square\)

5. **Cofinal extensions and $\kappa$-saturation**

Kotlarski [8] asked, generalizing Corollary 1.3(iii), if a simple cofinal extension of a $\kappa$-saturated model of PA must also be $\kappa$-saturated. He had already shown a very special case: if $\mathcal{R}$ is saturated, then $\mathcal{R}$ has a proper saturated simple extension. In general, his question has a negative answer as we will see in this section. It turns out that nearly all questions of this sort can be answered using Keisler's work on good ultrafilters. The bridge between simple extensions and ultrapowers is the following proposition.

5.1. **Proposition.** Suppose $\mathcal{R}$ is $\kappa$-saturated and $D$ is an ultrafilter over $I$, where $|I| < \kappa$. Then $\mathcal{R}^I/D$ is a simple, cofinal extension of $\mathcal{R}$.

**Proof.** For any function $f : I \to N$, let $\hat{f}$ denote the equivalence class of $N^I/D$ to which $f$ belongs. Identify $a \in N$ with $\hat{f}_a$, where $f_a : I \to N$ is the constant function with range $\{a\}$. Now let $g : I \to N$ be any one–one function, and let $f : I \to N$ be any function. By the $\kappa$-saturation of $\mathcal{R}$, there is an element $a \in N$ such that
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(a)\(g(i) = f(i)\) for each \(i \in I\). Thus, in \(\mathfrak{N}/D\), \((a)_{g_i} = \hat{f}\). Thus, \(\mathfrak{N}/D = \mathfrak{N}(\hat{g})\), so the extension is simple. To see that it is cofinal, use the \(\kappa\)-saturation of \(\mathfrak{N}\) to find \(b \in \mathbb{N}\) such that \(b > g(i)\) for each \(i \in I\). Then \(b > \hat{g}\) in \(\mathfrak{N}/D\). □

The reader can verify that if one takes \(I = \omega\) in Proposition 5.1, then the standard cut in \(\mathfrak{N}/D\) has downward cofinality at most \(2^{\aleph_0}\), and hence \(\mathfrak{N}/D\) is not \((2^{\aleph_0})^+\)-saturated. By Proposition 5.1 this gives a negative answer to Kotlarski's question. However, we prove a sharper result in Corollary 5.7 below.

We will need the notion of a \(\kappa\)-good ultrafilter. (See Chapter VI of [2].) Keisler [5] proved that such ultrafilters produce \(\kappa\)-saturated ultrapowers.

5.2. Lemma (Keisler). Suppose \(\mathfrak{N}\) is any structure and that \(D\) is a countably incomplete, \(\kappa\)-good ultrafilter over \(I\). Then \(\mathfrak{N}/D\) is \(\kappa\)-saturated. □

The existence of these ultrafilters was shown by Keisler using the GCH. Later Kunen eliminated the need for any additional set-theoretic hypothesis.

5.3. Lemma (Kunen [9]). For each \(\kappa\) there is a countably incomplete, \(\kappa^+\)-good ultrafilter over \(\kappa\) (which, consequently, is not \(\kappa^{++}\)-good).

Keilser [6] proved that under certain conditions there is a converse to Lemma 5.2. The following lemma can be proved by making only a minor adjustment to the proof of Theorem 3.4 of [6].

5.4. Lemma. If \(\mathfrak{N}\) is a model of \(PA\) with \(|\mathbb{N}| \geq \kappa\) and \(D\) is a countably incomplete ultrafilter over \(I\) such that \(\mathfrak{N}/D\) is \(\kappa^+\)-saturated, then \(D\) is \(\kappa^+\)-good.

Proof (Hint). Proceed as in the proof of Theorem 3.4 of [6]. The formula \(v_0 \in W_v\), asserting "\(v_0\) is in the \(v_1\)th \(\Sigma_1\)-set" is the key. First notice that the formula is versatile. For if \(T\) is a weak ideal over \(\{1, 2, \ldots, n\}\), then let \(a_1, a_2, \ldots, a_n\) be distinct elements such that \(W_{a_i} = \{t \in T : i \in t\}\). Then for \(t \subseteq \{1, 2, \ldots, n\}\), \(\mathfrak{N} \vDash \exists v_0 \land_{i \in t} (v_0 \in W_{a_i})\) iff \(t \in T\). Next, notice that the formula \(v_0 \in W_{a_t}\) satisfies (*) in Lemma 3.4a of [6]. Let \(n < \omega\), and let \(f : \mathbb{N} \to \mathbb{N}\) be the definable function such that

\[f(x) = \mu y[\forall s (s \in W_y \iff |s| = n \land x \in s)].\]

Let \(Y\) be the range of \(f\), so that \(|Y| \geq k\). Clearly, for \(t \subseteq Y\), \(\mathfrak{N} \vDash \exists v_0 \land_{y \vDash t} v_0 \in W_y\) iff \(|t| \leq n\). The remainder of the proof of Theorem 5.4 is the same as the proof of Theorem 3.4 in [6]. □

5.5. Lemma. Let \(\mathfrak{M}\) be a \(\lambda^+\)-saturated model of \(PA\) and \(\mathfrak{N} > \mathfrak{M}\) a cofinal extension generated over \(\mathfrak{M}\) by no more than \(\lambda\) elements. Then \(\mathfrak{N}\) has a simple, cofinal \(\lambda^+\)-saturated extension which is not \(\lambda^{++}\)-saturated.
Proof. Let \( I \subseteq M \) have cardinality \( \lambda \), and let \( \{ b_i : i \in I \} \) be a set of generators for \( N \) over \( M \). Let \( N(b) \) be a simple, cofinal extension of \( N \) such that for each \( i \in I \), \( N(b) \models (b)_i = b_i \). Thus, \( N(b) = M(b) \).

Using Lemma 5.3 let \( D \) be a countably incomplete, \( \lambda^+ \)-good ultrafilter over \( I \). Let \( f : I \rightarrow I \) be the identity function, and let \( \hat{f} \) be the equivalence class of \( M(b)D/I \) to which \( f \) belongs. Let \( g : I \rightarrow M(b) \) be arbitrary. For each \( i \in I \), there are \( a_i \in M \) and a term \( \tau_i(x, v) \) such that \( M(b) \models \tau_i(a_i, b) = g(i) \). Let \( c \in M \) be such that \( c > b \). There is \( d_i \in M \) such that \( M(b) \models \forall v < c \ (d_i)_v = \tau_i(a_i, v) \), so the same holds in \( M(b) \). Thus \( M(b) \models (d_i)_b = \tau(a_i, b) \); that is \( M(b) \models (d_i)_b = g(i) \). Using the \( \lambda^+ \)-saturation of \( M \), there is an element \( d \in M \) such that \( (d)_i = d_i \) for each \( i \in I \). But then \( \hat{d} = ((d)_i)_M \), so that \( M(b)D/I = M(b)(\hat{g}) \). Thus \( M(b)D/I \) is a simple, cofinal extension of \( M \), and thus also of \( N \). Also, \( M(b)D/I \) is \( \lambda^+ \)-saturated, by Lemma 5.2, and not \( \lambda^{++} \)-saturated, by Lemma 5.4. \( \square \)

The following corollaries are now immediate.

5.6. Corollary. Suppose \( N \) is \( \lambda^+ \)-saturated. Then \( N \) has a proper, simple cofinal extension which is \( \lambda^+ \)-saturated but not \( \lambda^{++} \)-saturated. \( \square \)

5.7. Corollary. Every model of PA has a proper, simple cofinal extension which is not \( \aleph_2 \)-saturated.

Proof. Let \( M \) be a model of PA and \( N \) a proper, simple cofinal extension. If \( N \) is not \( \aleph_2 \)-saturated, then we are done. Otherwise, use Corollary 5.6 to obtain a simple cofinal extension of \( M \) which is not \( \aleph_2 \)-saturated. \( \square \)

5.8. Corollary. Suppose \( \kappa \geq \aleph_1 \) is regular and \( \lambda \geq \aleph_0 \). Then there is \( N \) which is \( \kappa \)-saturated and not \( \kappa^+ \)-saturated such that \( N \) has a simple, cofinal extension which is \( \lambda^+ \)-saturated but not \( \lambda^{++} \)-saturated.

The condition that \( \kappa \) is regular is necessary in this corollary. For, if \( \kappa \) is singular and \( N \) is \( \kappa \)-saturated, then \( (N, <) \) is \( \kappa \)-saturated, and therefore is \( \kappa^+ \)-saturated. But then by Theorem 2.7(ii) \( N \) is also \( \kappa^+ \)-saturated.

Proof. If \( \kappa > \lambda \), then by Corollary 5.6 any \( \kappa \)-saturated model which is not \( \kappa^+ \)-saturated will do. So suppose \( \lambda \geq \kappa \), and let \( M \) be \( \lambda^+ \)-saturated. By Lemma 5.5 it suffices to find a cofinal extension \( M \succ M \) generated by at most \( \lambda \) generators which is \( \kappa \)-saturated but not \( \kappa^+ \)-saturated. Construct an elementary chain \( \langle M_\alpha : \alpha \leq \kappa \rangle \) as follows. Let \( M_0 = M \), and let \( M_\alpha = \bigcup \{ M_\beta : \beta < \alpha \} \) for limit \( \alpha \). If \( \alpha \) is odd, let \( M_{\alpha + 1} \) be a simple, cofinal extension of \( M_\alpha \) which fills the standard cut. If \( \alpha \) is even, use Lemma 5.5 to obtain a \( \lambda^+ \)-saturated, simple cofinal extension \( M_{\alpha + 1} \) of \( M_\alpha \). Then let \( N = M_\kappa \). The odd stages guarantee that \( N \) is not \( \kappa^+ \)-saturated, whereas the even stages guarantee that \( N \) is \( \kappa \)-saturated. \( \square \)
6. Short and tall models and PC$_5$* classes

It is a general fact of model theory that the class of recursively saturated models for a given finite language $L$ is a PC$_5$ class. For example, an $L$-structure $\mathfrak{A}$ is recursively saturated if and only if it is the set of urelements of a model $\mathfrak{A}^+$ of KPU$^+ + \neg[\text{infinity axiom}]$, (cf. Barwise–Schlipf [1]). In fact, if $\mathfrak{A} \models PA$ we may also require that $\mathfrak{A}^+$ satisfy the induction schema for its urelements $\mathfrak{A}$, using all $\Delta_0$ formulas of the language of KPU$^+$. A notion of satisfaction class in, for example, Schmerl [14] gives a natural theory whose countable reducts to the language of PA are exactly the countable recursively saturated models of PA; see Proposition 6.2. Are there similar characterizations for the countable models in other classes, such as the class of tall models, of lofty models, or of short models? The main result in this section answers this question negatively, using a characterization of recursive saturation similar to previous ideas of Smoryński–Stavi [16] and Paris–Harrington [12].

6.1. Definitions. For the remainder of this section, we let $L'$ range over countable languages that extend the language of PA.

(i) The theory PA($L$) is obtained from PA by adding the induction schema for all $L$-formulas.

(ii) A class $\mathcal{K}$ of models of PA is countably PC$_5^*$ if for some $L'$ there is an $L'$-theory $T \supseteq PA(L)$ such that for every countable model $M$ of PA, $M \in \mathcal{K}$ iff $M$ is expandable to a model of $T$.

6.2. Proposition. The class of recursively saturated models is countably PC$_5^*$.

Proof (Sketch, [14]). Let $L$ extend the language of PA by adding a new binary relation symbol $S$ and a constant symbol $c$. Then let $T$ be the theory obtained by adding to PA($L$) all sentences $c > n$ ($n < \omega$) together with the natural axioms that assert that $S$ is a partial non-standard satisfaction class, which means that $S$ satisfies the schema $\forall x [\varphi(x) \iff S([\varphi], x)]$. Notice that in any model $M$ of $T$, the induction schema guarantees that $S$ satisfies the inductive definition of satisfaction for all formulas up to some non-standard length $e$. It follows by familiar reasoning that $M$ is recursively saturated. Conversely, if $M$ is countable and recursively saturated, then $M$ is resplendent (cf. Barwise–Schlipf [1]) and hence it's easy to see that $M$ is expandable to a model of $T$. □

Our interest in countably PC$_5^*$ classes stems from their usefulness in constructing uncountable models, as illustrated in Schmerl [14] in the construction of large recursively saturated models which are rather classless, a notion recalled in the remark below. (We ask about uncountable models in Questions 7.1 and 7.2.)

6.3. Remark. The restriction to countable models is essential for Proposition 6.2.
To see this we recall a couple of definitions. A class of $M$ is a subset $X$ for which \( \{ x \in X : x < a \} \) is parametrically definable in $M$, for all $a \in M$. A model $M$ is rather classless if every class of $M$ is parametrically definable in $M$. A theorem of Kaufmann [3] and Shelah [15] shows that there exist rather classless, recursively saturated models of PA. However, no such model is expandable to any countable $\mathcal{L}$-theory $T \supseteq PA(\mathcal{L})$ all of whose countable models have recursively saturated reducts to the language of PA. For if $M^+ = (M, R_0, R_1, \ldots)$ is a rather classless model of such a theory $T$, where without loss of generality (by using a pairing function) each $R_i$ is unary, then (by the induction schema) each $R_i$ is a class, so is parametrically definable in $M$. It follows that $M^+$ has a countable elementary submodel $M^+$ whose reduct to the language of PA is short — a contradiction.

Now we show that even for countable models, there is (unfortunately) no analogue of Proposition 6.2 for other natural saturation and non-saturation notions such as loftiness and shortness.

The proof of the following theorem is given after some of its corollaries are drawn in 6.5.

**6.4. Theorem.** Let $T \supseteq PA(\mathcal{L})$ be an $\mathcal{L}$-theory. Then the following are equivalent.

(i) There is a countable tall model of PA that is not recursively saturated, but is expandable to a model of $T$.

(ii) There is a countable nonstandard short model of PA that is not bdd recursively saturated, but is expandable to a model of $T$.

(iii) There is a countable nonstandard short model of PA that is expandable to a model of $T$.

**6.5. Corollary.** The following classes of models of PA are not countably $\text{PC}^*_5$.

(a) The class of tall models.

(b) The class of short models.

(c) The class of bdd recursively saturated models; the class of bdd recursively saturated models which are short.

(d) The class of lofty models; of $\omega$-lofty models; of uniformly $\omega$-lofty models.

**Proof of Corollary 6.5.** Cases (a) and (b) follow from the implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) of Theorem 6.4, respectively. (Actually, (b) can be proved directly without difficulty.) The implication (iii) $\Rightarrow$ (i) of Theorem 6.4 yields case (c), since if a tall model is not recursively saturated then it is not bdd recursively saturated. Finally, case (d) follows from Theorem 4.11, which says that even the smallest of these classes (the uniformly $\omega$-lofty models) contains a countable model which is not recursively saturated; so since all of these classes are contained in the class of tall models, case (d) follows from the implication (i) $\Rightarrow$ (iii) of Theorem 6.4.

In order to prove the implication (i) $\Rightarrow$ (ii), we first present three lemmas. The second is similar to Theorem 2.2 of Smoryński–Stavi [16].
6.6. Lemma. Suppose $\mathcal{M}^*$ is a nonstandard model of $\text{PA}(\mathcal{L})$, and let $\mathcal{N}^*$ be a conservative extension of $\mathcal{M}^*$ such that the reduct $\mathcal{N}$ of $\mathcal{N}^*$ to the language of $\text{PA}$ is recursively saturated. Then the reduct $\mathcal{M}$ of $\mathcal{M}^*$ to the language of $\text{PA}$ is recursively saturated.

**Proof.** The proof of Proposition 6.2 shows that $\mathcal{N}$ has a partial non-standard satisfaction class $S$, which by means of a pairing function may be viewed as a unary relation on $\mathcal{N}$. Since $\mathcal{M}^*$ is a conservative extension of $\mathcal{M}^*$, $S \cap M$ is parametrically definable in $\mathcal{M}^*$ and hence the induction schema holds for $(\mathcal{M}, S \cap M)$. Since $\mathcal{M} < \mathcal{N}$, it follows that $S \cap M$ is a partial non-standard satisfaction class for $\mathcal{M}$, and hence $\mathcal{M}$ is recursively saturated. □

6.7. Lemma. Let $T \supseteq \text{PA}(\mathcal{L})$ be an $\mathcal{L}$-theory for some countable language $\mathcal{L}$, and suppose $\mathcal{M}$ is a model of $\text{PA}$ which is expandable to a model $\mathcal{M}^*$ of $T$. Let $f$ be a definable function of $\mathcal{M}^*$ which eventually dominates every definable function $g$ in $\mathcal{M}$, i.e.

$$\mathcal{M}^* \models \exists x \forall y > x (y \in X \rightarrow f(y) > g(y)).$$

Then $\mathcal{M}$ is recursively saturated.

**Proof.** Let $\mathcal{N}^*$ be a conservative extension of $\mathcal{M}^*$, and let $\bar{f}$ extend $f$ to $\mathcal{N}^*$ (using the same definition). Then $\bar{f}$ dominates every definable function $g$ in $\mathcal{N}$, so by Theorem 2.2 of [16], $\mathcal{N}$ is recursively saturated. Hence by Lemma 6.6, $\mathcal{M}$ is recursively saturated. □

6.8. Lemma. Suppose $\mathcal{M}^*$ is a model of $\text{PA}(\mathcal{L})$, whose reduct to the language of $\text{PA}$ is not recursively saturated. Then there is an end extension $\mathcal{M}^*(b)$ of $\mathcal{M}^*$ such that for every definable function $f$ of $\mathcal{M}^*$, $\mathcal{M}^*(b) \models f(b) < g(b)$ for some function $g$ that is definable in $\mathcal{M}$.

**Proof.** The proof expands on the ideas in proving the MacDowell-Specker Theorem. Let $\langle f_n : n < \omega \rangle$ enumerate all definable functions of $\mathcal{M}^*$. It suffices to define a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of unbounded subsets of $M$, each definable in $\mathcal{M}^*$, such that for all $n$ there is a definable function $g_n$ of $\mathcal{M}$ satisfying $f_n(x) < g_n(x)$ for all $x \in X_{n+1}$, and such that $f_n$ is constant or strictly increasing on $X_{n+1}$. Set $X_0 = M$. Having defined $X_n$, apply Lemma 6.7 to the function $f'_n(x) = f_n(\mu y : y \in [x \in X_n])$ to obtain a function $g_n$ definable in $\mathcal{M}$, such that $f'_n(x) < g_n(x)$ for arbitrarily large $x$. We may assume that $g_n$ is strictly increasing. Then set $X'_n = \{x \in X_n : f'_n(x) < g_n(x)\}$. Notice that if $f'_n(a) < g_n(a)$ and $b$ is the minimum of $\{x \in X_n : x \geq a\}$, then $g_n(b) > g_n(a) > f'_n(a) = f_n(b)$; so $X'_n$ is unbounded. Finally, choose $X_{n+1}$ to be a definable unbounded subset of $X'_n$ such that $f_n$ is constant or strictly increasing on $X_{n+1}$. □

**Proof of Theorem 6.4.** The direction (ii) $\Rightarrow$ (iii) is trivial. Now suppose $\mathcal{M}$ is a
model with expansion $\mathcal{M}^*$ that witnesses (i); we show that (ii) holds. Choose $\mathcal{M}^*(b)$ as in Lemma 6.8, and let $\mathcal{N}$ be the reduct of $\mathcal{M}^*(b)$ to the language of PA. The choice of $\mathcal{M}^*(b)$ guarantees that the elements of $\mathcal{R}$ which are definable from $b$ in $\mathcal{N}$ (i.e. using only the language of PA) are cofinal in $\mathcal{R}$; so, $\mathcal{R}$ is short. Since $\mathcal{N}$ is an end extension of $\mathcal{M}$, $\mathcal{N}$ is notbdd recursively saturated because $\mathcal{M}$ is not (since $\mathcal{M}$ is a tall model which is not recursively saturated).

Finally, for (iii) $\Rightarrow$ (i) suppose $\mathcal{M}$ is a countable non-standard short model of PA that is expandable to a model $\mathcal{M}^*$ of $T$. By iterating the MacDowell–Specker theorem (for the theory PA($\mathcal{L}$)) $\omega$-times (cf. Lemma 4.3), we obtain a countable conservative extension $\mathcal{M}^*$ of $\mathcal{M}^*$ which is tall. Then of course the reduct $\mathcal{N}$ of $\mathcal{M}^*$ to the language of PA is also tall, and $\mathcal{N}$ is not recursively saturated by Lemma 6.6 (because $\mathcal{M}$ is not recursively saturated). □

7. Questions

We know that if $\mathcal{M}$ has a recursively saturated simple extension, then $\mathcal{M}$ is lofty. The converse holds for countable models (Theorem 3.1). The same holds even if only the cofinality of $\mathcal{M}$ is countable, as was shown after Theorem 4.11. However for uncountable cofinality $\kappa$, even if $\kappa = \omega_1$, we do not know if every lofty model of cofinality $\kappa$ has a recursively saturated simple extension (though the answer is “Yes” if the model is not $\kappa$-like, by (∗) below). Indeed, we know of no model with cofinality greater than $\omega_1$ that is lofty but not recursively saturated.

7.1. Question. Are there any lofty models with cofinality $>\omega_1$ which are not recursively saturated?

7.2. Question. Are there any models with cofinality $>\omega_1$ which have recursively saturated simple extensions but which are not already recursively saturated?

The following observation together with Corollary 4.13 shows that Question 7.2 becomes true for cofinality $\omega_1$. Its proof is similar to the proof of (ii) $\Rightarrow$ (i) of Theorem 3.5.

(∗) If $\mathcal{M}$ is lofty but has no recursively saturated simple extension, then $\mathcal{M}$ is $\kappa$-life for some regular uncountable $\kappa$.

We do not know if there is an $\omega_1$-like model, or for that matter a $\kappa$-like model for some $\kappa$ of uncountable cofinality, satisfying the criteria of Question 7.2. Notice that Question 7.3 has an affirmative answer under CH (by Corollary 4.20).

7.3. Question (Without CH). Are there any models which have $\omega_1$-saturated simple extensions but are not $\omega_1$-saturated?

We know from Corollary 5.8 that if $\omega_1$ is replaced by any larger cardinal in
Question 7.3, then the answer is positive. Also, every $\kappa^+$-saturated model has a $\kappa^+$-saturated proper, simple extension, by Corollary 5.6.

7.4. **Question.** For inaccessible $\kappa$, does every $\kappa$-saturated model have a $\kappa$-saturated proper, simple extension?

Every countable uniformly $\omega$-lofty model has a recursively saturated simple extension generated by an element filling the standard cut.

7.5. **Question.** Is there a countable, uniformly $\omega$-lofty model which is not recursively saturated such that every extension filling the standard cut is recursively saturated?

7.6. **Question.** Is there a countable, uniformly $\omega$-lofty model having an extension which fills the standard cut and which is not recursively saturated?

The results of Section 6 were developed in an attempt to answer Question 7.1. Unfortunately, even the class of bdd $e$-lofty models is not countably PC$_3^a$ (nor is the class of bdd lofty models), as we have shown recently in [4]. However, the following question remains open. A positive answer would give an $\omega_1$-like lofty model, the existence of which is also open.

7.7. **Question.** Does every countable $e$-lofty model have an $e$-lofty proper end extension? (Similarly for bdd lofty models.)

The latter version of Question 7.7 has the following consequence, which is also open.

7.8. **Question.** In analogy to recursive saturation, does every countable bdd $e$-lofty model have an $e$-lofty end extension?

**Note added in proof**

Questions 7.4, 7.5 and 7.6 are answered positively in [4].

**References**


