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SYMMETRIC ALGEBRAS OF MODULES ARISING FROM A FIXED SUBMATRIX OF A GENERIC MATRIX

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We analyze symmetric algebras which arise from rather 'bad' ideals and modules. For example, the ideals are mixed, and every value $\neq 0$ occurs as the projective dimension of one of the modules. We are interested in the Cohen–Macaulay property, the canonical module, normality, and the divisor class group. The symmetric algebras under consideration can be defined as residue class rings modulo determinantal ideals covered by the theory of Hochster–Eagon. Part of the results can be regarded as an extension of work of Andrade and Simis.

Introduction

This work is concerned with the divisorial properties of symmetric algebras of modules and ideals arising from a generic matrix by fixing a subset of columns. To be precise, let $X=(X_{ij})$ be an $n\times m$-matrix $(n\leq m)$ of indeterminates over a field K. For a fixed integer r $(1\leq r\leq n)$, let X' denote the submatrix of X consisting of the first r columns.

We consider two basic situations. In the first we assume that $r=n-1$ and consider the ideal $I\subset R:=K[X]$ generated by the $n\times n$-minors of X involving the submatrix X'. Using different methods, we reprove half of [2, Theorem C, $(\mathrm{ii})_1$] concerning the symmetric and Rees algebras of I. We further compute the divisor class group of the symmetric algebra of I and its canonical module, and also the canonical module of the associated graded ring of I.

When $m=n+1$, the ideal M of all minors of X is just the cokernel of the map $(R^n)^*\to(R^m)^*$ given by the transpose X^* of X. The ideal I is generated by the

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images of those vectors in the canonical basis of $(R^m)^*$ which correspond to the rows of X^* complementary to (X').

In the second situation we let $m \geq n \geq r$ be arbitrary, obtaining, for all choices of m and n, submodules M_r of M (the ideal I corresponding to M_{n-1} if $m = n+1$). We give an explicit free minimal resolution of M_r which turns out to extend and clarify the resolution obtained in [2, §3]. We further show that the symmetric algebra of M_r is a Cohen–Macaulay normal domain. This leads us naturally to computing its divisor class group and canonical module. In the special case in which $m = n+1$, our considerations shed better light on the remark and the example mentioned in [2, §2].

The main tool for analyzing most of the present arithmetical properties is extracted from the Hochster–Eagon theory of determinantal ideals [10] and several other techniques for which we will refer to [4] and [6].

Notations. Capital X's and T's will denote indeterminates over a ground ring. For simplicity, we will assume the fixed ground ring to be a field K (although most of the results remain valid for more general ground rings). If R is a ring, X a $t \times s$-matrix with entries in R and $0 \leq u \leq \min\{t, s\}$, we will denote by $\mathrm{I}_u(X)$ the ideal of R generated by the $u \times u$-minors of X. The minor of X defined by the rows $j_1, \ldots, j_u$ and the columns $i_1, \ldots, i_u$ will be denoted $\Delta_{i_1 \ldots i_u}^{j_1 \ldots j_u}$. If $u = t$ or $u = s$, we will accordingly omit the row or column indices. For an R-module M, $\mathrm{S}(M)$ will stand for its symmetric algebra. If A is a normal domain, $\mathrm{Cl}(A)$ will denote its divisor class group. If S is a ring possessing a canonical module, ω_S will denote such a module. Any remaining notations, if not standard, will be explained in the text.

1. The symmetric algebra of the ideal of minors fixing $n-1$ columns viewed as a determinantal ring

Let $X = (X_{ij})$ be an $n \times m$-matrix of indeterminates over a field K, with $1 \leq n \leq m$. Set $R := K[X]$ and

$$I := (\Delta_{1 \ldots n-1\, n}, \Delta_{1 \ldots n-1\, n+1}, \ldots, \Delta_{1 \ldots n-1\, m}) \subset R\,.$$

The following enlarged matrix will play a major role in this section:

$$X|T := \begin{bmatrix} X_{11} & \cdots & X_{1,n-1} & X_{1n} & \cdots & X_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X_{n1} & \cdots & X_{n,n-1} & X_{nn} & \cdots & X_{nm} \\ 0 & \cdots & 0 & T_n & \cdots & T_m \end{bmatrix}.$$

Here $T_n, \ldots, T_m$ are indeterminates over R. The key observation to the results that will follow is given by

Lemma 1.1. *If $m > n$, the symmetric algebra $S(I)$ admits the R-algebra presentation*

$$R[T_n, \ldots, T_m]/\mathrm{I}_{n+1}(X \mid T)\,.$$

Proof. We have a surjective R-algebra homomorphism

$$R[T_n, \ldots, T_m] \to \mathrm{S}(I)\,, \quad T_i \to \Delta_{1 \ldots n-1\,i}\ (i = n, \ldots, m)\,.$$

The fact that the kernel of this map is $\mathrm{I}_{n+1}(X \mid T)$ follows from the description of the first syzygy module of I as in [1]. Namely, the generators of this module are the distinct Plücker relations among those of the form

$$\sum_k \varepsilon_{i_k} \Delta_{1 \ldots n-1\, i_k} \Delta_{j_1 \ldots j_s\, i_1 \ldots i_{k-1}\, n\, i_{k+1} \ldots i_{n-s}} = 0 \quad (\varepsilon_{i_k} = \pm 1)$$

for choices of ordered subsets $\{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n-1\}$ and $\{i_1, \ldots, i_{n-s}\} \subseteq \{n, \ldots, m\}$. Any such relation yields, upon substitution of T_i for $\Delta_{1 \ldots n-1\, i}$, the expansion of an $(n+1) \times (n+1)$-minor of $X \mid T$ along the last row. $\square$

A particular case of this presentation had been pointed out earlier [15, §3, Remark 1].

For the reader's convenience we now collect the results from [10] that most suit our purposes. For systematic reference we fix the following notation which differs from that in [10] only in minor changes. Let $Z = (Z_{ij})$, $1 \le i \le t$, $1 \le j \le s$, be a matrix with entries in a noetherian ring B, where $t \le s$. Let $H := s_0 = 0 < s_1 < \cdots < s_l = s$ be a strictly increasing sequence of integers, where $1 \le l < t$ is a fixed integer. Fix another integer k such that $0 \le k \le s$. Then $\mathrm{I}(H, k) = \mathrm{I}(H, k, Z)$ denotes the ideal of B generated by the $(q+1) \times (q+1)$-minors of the first s_q columns of Z, for every value of q $(1 \le q \le l)$, and by the elements $Z_{t1}, \ldots, Z_{tk}$ of the last row of Z.

Finally, let $h := \min_{1 \le q \le l} \{q \mid s_q \ge k\}$. Set

$$\mathrm{g}(H, k) := ts - (t+s)l + h + \binom{l+1}{2} + \sum_{q=1}^{l-1} s_q$$

Proposition 1.2. (i) $\operatorname{grade} \mathrm{I}(H, k) \le \mathrm{g}(H, k)$. *If either $k = s_q$, or $k = s_q + 1$ for some q $(0 \le q \le l)$ and* $\operatorname{grade} \mathrm{I}(H, k) = \mathrm{g}(H, k)$, *then $\mathrm{I}(H, k)$ is a perfect ideal. Suppose, moreover, that Z is a matrix of indeterminates over a field K. Then*

(ii) $\mathrm{I}(H, k)$ *is a radical ideal of height* $\mathrm{g}(H, k)$;

(iii) *If either* $k = s_q$ *or* $k = s_q + 1$, *for some* q $(0 \le q \le l)$, *then* $\mathrm{I}(H, k)$ *is a perfect ideal (hence the factor ring* $K[Z]/\mathrm{I}(H, k)$ *is Cohen–Macaulay)*;

(iv) *If* $k = s_q$, *for some* q $(0 \le q \le l)$, *then* $\mathrm{I}(H, k)$ *is a prime ideal and the factor ring* $K[Z]/\mathrm{I}(H, k)$ *is normal*;

(v) *Assume* $s_q < k < s_{q+1}$. *Define* $H' := s_0, \ldots, s_{q-1}, k, s_{q+1}, \ldots, s_l$ *and* $k' := s_{q+1}$. *Then* $\mathrm{I}(H, k) = \mathrm{I}(H', k) \cap \mathrm{I}(H, k')$ *is the primary decomposition of the radical ideal* $\mathrm{I}(H, k)$. □

The statements above and their proofs can be found in [10, Theorem 1, Corollary 3, Proposition 31]. We will in the sequel refer freely to these results as the 'Hochster–Eagon theory'. In our present situation this leads to

Corollary 1.3. $\mathrm{S}(I)$ *is a Cohen–Macaulay normal domain.*

Proof. Consider the fully enlarged generic matrix

$$X \,|\, \tilde{T} := \begin{bmatrix} X \\ \tilde{T} \end{bmatrix}, \quad \tilde{T} := T_1, \ldots, T_m\,.$$

By Proposition 1.1, one has

$$\mathrm{S}(I) \simeq R[T_1, \ldots, T_m]/(\mathrm{I}_{n+1}(X \,|\, \tilde{T}) + (T_1, \ldots, T_{n-1}))\,.$$

We now apply Proposition 1.2 with the following data:

$$\begin{aligned} &Z = X \,|\, \tilde{T}\,, \qquad s = m\,, \qquad t = n + 1\,, \qquad l = n\,, \\ &k = n - 1\,, \qquad H = (0, 1, 2, \ldots, n-1, m)\,, \\ &Z_{t1} = T_1, \ldots, Z_{tk} = T_{n-1}\,. \end{aligned}$$

The net result is that $\mathrm{I}_{n+1}(X \,|\, \tilde{T}) + (T_1, \ldots, T_{n-1}) = \mathrm{I}(H, n-1)$ is a perfect prime ideal and the corresponding factor ring is normal. □

Remark. A consequence is that $\mathrm{S}(I) = \mathrm{R}(I)$, where $\mathrm{R}(I)$ stands for the Rees algebra of I. In [2] this equality was the departing point to proving the preceding corollary.

2. The divisor class group

Keeping the notation of Section 1, we moreover set $A := \mathrm{S}(I)$. Unless explicitly stated, all ideals are taken in A. Small letters will usually denote residue classes in A of elements of $R[T_n, \ldots, T_m]$.

Lemma 2.1. (i) *If* $n \geq 3$, *the ideal* (x_{11}) *is prime.*

(ii) *If* $n = 1$, *then* $(x_{11}) = \mathfrak{p}_0 \cap \mathfrak{q}_0$, *where* $\mathfrak{p}_0 = (x_{11}, t_1)$ *and* $\mathfrak{q}_0 = (x_{11}, \ldots, x_{1m})$. *The ideals* $\mathfrak{p}_0$ *and* $\mathfrak{q}_0$ *are prime.*

(iii) *If* $n = 2$, *then* $(x_{11}) = \mathfrak{p} \cap \mathfrak{q}$, *where* $\mathfrak{p} = (x_{11}, x_{21})$ *and* $\mathfrak{q} = (x_{11}) + \mathrm{I}_2$ *(submatrix formed by first and third rows of* $X \mid T$*). The ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime.*

Proof. (i) Consider the ideal $\mathfrak{x} = (x_{11}, \ldots, x_{n1}) \subset A$. Clearly, $\mathfrak{x} = (\mathrm{I}_{n+1}(X_0 \mid T) + (X_{11}, \ldots, X_{n1}))/\mathrm{I}_{n+1}(X \mid T)$, where X_0 is the matrix obtained from X by deletion of the first column. It follows that $\operatorname{grade} \mathfrak{x} = (m-1-n) + n - (m-n) = n-1$.

In order to prove that $A/(x_{11})$ is a domain, we will apply [6, Lemma 2.4] using $\mathfrak{x}/(x_{11}) \subset A/(x_{11})$ as the 'test ideal'. Then, what is needed to show is that

(1) $x_{21}x_{i1} \not\in (x_{11})$, $i = 3, \ldots, n$;

(2) $(x_{21}, \ldots, x_{n1}) \not\subset P$, for every prime P associated to $A/(x_{11})$;

(3) $A/(x_{11})[x_{i1}^{-1}]$ is a domain for $i = 2, \ldots, n$.

Now, (1) is clear by arguing with degrees. As for (2), since x_{11} is not a zero-divisor on A, one has $\operatorname{grade} \mathfrak{x}/(x_{11}) = \operatorname{grade}(\mathfrak{x}) - 1 = n - 2 \geq 1$ (under the present assumption $n \geqq 3$). Finally, to prove (3) it suffices, by an evident symmetry, to show that x_{21} is a prime element in $A[x_{11}^{-1}]$. This is accomplished by means of the well-known inversion-elementary transformation trick which yields an isomorphism

$$(*) \qquad A[x_{11}^{-1}] \simeq (S[U_{n-1}, \ldots, U_{m-1}]/\mathrm{I}_n(Y \mid U)[X_{11}, \ldots, X_{1m}; \\ X_{21}, \ldots, X_{n1}; X_{11}^{-1}],$$

where $S := K[Y]$ and

$$Y \mid U := \begin{bmatrix} Y_{11} & \cdots & Y_{1,n-2} & Y_{1,n-1} & \cdots & Y_{1,m-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ Y_{n-1,1} & \cdots & Y_{n-1,n-2} & Y_{n-1,n-1} & \cdots & Y_{n-1,m-1} \\ 0 & \cdots & 0 & U_{n-1} & \cdots & U_{m-1} \end{bmatrix}.$$

Through this isomorphism, the element $x_{21} \in A[x_{11}^{-1}]$ is mapped onto X_{21}, an indeterminate over the coefficient ring $S[U_{n-1}, \ldots, U_{m-1}]/\mathrm{I}_n(Y \mid U)$. By Corollary 1.3 the latter is a domain. Thus, we are through.

(ii) For $n = 1$,

$$X \mid T = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ T_1 & \cdots & T_m \end{bmatrix}.$$

The contention is then a special case of the more general result, Proposition 1.2(v). It can, at any rate, be readily checked.

(iii) For $n=2$, we have

$$X\,|\,T=\begin{bmatrix} X_{11} & X_{12} & \dots & X_{1m} \\ X_{21} & X_{22} & \dots & X_{2m} \\ 0 & T_2 & \dots & T_m \end{bmatrix}.$$

Firstly, $\mathfrak{p}$ and $\mathfrak{q}$ are indeed prime ideals. For,

$$\mathfrak{p}=(\mathrm{I}_3(X\backslash X_1\,|\,\tilde{T}\backslash T_1)+(X_{11},X_{21}))/\mathrm{I}_3(X\,|\,T)$$

where

$$X\backslash X_1\,|\,\tilde{T}\backslash T_1:=\begin{bmatrix} X_{12} & \dots & X_{1m} \\ X_{22} & \dots & X_{2m} \\ T_2 & \dots & T_m \end{bmatrix},$$

and $\mathrm{I}_3(X\backslash X_1\,|\,\tilde{T}\backslash T_1)+(X_{11},X_{21})$ is clearly a prime ideal in $R[T]$. A similar remark applies to $\mathfrak{q}$. Finally, to prove the equality $(x_{11})=\mathfrak{p}\cap\mathfrak{q}$ it suffices, as (x_{11}) is a divisorial ideal, to note that $\mathfrak{p}\mathfrak{q}\subset(x_{11})\subset\mathfrak{p}\cap\mathfrak{q}$. □

The preceding lemma provides the tool for the initial inductive step in the proof of Theorem 2.3. The next lemma deals with the height one primes that appear in the generation of the divisor class group. Set, namely:

$\mathfrak{a}:=$ (residue class of) the ideal I_n(first n columns of $X\,|\,T$),

$\mathfrak{p}:=$ (residue class of) the ideal $\mathrm{I}_{n-1}(X')R[T]$, where X' is formed by the first $n-1$ columns of X,

$\mathfrak{r}:=$ (residue class of) the ideal $(T_n,\Delta^{1\,\dots\,n}_{1\,\dots\,n})$,

$\mathfrak{c}:=$ (residue class of) the ideal $\mathrm{I}_n(X)R[T]$.

Lemma 2.2.(i) *The ideals $\mathfrak{p}$ and $\mathfrak{r}$ are prime of height one and $\mathfrak{a}=\mathfrak{p}\cap\mathfrak{r}$ is the primary decomposition of the radical ideal $\mathfrak{a}$.*

(ii) *The ideal $\mathfrak{c}$ is prime of height one and $IA=\mathfrak{p}\cap\mathfrak{c}$ is the primary decomposition of the radical ideal IA.*

(iii) $\mathfrak{r}\cap A_+=(t_n)$, *where* $A_+=(t_n,\dots,t_m)$.

Proof. (i) We will apply Proposition 1.2 relative to the generic matrix $X\,|\,\tilde{T}=[\frac{X}{\tilde{T}}]$, $\tilde{T}=T_1\dots T_m$, with the following prescription:

$$H=(0,1,\dots,n-2,n,m)\,,\qquad l=n\,,\qquad k=n-1\,,$$

$$Z_{n+1\,1}=T_1,\dots,Z_{n+1\,n-1}=T_{n-1}\,.$$

Then, as one readily checks, the corresponding ideal is $\mathrm{I}(H,k)=\mathrm{I}_n$(first n columns of $X\,|\,\tilde{T})+\mathrm{I}_{n+1}(X\,|\,\tilde{T})+(T_1,\dots,T_{n-2})$. In other words, we recover $\mathfrak{a}$ as $\mathrm{I}(H,n-1)A$, showing that $\mathfrak{a}$ is a radical ideal. On the other hand,

$n-2(=s_{n-2})<n-1(=k)<n(=s_{n-1})$. Thus, if we let $H':=(0,1,\ldots,n-3, n-1, n, m)$ and $k':=n$, we get

$$\mathrm{I}(H, n-1)=\mathrm{I}(H', n-1)\cap \mathrm{I}(H, n)$$

and, moreover, $\mathrm{I}(H', n-1)$ and $\mathrm{I}(H, n)$ are prime ideals of height 1, respectively,

$$\begin{aligned} g(H', n-1) &= (n+1)m-(n+1+m)n+n-2+\binom{n+1}{2}+1+2 \\ &\quad +\cdots+n-3+n-1+n \\ &= m \end{aligned}$$

and

$$\begin{aligned} g(H, n) &= (n+1)m-(n+1+m)n+n-1+\binom{n+1}{2}+1+2 \\ &\quad +\cdots+n-2+n \\ &= m \end{aligned}$$

(cf. Proposition 1.2). Now, we have

$$\begin{aligned} \mathrm{I}(H', n-1) &= \mathrm{I}_{n-1}(\text{first } n-1 \text{ columns of } X \mid \tilde{T})+\mathrm{I}_{n+1}(X\mid \tilde{T}) \\ &\quad +(T_1,\ldots,T_{n-1}) \end{aligned}$$

(since I_n(first n columns) $\subset \mathrm{I}_{n-1}$(first $n-1$ columns)),

$$\begin{aligned} \mathrm{I}(H, n) &= \mathrm{I}_n(\text{first } n \text{ columns of } X\mid\tilde{T})+\mathrm{I}_{n+1}(X\mid\tilde{T}) \\ &\quad +(T_1,\ldots,T_{n-1},T_n) \\ &= (\Delta^{1\,\cdots\, n}_{1\,\cdots\, n})\mathrm{I}_{n+1}(X\mid\tilde{T})+(T_1,\ldots,T_{n-1},T_n)\,. \end{aligned}$$

Thus, we recover $\mathfrak{p}=\mathrm{I}(H', n-1)A$ and $\mathfrak{r}=\mathrm{I}(H,n)A$. Since $\mathrm{I}_{n+1}(X\mid\tilde{T})+(T_1,\ldots,T_{n-1})$ has height $m-(n+1)+1+n-1=m-1$, we derive that $\mathfrak{p}$ and $\mathfrak{r}$ are indeed (prime) ideals of height 1, as was to be shown.

(ii) Clearly, $\mathrm{I}_n(X)R[T]\supset \mathrm{I}_{n+1}(X\mid T)$. Therefore, $\mathfrak{c}$ is a prime ideal of height $m-1+1-(m-n)=1$. On the other hand, one has $\mathrm{I}_{n-1}(X')\mathrm{I}_n(X)\subset I$ (cf., e.g. [2], where the equality $\mathrm{I}_r(X')\cap \mathrm{I}_n(X)=I$ is proved for any set of r columns). Consequently, $\mathfrak{p}\mathfrak{c}\subset \mathrm{IA}\subset \mathfrak{p}\cap\mathfrak{c}$. But $IA\simeq A_+$, a height one prime. Therefore, IA is divisorial and the conclusion is that $IA=\mathfrak{p}\cap\mathfrak{c}$.

(iii) Using the Koszul-type generators of $\mathrm{I}_{n+1}(X\mid T)$, one easily sees that $\mathfrak{r}A_+\subset(t_n)$. Clearly, $(t_n)\subset\mathfrak{r}\cap A_+$. Since (t_n) is divisorial, again we must have $(t_n)=\mathfrak{r}\cap A_+$. $\square$

The next result describes the divisor class group of A. We will assume that $n\geq 2$ as otherwise it is well known that $\mathrm{Cl}(A)\simeq\mathbb{Z}$ (cf., e.g., [4]), generated by $\mathrm{cl}(\mathfrak{r})$.

Theorem 2.3. $(n \geq 2)$ $\mathrm{Cl}(A) \simeq \mathbb{Z} \oplus \mathbb{Z}$, *where the direct summands are generated by* $\mathrm{cl}(\mathfrak{p})$ *and* $\mathrm{cl}(\mathfrak{r})$, *respectively.*

Proof. We proceed by induction on $n \geq 2$. Assume $n = 2$. By the well-known lemma of Nagata, we have an exact sequence

$$0 \to U := \ker(\pi) \to \mathrm{Cl}(A) \xrightarrow{\pi} \mathrm{Cl}(A[x_{11}^{-1}]) \to 0\,,$$

where π is induced by the canonical map $A \to A[x_{11}^{-1}]$. But, using the isomorphism (*) and the case $n = 1$, we see that $\mathrm{Cl}(A[x_{11}^{-1}]) \simeq \mathbb{Z}$, generated by $\pi(\mathrm{cl}(\mathfrak{r}))$. On the other hand, U is generated by the classes of the height one primes of A containing x_{11}. Since $(x_{11}) = \mathfrak{p} \cap \mathfrak{q}$ (Lemma 2.1(iii)), U is generated, say, by $\mathrm{cl}(\mathfrak{p})$. Therefore, $\mathrm{Cl}(A) = \mathbb{Z}\mathrm{cl}(\mathfrak{p}) \oplus \mathbb{Z}\mathrm{cl}(\mathfrak{r})$, with $\mathbb{Z}\mathrm{cl}(\mathfrak{r}) \simeq \mathbb{Z}$. It remains to be shown that $\mathrm{cl}(\mathfrak{p})$ is not a torsion element of $\mathrm{Cl}(A)$.

We use a device as in [6, Proof of (3.2)]. Thus, assume $\nu\mathrm{cl}(\mathfrak{p}) = 0$, for some $\nu \in \mathbb{Z}$, $\nu \geq 0$. In other words, $\nu\mathrm{div}(\mathfrak{p}) = \mathrm{div}(Af)$, for some $f \in A$. Applying the map $\tilde{\pi}: \mathrm{Div}(A) \to \mathrm{Div}(A[x_{11}^{-1}])$, we obtain $\mathrm{div}(A[x_{11}^{-1}]f) = 0$, i.e., f is a unit in $A[x_{11}^{-1}]$. Using the isomorphism (*), one sees that $f = \alpha x_{11}^{\mu}$, for some $\alpha \in K$ and some non-negative integer μ. But then $\mathrm{div}(Af) = \mu\mathrm{div}(x_{11}) = \mu(\mathrm{div}(\mathfrak{p}) + \mathrm{div}(\mathfrak{q}))$. Equating this to $\nu\mathrm{div}(\mathfrak{p})$, one obtains $\nu - \mu = \mu = 0$, hence $\nu = 0$.

To complete the induction, we now use Lemma 2.1(i) via Nagata's exact sequence again. □

Remark. $(n \geq 2)$ $\mathrm{Cl}(A) \simeq \mathbb{Z} \oplus \mathbb{Z}$, where the direct summands are generated by $\mathrm{cl}(\mathfrak{p})$ and $\mathrm{cl}(\mathfrak{c})$.

Proof. It suffices to show that $\mathrm{cl}(\mathfrak{c}) = -(\mathrm{cl}(\mathfrak{p}) + \mathrm{cl}(\mathfrak{r}))$. For this, note that $\mathrm{cl}(A_+) = \mathrm{cl}(IA) = \mathrm{cl}(\mathfrak{p}) + \mathrm{cl}(\mathfrak{c})$ from Lemma 2.2(ii) and that $\mathrm{cl}(A_+) = -\mathrm{cl}(\mathfrak{r})$ from Lemma 2.2(iii) □

This can also be established by using the so called 'exact sequence of the Rees algebra divisor class group' (cf. [14]) which, in the case above, reads as follows:

$$0 \to (\mathbb{Z}\mathrm{cl}(\mathfrak{p}) + \mathbb{Z}\mathrm{cl}(\mathfrak{c})) \to \mathrm{Cl}(\mathrm{A}) \xrightarrow{\iota} \mathrm{Cl}(R) \to 0\,,$$

$R = K[X]$. To show linear independence here is even more straightforward since ι is given by the inclusion $A = R[It] \subset R[t]$.

3. The canonical module of A and A/IA

Let $S := R[T_1, \ldots, T_m]/\mathrm{I}_{n+1}(X \mid \tilde{T})$. Then the canonical module ω_S is well known (cf., e.g., [6]). Since $T_1, \ldots, T_{n-1}$ is an S-sequence, it follows that the canonical module $\omega_A \simeq \omega_S/(T_1, \ldots, T_{n-1})\omega_S \simeq \mathfrak{a}^{m-n-1}$, where $\mathfrak{a}$ is, as in Section 2, the ideal in A generated by the $n \times n$-minors of the first n columns of $X|T$. Here it is more convenient to represent ω_A by a slightly different ideal, primarily to obtain a good representation of the canonical module of A/IA. The type of a graded Cohen–Macaulay ring $\bigoplus_{i\geq 0} A_i$, $A_0 = K$, simply is the type of its localization with respect to the irrelevant maximal ideal. Since the canonical module is graded, its minimal number of generators gives the type.

Proposition 3.1. *Let $\mathfrak{b}$ denote the ideal generated by the $n \times n$-minors of the last n columns of $X|T$. Then*

(i) *$\mathfrak{b}$ is a prime ideal of height one*;

(ii) $\omega_A \simeq \mathfrak{b}^{m-n-1}$;

(iii) *The type of A is* $\binom{m-1}{m-n-1}$.

Proof. (ii) The proof is essentially contained in the preceding remark. Thus, if $\tilde{\mathfrak{b}}$ stands for the ideal in S generated by the $n \times n$-minors of the last n columns of $X|T$, then $\omega_S \simeq \tilde{\mathfrak{b}}^{m-n-1}$ and so

$$\omega_A \simeq (\tilde{\mathfrak{b}}^{m-n-1} + (t_1, \ldots, t_{n-1}))/(t_1, \ldots, t_{n-1}) \simeq \mathfrak{b}^{m-n-1}.$$

(i) Set $\bar{S} := S/\tilde{\mathfrak{b}}$, $\bar{A} := A/\mathfrak{b}$. As $\mathfrak{b} \simeq \mathfrak{a}$, $\mathfrak{b}$ is divisorial. Since A is Cohen–Macaulay, we get $\dim \bar{A} = \dim A - 1 = \dim R + 1 - 1 = \dim R$. But also S is Cohen–Macaulay and $\tilde{\mathfrak{b}}$ is divisorial. Therefore, $\dim \bar{A} = \dim S - n = \dim \bar{S} + 1 - n = \dim \bar{S} - (n-1)$. Since $\bar{A} \simeq S/(\tilde{\mathfrak{b}} + (t_1, \ldots, t_{n-1}))$, we must conclude that $t_1, \ldots, t_{n-1}$ is an $\bar{S}$-sequence. It follows that $\bar{A}$ itself is Cohen–Macaulay.

To show $\bar{A}$ is a domain, we will verify that t_m is not a zero-divisor in $\bar{A}$ and that $\bar{A}[t_m^{-1}]$ is a domain. The first part will follow, since $\bar{A}$ is Cohen–Macaulay, provided we can show $\dim \bar{A}/\bar{A}t_m = \dim \bar{A} - 1$. For this, let $\mathfrak{c} = (t_m, \Delta^{1\,\ldots\,n}_{1\,\ldots\,n-1\,m})A$. As in Lemma 2.2, $\mathfrak{c}$ is a prime ideal (of height 1) and $\mathfrak{c}A_+ \subset (t_m)$. Clearly, then $\bar{\mathfrak{c}}\bar{C}_+ \subset \bar{A}t_m$ and $\dim \bar{A}/\bar{A}t_m \leq \max\{\dim \bar{A}/\bar{\mathfrak{c}}, \dim \bar{A}/\bar{A}_+\} < \max\{\dim A/\mathfrak{c}, \dim A/A_+\} = \dim A - 1 = \dim \bar{A}$, as required.

To show that $\bar{A}[t_m^{-1}]$ is a domain one uses again the inversion-elementary transformation trick, getting an isomorphism similar to (*), which takes one back to the usual determinantal case of Hochster-Eagon.

(iii) It follows immediately from (ii) that $\binom{m-1}{m-n-1}$ is an upper bound of the type. As will be seen in Proposition 3.2(ii) and (iii), it is also a lower bound. □

Since the symmetric algebra $A = \mathrm{S}(I)$ coincides with the Rees ring $\mathrm{R}(I)$ (cf. the remark following Corollary 1.3), the residue class ring $A/IA = A \otimes R/I = \mathrm{S}_{R/I}(I/I^2)$ is the associated graded ring of R with respect to I. It follows from Lemma 2.2(ii) that A/IA is reduced. Since A and R are Cohen–Macaulay, A/IA is Cohen–Macaulay, too (cf. [13], for example, or [2]). Furthermore the canonical module $\omega_A \cong \mathfrak{b}^{m-n-1}$ has been embedded such that its single minimal prime does not contain IA or A_+; therefore we conclude directly from [7] that

$$\omega_{A/IA} \cong (\mathfrak{b}^{m-n-1} + IA)/IA\,.$$

So only the last statement in the following proposition needs still to be proved. It finishes the proof of Proposition 3.1 since the type of A/IA is obviously a lower bound for the type of A here.

Proposition 3.2. ($n \geq 2$) (i) *A/IA is a reduced Cohen–Macaulay ring.*
(ii) $\omega_{A/IA} \cong (\mathfrak{b}^{m-n-1} + IA)/IA$.
(iii) *The type of A/IA is $\binom{m-1}{m-n-1}$, too.*

Proof. Let $J \subset \bar{A}$ be the ideal generated by the x_{ij}, t_j, $j \leq m-n$. Then $IA \subset J$ and it suffices to show that the ideal $(\mathfrak{b}^{m-n-1} + J)/J$ needs $\binom{m-1}{m-n-1}$ generators. A/J is isomorphic to the polynomial ring over K in the indeterminates appearing in the matrix

$$\begin{bmatrix} T_m & X_{nm} & \cdots & X_{1m} \\ \vdots & & & \\ T_n & \vdots & & \vdots \\ 0 & & & \\ \vdots & & & \\ 0 & X_{n,m-n+1} & \cdots & X_{1,m-n+1} \end{bmatrix}$$

and $(\mathfrak{b}^{m-n-1} + J)/J \cong \bar{\mathfrak{b}}^{m-n-1}$ where $\bar{\mathfrak{b}}$ is the ideal generated by the maximal minors of the matrix above. It suffices to show that these maximal minors are algebraically independent. From the theory of Grassmannians this is known to hold for the maximal minors of a full $n \times (n+1)$-matrix of indeterminates Y_{ij} and even to remain valid when (Y_{ij}) is specialized to

$$\begin{bmatrix} Y_{11} & Y_{12} & \cdots & & Y_{1,n+1} \\ 0 & Y_{22} & & & \\ \vdots & 0 & & & \vdots \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots\ 0 & Y_{nn} & Y_{n,n+1} \end{bmatrix},$$

cf. [11, Chapter XIV, 9.]. □

4. Basic submodules of a generic module of projective dimension 1: their free resolutions

Herein we keep the general notation of the first section. Thus, $X := (X_{ij})$ is a $n \times m$-matrix of indeterminates over a field K, where $1 \leq n \leq m$, and an integer r

is fixed such that $0 \leq r \leq n$. We set $R := K[X]$ as before. Consider $M := \operatorname{coker}(X^* : (R^n)^* \to (R^m)^*)$. Fix a basis $e_1^*, \ldots, e_m^*$ of $(R^m)^*$ and let y_k denote the image of e_k^* in M, for $k = 1, \ldots, m$.

Definition 4.1 $M_r = M(n, m; r) := \sum_{k=r+1}^m Ry_k$. We thus obtain a chain of R-submodules

$$M(n, m; n) \subset M(n, m; n-1) \subset \cdots \subset M(n, m; 0) = M\,.$$

Let now $F_r := \sum_{k=r+1}^m Re_k^*$, a free module mapping onto M_r by means of $e_k^* \to y_k$. We want to claim that this map is the augmentation of a free complex

$$\begin{aligned} \mathscr{C}_r\colon\quad & 0 \to S_{n-r-1}(F_r') \otimes \bigwedge^{n} G^* \to S_{n-r-2}(F_r') \otimes \bigwedge^{n-1} G^* \to \cdots \\ & \cdots \to S_1(F_r') \otimes \bigwedge^{r+2} G^* \xrightarrow{\eta} \bigwedge^{r+1} G^* \xrightarrow{\zeta} F_r\,, \end{aligned}$$

where $F_r' := \sum_{k=1}^r Re_k$ and $G = R^n$. This complex is defined as follows: Firstly, consider the complex of Buchsbaum–Rim resolving $\operatorname{coker}(G^* \xrightarrow{X'^*} F_r'^*)$:

$$\begin{aligned} & 0 \to S_{n-r-1}(F_r') \otimes \bigwedge^{n} G^* \to \cdots \to S_1(F_r') \otimes \bigwedge^{r+2} G^* \\ & \xrightarrow{\eta} \bigwedge^{r+1} G^* \xrightarrow{\varepsilon} G^* \xrightarrow{X'^*} F_r'^* \end{aligned}$$

[8]. We then define the map $\zeta : \wedge^{r+1} G^* \to F_r$ as the composite of two maps

$$\bigwedge^{r+1} X^* : \bigwedge^{r+1} G^* \to \bigwedge^{r+1} F^*\,, \quad F^* := (R^m)^*$$

and

$$\xi : \bigwedge^{r+1} F^* \to F_r\,,$$

where $\xi(e_1^* \wedge \cdots \wedge e_r^* \wedge e_k^*) = e_k^*$ if $k = r+1, \ldots, m$, and ξ(any other basis element) $= 0$.

All definitions being posed, we claim a little more, namely:

Proposition 4.2. *$\mathscr{C}_r$ is a free resolution of M_r. In particular, M_r has projective dimension $n - r$ for $r \geq 1$.*

Proof. Firstly, we will check that $\mathscr{C}_r$ is indeed a complex, in other words, that $\zeta \circ \eta = 0$. For this, recall the action of η on a typical basis element of $S_1(F_r') \otimes \wedge^{r+2} G^*$:

$$e_i \otimes f^*_{j_1} \wedge \cdots \wedge f^*_{j_{r+2}} \to \sum_k \pm X'^*(f^*_{j_k})(e_i) f^*_{j_1} \wedge \cdots \wedge \hat{f}^*_{j_k} \wedge \cdots \wedge f^*_{j_{r+2}}$$
$$= \sum_k \pm X_{j_k i} f^*_{j_1} \wedge \cdots \wedge \hat{f}^*_{j_k} \wedge \cdots \wedge f^*_{j_{r+2}}$$

where $V = \{j_1, \ldots, j_{r+2}\} \subseteq \{1, \ldots, n\}$, $i \in \{1, \ldots, r\}$, $\{f^*_1, \ldots, f^*_n\}$ a basis of G^*. Applying $\zeta = \xi \circ \wedge^{r+1} X^*$ to the resulting element of $\wedge^{r+1} G^*$, one obtains the element

$$\sum_{k=r+1}^{m} \Big(\sum_{j \in V} \pm X_{ji} \Delta^{V \setminus \{j\}}_{1, \ldots, r, k} \Big) e^*_k \in F_r \,.$$

Thus the vanishing of $\zeta \circ \eta$ is equivalent to the existence of the well-known linear relations of the $(r+1) \times (r+1)$-minors of the $(r+1) \times (r+2)$-matrix with columns $1, \ldots, n, k$ and rows $V = \{j_1, \ldots, j_{r+2}\}$, for each $k = r+1, \ldots, m$.

We now proceed to show that $\mathscr{C}_r$ is acyclic and resolves M_r. It suffices to show exactness at F_r and $\wedge^{r+1} G^*$.

(1) Exactness at F_r. We claim that $\ker(F_r \to M_r)$ is generated by the elements

$$\sum_{k=r+1}^{m} \Delta^{U}_{1, \ldots, r, k} e^*_k \,,$$

where U runs through the subsets of cardinality $r+1$ of $\{1, \ldots, n\}$. By the definition of ζ, it will then follow that $\ker(F_r \to M_r) = \zeta(\wedge^{r+1} G^*)$.

Thus, let $\sum_{k=r+1}^{m} \alpha_k e^*_k \in \ker(F_r \to M_r)$. One easily sees that

$$\alpha_k = \sum_{i=1}^{n} \beta_i X_{ik}, \quad k = r+1, \ldots, m \,,$$
$$\sum_{i=1}^{n} \beta_i X_{il} = 0, \quad l = 1, \ldots, r \,,$$
$$\text{for some } \beta_i \in R,\ i = 1, \ldots, n \,.$$

These homogeneous linear equations imply that $\sum_{i=1}^{n} \beta_i f^*_i \in \varepsilon(\wedge^{r+1} G^*)$, where $\varepsilon : \wedge^{r+1} G^* \to G^*$ is the so-called Cramer map in the complex of Buchsbaum–Rim as above. That is, we have

$$\sum_{i=1}^{n} \beta_i f^*_i = \varepsilon\Big(\sum_U \gamma_{j_1 \ldots j_{r+1}} f^*_{j_1} \wedge \cdots \wedge f^*_{j_t} \Big)$$
$$= \sum_U \gamma_{j_1 \ldots j_{r+1}} \sum_t \pm \Big(\bigwedge^r X'^* \Big) (f^*_{j_1} \wedge \cdots \wedge \hat{f}^*_{j_t} \wedge \cdots \wedge f^*_{j_{r+1}}) f^*_{j_t}$$
$$= \sum_{i=1}^{n} \Big(\sum_{\substack{U \\ i \in U}} \gamma_{j_1 \ldots j_{r+1}} (\pm \Delta^{U \setminus \{i\}}_{1, \ldots, n}) \Big) f^*_i \,,$$

where $U = \{j_1, \ldots, j_{r+1}\}$ runs through the ordered subsets of $r+1$ elements of $\{1, \ldots, n\}$ and $\gamma_{j_1 \ldots j_{r+1}} \in R$ are suitable coefficients. Using the values of α_k in terms of β_i and X_{ik} as found above, we easily arrive at the expressions

$$\begin{aligned}\sum_{k=r+1}^{m} \alpha_k e_k^* &= \sum_{k=r+1}^{m} \Big(\sum_U \gamma_{j_1 \ldots j_{r+1}} \sum_{i \in U} \pm X_{ik} \Delta_{1 \ldots r}^{U \setminus i}\Big) e_k^* \\ &= \sum_U \gamma_{j_1 \ldots j_{r+1}} \Big(\sum_{k=r+1}^{m} \Delta_{1 \ldots rk}^{U} e_k^*\Big),\end{aligned}$$

as was to be shown.

(2) Exactness at $\wedge^{r+1} G^*$. It suffices to show that im ζ and coker η have the same rank, since coker $\eta \cong$ im ε is torsion-free and mapped onto im ζ. We already have exactness at F_r, so

$$\operatorname{rk} \operatorname{im} \zeta = \operatorname{rk} F_r - \operatorname{rk} M_r = n - r ,$$

and

$$\operatorname{rk} \operatorname{coker} \eta = \operatorname{rk} \operatorname{im} \varepsilon = \operatorname{rk} G^* - \operatorname{rk} X'^* = n - r . \qquad \square$$

5. The symmetric algebra $S(M_r)$: its divisor class group

We proceed to study the arithmetic of the symmetric algebra of the module M_r. It turns out, as we will presently show, that $S(M_r)$ is a ring with good arithmetic properties, regardless of r.

One needs the following basic results about $M_r = M(n, m; r)$:

Lemma 5.1. $S(M_r) \simeq R[T_{r+1}, \ldots, T_m]/I_{r+1}(X'|L)$, *where*

$$X'|L := \begin{bmatrix} X_{11} & \ldots & X_{1r} & \sum_{\rho=r+1}^{m} X_{1\rho} T_\rho \\ \vdots & & \vdots & \vdots \\ X_{n1} & \ldots & X_{nr} & \sum_{\rho=r+1}^{m} X_{n\rho} T_\rho \end{bmatrix},$$

the T_i being indeterminates over R.

Proof. A typical generator of $I_{r+1}(X'|L)$ is given by

$$\sum_t \Big(\sum_{\rho=r+1}^{m} X_{j_t\rho} T_\rho\Big) \Delta_{1 \ldots r}^{j_1 \ldots \hat{j}_t \ldots j_{r+1}}(X') = \sum_k \Delta_{1 \ldots rk}^{j_1 \ldots j_{r+1}}(X) T_k ,$$

for some subset $\{j_1, \ldots, j_{r+1}\} \subset \{1, \ldots, n\}$. The result is now contained in Proposition 4.2. $\square$

Lemma 5.2. *Let $\nu_p(E)$ stand for the minimal number of generators of a module E locally at a prime ideal P. Then*

(i) $\nu_P(M_r) \leq \operatorname{ht} P + \operatorname{rk} M_r$ *for any prime* $P \subset R$;

(ii) *If, moreover,* $n < m$ *and* $P \neq (0)$, *the estimate in* (i) *can be sharpened to*

$$\nu_P(M_r) \leq \operatorname{ht} P + \operatorname{rk} M_r - 1 .$$

Proof. Assume first $P \supseteq \mathrm{I}_r(X')$. Then $\operatorname{ht} P \geq \operatorname{ht} \mathrm{I}_r(X') = n - r + 1 \geq 1$. On the other hand, $\nu_P(M_r)$ is certainly bounded by the number of generators of M_r itself. Therefore, $\nu_P(M_r) \leq m - r < m - r + 1 = (n - r + 1) + (m - n) \leq \operatorname{ht} P + \operatorname{rk} M_r$, so one is through in this case. If, on the other hand, $P \not\supseteq \mathrm{I}_r(X')$, then $\nu_P(M_r) = \nu_P(M)$ since $\operatorname{rad}(\operatorname{ann} M/M_r) = \mathrm{I}_r(X')$. But, for M itself and provided $n < m$, the sharpened estimate $\nu_P(M) \leq \operatorname{ht} P + \operatorname{rk} M - 1$ holds for a prime $P \neq (0)$ (cf. [12] or [15]). Since $\operatorname{rk} M_r = \operatorname{rk} M$, we are done. □

Lemma 5.3. *Let $A := \mathrm{S}(M_r)$ and let t_k(resp. x_{11}) be the residue class in A of T_k(resp. X_{11}). Then*

(i) $A[t_k^{-1}] \simeq K[T, T_k^{-1}][X]/\mathrm{I}_{r+1}(X'|X_k)$, *where*

$$X'|X_k := \begin{bmatrix} X_{11} & \cdots & X_{1r} & X_{1k} \\ \vdots & & \vdots & \vdots \\ X_{n1} & \cdots & X_{nr} & X_{nk} \end{bmatrix};$$

(ii) $A[x_{11}^{-1}] \simeq \mathrm{S}(M(n-1, m-1; r-1))[X_{11}, \ldots, X_{1m}; X_{21}, \ldots, X_{n1}; X_{11}^{-1}]$.

Proof. (i) Consider the following $K[T, T_k^{-1}]$-automorphism Ψ of $K[T, T_k^{-1}][X]$:

$$\Psi(X_{ij}) = X_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq m, \ j \neq k ,$$
$$\Psi(X_{ik}) = X_{ir+1}T_{r+1}T_k^{-1} + \cdots + X_{ik} + \cdots + X_{im}T_mT_k^{-1}, \quad 1 \leq i \leq n .$$

It is obvious that $\Psi(\mathrm{I}_{r+1}(X'|X_k)) = \mathrm{I}_{r+1}(X'|L)$.

(ii) This is clear by the inversion and elementary transformation trick. □

The last lemma we will need is a basic test of integrality for rings. We first note the following preliminary fact: *Let R be a reduced ring, let $\mathfrak{a}$, $\mathfrak{b}$ ($\mathfrak{b} \neq (0)$) be ideals such that $\mathfrak{a}\mathfrak{b} = (0)$. Let there be given a third ideal $\mathfrak{c}$ such that $\mathfrak{c}$ is not contained in any associated prime of $R/(\mathfrak{a} + \mathfrak{b})$ or any minimal prime of R containing $\mathfrak{a}$. Then $\mathfrak{c}$ is not contained in any associated prime of $R/\mathfrak{a}$ either.* The proof is easy and depends only on elementary properties of associated primes. Using this general fact together with [6, (2.4)] one obtains the following result which will be needed in the sequel:

Lemma 5.4. *Let R be a reduced ring, let $\mathfrak{a}$, $\mathfrak{b}$ ($\mathfrak{b} \neq (0)$) and $\mathfrak{c}$ be ideals such that $\mathfrak{a}\mathfrak{b} = (0)$ and $\mathfrak{c}$ is contained in no associated prime of $R/(\mathfrak{a} + \mathfrak{b})$ and no minimal prime of R containing $\mathfrak{a}$. Let $\mathfrak{c}$ admit a system of generators $x_1, \ldots, x_s$ such that $x_1 x_i \not\subseteq \mathfrak{a}$, $i = 2, \ldots, s$ and such that $\mathfrak{a}R[x_i^{-1}]$ is prime in $R[x_i^{-1}]$ for $i = 1, \ldots, s$. Then $\mathfrak{a}$ is a minimal prime ideal.* □

We are ready for the main result of this section.

Theorem 5.5. (i) *If $n < m$, $\mathrm{S}(M_r)$ is a Cohen–Macaulay normal domain.*

(ii) *If $r < n = m$, $\mathrm{S}(M_r)$ is a reduced Cohen–Macaulay ring with minimal primes $\mathrm{S}(M_r)_+ = (t_{r+1}, \ldots, t_m)$ and (Δ), where t_k is the residue class of T_k and Δ is the determinant of the square matrix X.*

Proof. A unified argument for (i) and (ii) shows that $\mathrm{S}(M_r)$ is Cohen–Macaulay. Namely, by Lemma 5.1, $\mathrm{S}(M_r)$ is determinantal; by Proposition 1.2(i), it will then be sufficient to check that $\mathrm{I}_{r+1}(X'|L)$ has the maximum possible grade $n - r$. For this, one can use the first part of Lemma 5.2 to derive $\dim \mathrm{S}(M_r) = \dim R + \operatorname{rk} M_r$ (cf. [15]; also [3, 12]), from which the desired value for the grade easily follows. We now proceed separately for the two cases.

(i) $n < m$. Set $A := \mathrm{S}(M_r)$. Let $J \subset A$ be the ideal generated by x_{ij} ($1 \leq i \leq n$, $1 \leq j \leq r$) and t_k ($r + 1 \leq k \leq m$). Clearly, $A/J \simeq K[X_{ij}\colon 1 \leq i \leq n,\ r + 1 \leq j \leq m]$, so in particular, $\dim A/J = (m - r)n = mn + m - n - (nr + m - n) = \dim A - (nr + m - n)$. As we have seen, A is Cohen–Macaulay. Therefore, $\operatorname{grade} J = nr + m - n$. Now one uses induction on $n \geq 0$. For $n = 0$, M_r is even a free module, so $A = \mathrm{S}(M_r)$ is certainly normal. Assume then $n \geq 1$. If $r = 0$, $M_r = M$ and the result is known (cf., e.g., [12]). If $r \geq 1$, then $\operatorname{grade} J = nr + m - n \geq 2$ (*as* $n < m$). Therefore, A will be normal along with the localizations $A[t_k^{-1}]$ and $A[x_{ij}^{-1}]$, $1 \leq i \leq n$, $1 \leq j \leq r$, $r + 1 \leq k \leq m$. The latter are indeed normal by Lemma 5.3 and the inductive hypothesis. Since A is normal and a graded algebra over a field, it has to be a domain. Note that the integrality also follows from the fact that A is Cohen–Macaulay, the inequalities of Lemma 5.2, and [15, Proposition 3.3].

(ii) $r < n = m$. Clearly, $A_+ = (t_{r+1}, \ldots, t_m)$ is a prime ideal of height 0 as $(T_{r+1}, \ldots, T_m) \supset \mathrm{I}_{r+1}(X'|L)$ and the latter has height $m - r$ as we have seen. As for (Δ), note Δ annihilates the generators of M, hence those of $M_r \subset M$. Thus, $\Delta A_+ = (0)$. Now, the ideal $(A_+, \Delta) \subset A$ is prime since $A/(A_+, \Delta) \simeq R/(\Delta)$. On the other hand, if we let $\mathfrak{c} \subset A$ be the ideal generated by x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r$, then $\dim A/\mathfrak{c} = n(m - r) + m - n = nm - ((n + 1)r - m) = n^2 - ((n + 1)r - n) = \dim R - ((n + 1)r - n) = \dim A - ((n + 1)r - n) \leq \dim A - 1$. Thus, $\mathfrak{c}$ is an ideal of height ≥ 1.

We can therefore apply Lemma 5.4, to conclude that $\mathfrak{a} := (\Delta)$ is a minimal prime ideal of A, with $\mathfrak{b} := A_+$ and $\mathfrak{c}$ as above, provided we show that A is reduced. For this, we proceed as in the proof of normality in (i). Namely, letting

$J \subset A$ be the ideal generated by x_{ij} and t_k (i.e., $J := \mathfrak{c} + \mathfrak{b}$), we have grade $J = nr + m - n = nr \geq 1$. Therefore, A is reduced along with the localizations $A[x_{ij}^{-1}]$ and $A[t_k^{-1}]$; the latter are reduced by Lemma 5.3 and the inductive hypothesis on $n \geq 1$. □

We next proceed to discuss the divisor class group of $A = \mathrm{S}(M_r)$ $(n < m)$. We exclude the case $r = 0$ as it is well known (cf., e.g., [4]). For completeness, we recall that if $r = 0$ and $m > n + 1$, then $\mathrm{Cl}(A) = 0$, while if $r = 0$ and $m = n + 1$, then $\mathrm{Cl}(A) \simeq \mathbb{Z}$ is generated by $\mathrm{cl}(A_+) = \mathrm{cl}(\mathrm{I}_n(X)A)$. Thus, assume $r \geq 1$. We first isolate the relevant prime ideals for the generation of $\mathrm{Cl}(A)$.

Proposition 5.6. $(r \geq 1)$ *Let* $A = R[T_{r+1}, \ldots, T_m]/\mathrm{I}_{r+1}(X'|L)$ *where* $X'|L$ *is the matrix described in Lemma* 5.1. *Let* $\mathfrak{p}$ (*resp.* $\mathfrak{q}$) *be the ideal of* A *generated by the* $r \times r$*-minors of the first r columns* (*resp. rows*) *of* $X'|L$. *Then* $\mathfrak{p}$ (*resp.* $\mathfrak{q}$) *is a prime ideal of height* 1.

Proof. We consider only $\mathfrak{p}$, the discussion for $\mathfrak{q}$ being entirely similar. First, one observes that the preimage of $\mathfrak{p}$ in $R[T]$ is one of the ideals appearing in the theory of Hochster–Eagon. Also its grade is the maximal possible as predicted in Proposition 1.2. Therefore, the preimage of $\mathfrak{p}$ in $R[T]$ is a perfect ideal by Proposition 1.2(i). So $A/\mathfrak{p}$ is Cohen–Macaulay. On the other hand, one has

$$\begin{aligned} \mathrm{ht}\, A_+(A/\mathfrak{p}) &= \mathrm{ht}((\mathfrak{p}, A_+)/\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}, A_+) - \mathrm{ht}(\mathfrak{p}) \\ &= \dim A - \dim A/(\mathfrak{p}, A_+) - \mathrm{ht}(\mathfrak{p}) \\ &= \dim R + m - n - \dim R/\mathrm{I}_r(X') - \mathrm{ht}(\mathfrak{p}) \\ &= \dim R + m - n - \dim R + n - r + 1 - 1 \\ &= m - r \geq 1\,. \end{aligned}$$

Since $A_+ = (t_{r+1}, \ldots, t_m)$, we deduce as before that $A/\mathfrak{p}$ is a domain along with $(A/\mathfrak{p})[t_k^{-1}]$, $r + 1 \leq k \leq m$; the latter are domains by Lemma 5.3. □

Theorem 5.7. $(r \geq 1)$ *Let* $A = \mathrm{S}(M_r)$, $\mathfrak{p}$ *and* $\mathfrak{q}$ *be as above.*

(i) *If* $m > n + 1$, *then* $\mathrm{Cl}(A) \simeq \mathbb{Z}$, *generated by* $\mathrm{cl}(\mathfrak{p})$ *or* $\mathrm{cl}(\mathfrak{q}) = -\mathrm{cl}(\mathfrak{p})$.

(ii) *If* $m = n + 1$, *then* $\mathrm{Cl}(A) \simeq \mathbb{Z} \oplus \mathbb{Z}$, *the summands being generated by* $\mathrm{cl}(\mathfrak{p})$ *and* $\mathrm{cl}(A_+)$ *respectively.*

Proof. (i) We claim that t_m is a prime element in A. In fact, one has

$$A/(t_m) \simeq \mathrm{S}(M(n, m-1; r))[X_{1m}, \ldots, X_{nm}]\,,$$

where $\mathrm{S}(M(n, m-1; r))$ is a domain by virtue of Theorem 5.5(i). Therefore,

$\mathrm{Cl}(A) \simeq \mathrm{Cl}(A[t_m^{-1}])$ by the lemma of Nagata. Now, using Lemma 5.3(i) and the remark before Proposition 5.6, one has $\mathrm{Cl}(A[t_m^{-1}]) \simeq \mathbb{Z}$, generated by $\mathrm{cl}(\mathfrak{p}A[t_m^{-1}])$.

(ii) Here, t_m is no longer a prime element. However, letting Δ_m denote the determinant of the square matrix obtained from X by deletion of the mth column, one sees from the isomorphism

$$A/(t_m) \simeq \mathrm{S}(M(n, n; r))[X_{1n}, \ldots, X_{nn}]$$

and from Theorem 5.5(ii), that $(t_m) = A_+ \cap (t_m, \Delta_m)$, where $A_+ = (t_{r+1}, \ldots, t_m)$ and (t_m, Δ_m) are primes of height 1. Applying Nagata's lemma, we see that $\mathrm{Cl}(A)$ is generated by $\mathrm{cl}(A_+)$ and $\mathrm{cl}(\mathfrak{p})$. Finally, the same 'unit trick' as used in the proof of Theorem 2.3 can be applied to show that $\mathrm{cl}(A_+)$ and $\mathrm{cl}(\mathfrak{p})$ are $\mathbb{Z}$-linearly independent generators. $\square$

Remark. An observation, similar to the one after Theorem 2.3, can be made here to the effect that Theorem 5.7(ii) is a consequence of the results developed in [14] for computing the class group of Rees algebras. In fact, in the special case where $m = n + 1$, M_r is an ideal in R generated by the maximal minors of X involving the first r columns and $\mathrm{S}(M_r)$ becomes the Rees algebra of M_r. These ideals were dealt with in [2].

6. The canonical module of $\mathrm{S}(M_r)$ ($n < m$)

Throughout this section we assume $r < n < m$ (note M_n is free). The case where $r = 0$ being well known, we will grant $r \geq 1$ as well. As before, ω_B will denote the canonical module of the ring B, provided it exists – which is certainly the case for the rings we will consider. The notation of the preceding section will prevail here.

Proposition 6.1. $(1 \leq r < n < m)$

(i) $\omega_{\mathrm{S}(M_r)} \simeq \mathfrak{q}^{n-r-1}$.

(ii) *The type of* $\mathrm{S}(M_r)$ *is* $\binom{n-1}{n-r-1}$. *In particular*, $\mathrm{S}(M_r)$ *is a Gorenstein ring if and only if* $n = r + 1$.

Proof. (i) We had $\mathrm{S}(M_r) \simeq R[T_{r+1}, \ldots, T_m]/\mathrm{I}(X'|L)$. Consider $X'|L$ as a specialization of the completely generic matrix Y of the same size. Take any free resolution of $\mathrm{I}_{r+1}(Y)$ and dualize it as usual to obtain a resolution of the corresponding canonical module; then specialize; this is the same as first specializing the generic resolution and then dualizing to obtain a resolution of $\omega_{\mathrm{S}(M_r)}$. For the result in the generic case we refer to [5].

(ii) The number given is obviously an upper bound. Similar to the argument for Proposition 3.2(ii) it is enough to show that the maximal minors of the matrix

$$\begin{bmatrix} X_{11} & \dots & X_{1r} & \sum_{k=r+1}^{m} X_{1k}T_k \\ \vdots & & \vdots & \vdots \\ X_{r1} & \dots & X_{rr} & \sum_{k=r+1}^{m} X_{rk}T_k \end{bmatrix}$$

are algebraically independent over K. This holds since it is certainly true after inversion of a T_i. □

We now turn to the case $m = n + 1$. As observed in the preceding section, M_r is an ideal in R generated by the maximal minors of X involving the first r columns.

Proposition 6.2. ($1 \le r < n = m - 1$) *Set* $I := M_r \subset R$, $A := \mathrm{S}(M_r)$ *and* $G := A/IA$, *the associated graded ring of* R *with respect to* I. *Then*

(i) *G is a reduced Cohen–Macaulay ring*;
(ii) *The primary decomposition of IA is* $IA = \mathfrak{p} \cap \mathrm{I}_n(X)A$;
(iii) $\omega_G \cong (\mathfrak{q}^{n-r-1} + IA)/IA$;
(iv) *The type of G is* $\binom{n-1}{n-r-1}$, *too.*

Proof. (i) Since A is Cohen–Macaulay, it is well known (and easy to show) that G is Cohen–Macaulay as well. To show that G is reduced is more involved. We first observe that, by the usual argument, if suffices to prove that $G[t_k^{-1}]$ is reduced for $k = r + 1, \dots, m$. Indeed, $G_+ = (t_{r+1}, \dots, t_m)$ is such that $\dim G/G_+ < \dim G$. Now, set $P := K[T][T_i^{-1}]$ (for a fixed arbitrary i) and

$$\tilde{X} := \begin{bmatrix} X_{11} & \dots & X_{1r} & \sum_{k=r+1}^{m} X_{1k}T_k & X_{1,r+2} & \dots & X_{1m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ X_{n1} & \dots & X_{nr} & \sum_{k=r+1}^{m} X_{nk}T_k & X_{n,r+2} & \dots & X_{nm} \end{bmatrix}.$$

Clearly, the entries of $\tilde{X}$ generate the polynomial ring $P[X]$ as a P-algebra. On the other hand, $G[t_i^{-1}] \simeq P[X]/\mathfrak{g}$, where $\mathfrak{g}$ is the ideal of $P[X]$ generated by the $(r+1) \times (r+1)$-minors of the first $r+1$ columns of $\tilde{X}$ and by the maximal minors of $\tilde{X}$ involving the first r columns. We then conclude by an argument of Hodge algebras; namely, it is easy to check that $\mathfrak{g}$ is generated by an 'ideal' of the poset of all minors of $\tilde{X}$, with respect to which $P[X]$ is an ordinal Hodge algebra. It follows that $P[X]/\mathfrak{g}$ is an ordinal Hodge algebra itself over P. Since P is reduced, so is $P[X]/\mathfrak{g}$, cf. [9].

(ii) By (i), IA is a radical ideal. Since $\mathfrak{p}\mathrm{I}_n(X) \subset IA$ (direct argument or see [2]), it then suffices to check that $\mathfrak{p}$ and $\mathrm{I}_n(X)A$ are minimal primes of G. For $\mathfrak{p}$

this is clear (Proposition 5.6). As for $\mathrm{I}_n(X)A$ we apply Lemma 5.4 with $R := G$, $\mathfrak{a} := \mathfrak{p}\mathrm{I}_n(X)G$, $\mathfrak{b} := \mathfrak{p}G$ and $\mathfrak{c} := G_+$. Let us verify whether the hypotheses of that lemma hold in our setting. First, $\mathfrak{a}+\mathfrak{b}$ is a prime ideal as $A/(\mathrm{I}_n(X), \mathfrak{p}) \simeq K[X, T]/(\mathrm{I}_n(X), \mathrm{I}_r(X_r))$, X_r consisting of the first r columns of X, and the latter is a domain by the theory of Hochster–Eagon (cf. Proposition 1.2). It is clear that $G_+ \not\subset \mathfrak{a}+\mathfrak{b}$ and that height $G_+ \geq 1$. It is also clear that $t_{r+1}t_k \not\in \mathrm{I}_n(X)A$, $k = r+2, \ldots, m$. Thus, it remains to show that $(G/\mathfrak{a})[t_k^{-1}]$ is a domain, which is done by means of an argument similar to the one in part (i).

(iii) and (iv) The assertion on the canonical module follows from Proposition 6.1(i) as Proposition 3.2(ii) followed from Proposition 3.1(ii) by virtue of [7]. The type is calculated as above.

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