Standard Young Tableaux in the Weyl Group Setting

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A generalization of the notion of standard Young tableau has recently arisen from work on the representation theory of affine Hecke algebras. In the generalized setting, a standard tableau is defined to be any element of a finite Weyl group whose inversion set satisfies a certain pair of intersection conditions. In this paper, we prove that the set of generalized standard tableaux of fixed shape, when nonempty, is a certain interval in the weak ordering. In addition, we establish a nonemptiness criterion for the set of standard tableaux of prescribed shape. These results are obtained for shapes that satisfy an integrality condition.

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1. INTRODUCTION AND DEFINITIONS

Let \( \Phi \) be a finite crystallographic root system, spanning a real Euclidean space \( V \), and let \( W \) be the corresponding Weyl group. The basic facts concerning reflection groups and root systems that are used in this paper can be found in \([3, 4]\). Choose a system \( \Phi^+ \) of positive roots and let \( \Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) be the unique system of simple roots contained in \( \Phi^+ \). Recall that any root \( \beta \) can be written uniquely as a sum \( \sum_i m_i \alpha_i \), where the \( m_i \) are integers that are either all nonnegative or all nonpositive, and the height of \( \beta \) is the integer \( \sum_i m_i \).

One has, for each \( w \in W \), an inversion set \( \Phi(w) \) consisting of those \( \alpha \in \Phi^+ \) such that \( \langle w \alpha, \alpha \rangle \neq 0 \), where \( \Phi^- = -\Phi^+ \).

It is known that the simple system \( \Delta \) determines a fundamental domain \( D \) for the natural action of \( W \) on \( V \). The set \( D \) consists of those vectors \( \gamma \in V \) such that \( \langle \gamma, \alpha_i \rangle \geq 0 \) for all \( 1 \leq i \leq n \). Following \([5]\), we define, for each \( \gamma \in D \), two sets \( Z(\gamma) = \{ \alpha \in \Phi^+ | \langle \alpha, \gamma \rangle = 0 \} \) and \( P(\gamma) = \{ \alpha \in \Phi^+ | \langle \alpha, \gamma \rangle > 0 \} \).
\[ \langle \alpha, \gamma \rangle = 1 \]. In the sequel, we will sometimes denote these sets simply by \( Z \) and \( P \) when \( \gamma \) has been chosen and will remain fixed in a discussion. A **placed shape** is a pair \( (\gamma, J) \), where \( \gamma \in D \) and \( J \subseteq P(\gamma) \). A **standard tableau of shape** \( (\gamma, J) \) is an element \( w \in W \) such that \( \Phi(w) \cap Z(\gamma) = \emptyset \) and \( \Phi(w) \cap P(\gamma) = J \). We denote the collection of all standard tableaux of shape \( (\gamma, J) \) by \( \mathcal{T}(\gamma, J) \).

Some motivation for the above definitions of shape and tableau comes from the representation theory of affine Hecke algebras. Let \( H \) be the affine Hecke algebra corresponding to \( \Phi \). In [6], a type of placed shape, called a placed skew shape, is defined, and it is shown that there is a one-to-one correspondence between placed skew shapes \( (\gamma, J) \) and irreducible really calibrated representations \( H^{(\gamma, J)} \) of \( H \). Furthermore, the dimension of \( H^{(\gamma, J)} \) equals the number of standard tableaux of shape \( (\gamma, J) \). We remark that in type \( A \), one can convert placed shapes into placed configurations of boxes, and when a placed shape \( (\gamma, J) \) converts into a placed configuration of boxes of skew shape, the standard tableaux in \( \mathcal{T}(\gamma, J) \) convert into the classical standard tableaux of skew shape. The details of this conversion are explained in [5].

We impose the simplifying assumption of integrality on \( \gamma \in D \), meaning that \( \langle \alpha, \gamma \rangle \in \mathbb{Z} \) for all \( \alpha \in \Phi \). In [5], there is a discussion concerning the relationship between the general case and that of integral \( \gamma \).

In this paper, we prove that \( \mathcal{T}(\gamma, J) \), when nonempty, is a certain closed interval in the weak ordering of \( W \) (Theorem 3.2). We also establish a nonemptiness criterion for \( \mathcal{T}(\gamma, J) \) (Theorem 3.4). These results address conjectures posed by Ram in [5].

### 2. Sums of Positive Roots

The following two propositions establish some general properties concerning sums of positive roots and will play a role in our description of the set of standard tableaux of fixed shape. In their proofs, we freely use the following known property of roots: if \( \alpha, \beta \in \Phi \) are nonproportional, then \( \langle \alpha, \beta \rangle > 0 \) implies \( \alpha - \beta \in \Phi \), and \( \langle \alpha, \beta \rangle < 0 \) implies \( \alpha + \beta \in \Phi \).

**Proposition 2.1.** Let \( \alpha, \beta, \delta_1, \delta_2, \ldots, \delta_m \) be positive roots such that \( \alpha + \beta = \delta_1 + \delta_2 + \cdots + \delta_m \). Then, reindexing the \( \delta_i \) if necessary, we have \( \alpha = \delta_1 + \delta_2 + \cdots + \delta_k + \varepsilon \), where either \( \varepsilon = 0 \) or else both \( \varepsilon \) and \( \delta_{k+1} - \varepsilon \) are positive roots.

**Proof.** Observe first that the conclusion is symmetric in \( \alpha \) and \( \beta \) since the equation involving \( \alpha \) is equivalent to \( \beta = \delta_m + \delta_{m-1} + \cdots + \delta_{k+2} + (\delta_{k+1} - \varepsilon) \).
We proceed by induction on \( m \). The proposition is true for \( m = 1 \). Suppose \( m > 1 \). Since \( 0 < \langle \alpha + \beta, \alpha + \beta \rangle = \sum_j \langle \alpha + \beta, \delta_j \rangle \), we have \( \langle \alpha + \beta, \delta_j \rangle > 0 \) for some \( j \). Thus, \( \langle \alpha, \delta_j \rangle > 0 \) or \( \langle \beta, \delta_j \rangle > 0 \). By the symmetry noted above, we may assume that \( \langle \alpha, \delta_j \rangle > 0 \).

If \( \alpha = \delta_j \), then set \( \epsilon = 0 \) and reindex the \( \delta_j \) by the permutation \( \pi \) given by \( 1 \mapsto j, j \mapsto 1 \). If \( \alpha \neq \delta_j \), then \( \alpha - \delta_j \) is a root. If \( \alpha - \delta_j \in \Phi^- \) then \( \delta_j - \alpha \in \Phi^+ \). In this case, reindex the \( \delta_j \) by \( \pi \) and set \( \epsilon = \alpha \) (here, \( k = 0 \)). If \( \alpha - \delta_j \in \Phi^+ \), reindex the \( \delta_j \) by the permutation \( \tau \) given by \( j \mapsto m, m \mapsto j \), and then apply induction to the positive roots \( \alpha - \delta_m, \beta, \delta_1, \delta_2, \ldots, \delta_{m-1} \).

For the following proposition, we introduce a partial ordering on \( V \). Given \( \alpha, \beta \in V \), we write \( \alpha \leq \beta \) if and only if \( \beta - \alpha \) is a linear combination of simple roots with all coefficients nonnegative. Proposition 2.2. Let \( \alpha, \beta \in V \) be vectors such that \( \beta \in \Phi^+ \) and either \( \alpha = 0 \) or \( \alpha \in \Phi^- \). Then there exists a sequence of simple roots \( \alpha_1, \alpha_2, \ldots \) satisfying \( \beta - \alpha \) by 1. If \( \alpha \neq \delta_j \), then \( \alpha - \delta_j \) is a root. If \( \alpha - \delta_j \in \Phi^- \) then \( \delta_j - \alpha \in \Phi^+ \). In this case, reindex the \( \delta_j \) by \( \pi \) and set \( \epsilon = \alpha \) (here, \( k = 0 \)). If \( \alpha - \delta_j \in \Phi^+ \), reindex the \( \delta_j \) by the permutation \( \tau \) given by \( j \mapsto m, m \mapsto j \), and then apply induction to the positive roots \( \alpha - \delta_m, \beta, \delta_1, \delta_2, \ldots, \delta_{m-1} \).

Proposition 2.2. Let \( \alpha, \beta \in V \) be vectors such that \( \beta \in \Phi^+ \) and either \( \alpha = 0 \) or \( \alpha \in \Phi^- \). Then there exists a sequence of simple roots \( \alpha_1, \alpha_2, \ldots \) satisfying \( \beta - \alpha \) by 1. If \( \alpha \neq \delta_j \), then \( \alpha - \delta_j \) is a root. If \( \alpha - \delta_j \in \Phi^- \) then \( \delta_j - \alpha \in \Phi^+ \). In this case, reindex the \( \delta_j \) by \( \pi \) and set \( \epsilon = \alpha \) (here, \( k = 0 \)). If \( \alpha - \delta_j \in \Phi^+ \), reindex the \( \delta_j \) by the permutation \( \tau \) given by \( j \mapsto m, m \mapsto j \), and then apply induction to the positive roots \( \alpha - \delta_m, \beta, \delta_1, \delta_2, \ldots, \delta_{m-1} \).

Proof. The proof is simply that of the well known case \( \alpha = 0 \).

We proceed by induction on the height of \( \beta - \alpha \), assuming it to be greater than 1 since the lower cases are clear. Since \( \langle \beta - \alpha, \beta - \alpha \rangle > 0 \) and \( \beta - \alpha > 0 \), we have \( \langle \beta - \alpha, \alpha_i \rangle > 0 \) for some simple root \( \alpha_i \). Thus, either \( \langle \beta, \alpha_i \rangle > 0 \) or \( \langle - \alpha, \alpha_i \rangle > 0 \). We conclude by applying induction to the pair \( \alpha, \beta - \alpha \) in the first case and the pair \( \alpha + \alpha_j, \beta \) in the second case.}

3. A DESCRIPTION OF \( \mathcal{F}^{(\gamma, J)} \)

The set of standard tableaux of a given shape will be described in terms of the weak ordering of \( W \). Given elements \( v, w \in W \), we write \( v \leq w \) if and only if \( \Phi(v) \subseteq \Phi(w) \). This definition is equivalent to the usual definition of the (left) weak ordering [1, Proposition 2].

Our description of \( \mathcal{F}^{(\gamma, J)} \) as an interval in the weak ordering involves specifying its endpoints, and this is accomplished by proving that two particular subsets of positive roots are inversion sets for standard tableaux. Thus, the following known characterization of inversion sets will be useful.

Let \( T \subseteq \Phi^+ \). We say that \( T \) is closed if, whenever \( \alpha, \beta \in T \) and \( \alpha + \beta \in \Phi^+ \), we have \( \alpha + \beta \in T \). It is known (cf. [1, Proposition 3]; [2, (2.1)]) that a subset \( T \) of \( \Phi^+ \) is the inversion set of some \( w \in W \) if and only if both \( T \) and \( \Phi^+ \setminus T \) are closed (recall that \( \Phi \) is finite crystallographic).
The closure $\overline{T}$ of a subset $T$ of $\Phi^+$ is defined to be the smallest closed subset of $\Phi^+$ containing $T$. We remark that $\overline{T}$ equals the set of all positive roots $\beta$ such that $\beta$ can be written as a sum of elements from $T$. Given any subset $T$ of $\Phi^+$, we let $T^c$ denote the complement in $\Phi^+$ of $T$.

In [5, (1.3)], Ram proposes the following nonemptiness condition for $\mathcal{S}(\gamma, J)$:

(ne) \quad \text{If } \beta \in J, \alpha \in Z, \text{ and } \beta - \alpha \in \Phi^+ \text{ then } \beta - \alpha \in J.

The necessity of condition (ne) is evident. One of the results of this section establishes the sufficiency of condition (ne) for integral $\gamma$.

**Proposition 3.1.** Let $(\gamma, J)$ be a placed shape such that $\gamma$ is integral and condition (ne) is satisfied. Then the sets $\overline{T}$ and $Z \cup (P \setminus J)$ are closed.

**Proof.** Let $\beta := \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in \Phi^+$. There are two statements to verify:

1. If $\beta \in \overline{T}$ then $\beta_1 \in \overline{T}$ or $\beta_2 \in \overline{T}$.
2. If $\beta \in Z \cup (P \setminus J)$ then $\beta_1 \in Z \cup (P \setminus J)$ or $\beta_2 \in Z \cup (P \setminus J)$.

For statement 1, observe that $\beta \in \overline{T}$ implies $\beta = \delta_1 + \delta_2 + \cdots + \delta_m$, where each $\delta_i \in J$. By Proposition 2.1, we may take $\delta_1 = \delta_1 + \delta_2 + \cdots + \delta_k + \epsilon$, where either $\epsilon = 0$ or else $\epsilon, \delta_{k+1} - \epsilon \in \Phi^+$. If $\epsilon = 0$ then $\beta_1 = \delta_1 + \delta_2 + \cdots + \delta_k \in \overline{T}$.

Suppose $\epsilon \neq 0$. We now prove that either $\epsilon \in J$ or $\delta_{k+1} - \epsilon \in J$. By Proposition 2.2, there exists a sequence of simple roots $\alpha_i, \alpha_j, \ldots, \alpha_p$ such that $\epsilon = \delta_{k+1} - \alpha_i - \alpha_j - \cdots - \alpha_p$ and $\delta_{k+1} - \alpha_i - \alpha_j - \cdots - \alpha_p \in \Phi^+$ for all $1 \leq i \leq p$. Similarly, there exists a sequence of simple roots $\alpha_i, \alpha_j, \ldots, \alpha_q$ such that $\delta_{k+1} - \epsilon = \delta_{k+1} - \alpha_l - \alpha_j - \cdots - \alpha_q$ and $\delta_{k+1} - \alpha_l - \alpha_j - \cdots - \alpha_q \in \Phi^+$ for all $1 \leq j \leq q$.

Since $\gamma$ is integral and $\delta_{k+1} \in J \subseteq P(\gamma)$, we have either (a) $\alpha_i, \alpha_j, \ldots, \alpha_p \in Z$, or else (b) $\alpha_i, \alpha_j, \ldots, \alpha_q \in Z$. In case (a), repeated application of condition (ne), starting with $\delta_{k+1}$ in the role of $\beta$ and $\alpha_j$, in the role of $\gamma$, gives $\epsilon \in J$. Hence, $\beta_1 = \delta_1 + \delta_2 + \cdots + \delta_k + \epsilon \in \overline{T}$. In case (b), repeated application of condition (ne) gives $\delta_{k+1} - \epsilon \in J$, hence $\beta_2 = \delta_1 + \delta_2 + \cdots + \delta_k + (\delta_{k+1} - \epsilon) \in \overline{T}$. This completes the verification of statement 1.

Our approach to verifying statement 2 is similar. We write $\beta = \delta_1 + \delta_2 + \cdots + \delta_m$, where each $\delta_i \in Z \cup (P \setminus J)$. We use Proposition 2.1 to write $\beta_1 = \delta_1 + \delta_2 + \cdots + \delta_k + \epsilon$, where either $\epsilon = 0$ or else $\epsilon, \delta_{k+1} - \epsilon \in \Phi^+$.

If $\epsilon = 0$ then $\beta_1, \beta_2 \in Z \cup (P \setminus J)$. So suppose $\epsilon \neq 0$. Then integrality of $\gamma$ implies that either $\epsilon \in Z$ or $\delta_{k+1} - \epsilon \in Z$; in the former case $\beta_1 \in Z \cup (P \setminus J)$, and in the latter case $\beta_2 \in Z \cup (P \setminus J)$. This completes the verification of statement 2. \blacksquare
The following theorem generalizes a result concerning standard tableaux in type $A$ due to Ram [5, Theorem (4.5)] and answers the conjecture in [5, (11.1)] for all finite Weyl groups.

**Theorem 3.2.** Let $(\gamma, J)$ be a placed shape such that $\gamma$ is integral. If $\mathcal{F}(\gamma, J) \neq \emptyset$ then $\mathcal{F}(\gamma, J) = [w_{\min}, w_{\max}]$, an interval in the weak ordering of $W$, where $\Phi(w_{\min}) = \overline{J}$ and $\Phi(w_{\max}) = \overline{Z} \cup (P \setminus J)^\circ$.

**Proof.** Nonemptiness of $\mathcal{F}(\gamma, J)$ implies that condition (ne) holds. By Proposition 3.1 and the characterization of inversion sets that was stated earlier, there exist $w_{\min}, w_{\max} \in W$ such that $\Phi(w_{\min}) = \overline{J}$ and $\Phi(w_{\max}) = Z \cup (P \setminus J)^\circ$. If we can establish that $w \in \mathcal{F}(\gamma, J) = \Phi(w_{\min}) \subseteq \Phi(w) \subseteq \Phi(w_{\max})$, then the nonemptiness of $\mathcal{F}(\gamma, J)$ will give us the theorem. For the implication $\Rightarrow$, the first inclusion gives $J \subseteq \Phi(w)$ and the second inclusion gives $\Phi(w) \cap (Z \cup (P \setminus J)) = \emptyset$. Hence, $w \in \mathcal{F}(\gamma, J)$. For the implication $\Rightarrow$, we start with $\Phi(w) \cap P = J$ and $\Phi(w) \cap Z = \emptyset$. Hence, $\Phi(w) \subseteq (Z \cup (P \setminus J))^\circ$. Closedness of $\Phi(w)$ and the condition $\Phi(w) \cap P = J$ together imply $\overline{J} \subseteq \Phi(w)$.

Remark 3.3. In [5], a minimal element of $\mathcal{F}(\gamma, J)$ in the weak ordering is called a column reading tableau, and a maximal element of $\mathcal{F}(\gamma, J)$ is called a row reading tableau. Theorem 3.2 may be understood in this language as asserting the existence of a unique column reading tableau and a unique row reading tableau when $\mathcal{F}(\gamma, J)$ is nonempty and $\gamma$ is integral.

The following theorem addresses the sufficiency of the nonemptiness condition (ne).

**Theorem 3.4.** Let $(\gamma, J)$ be a placed shape such that $\gamma$ is integral. If condition (ne) is satisfied then $\mathcal{F}(\gamma, J)$ is nonempty.

**Proof.** We invoke Proposition 3.1 and the characterization of inversion sets stated earlier, thereby obtaining elements $w_{\min}, w_{\max} \in W$ such that $\Phi(w_{\min}) = \overline{J}$ and $\Phi(w_{\max}) = \overline{Z} \cup (P \setminus J)^\circ$. In the proof of Theorem 3.2, we showed that $w \in \mathcal{F}(\gamma, J)$ if and only if $\Phi(w_{\min}) \subseteq \Phi(w) \subseteq \Phi(w_{\max})$. It therefore suffices in the present situation to prove that $\Phi(w_{\min}) \subseteq \Phi(w) \subseteq \Phi(w_{\max})$.

We prove that if $\beta \in Z \cup (P \setminus J)$ then $\beta \in \overline{J}$ by induction on $m$, where $\beta = \delta_1 + \delta_2 + \cdots + \delta_m$ and each $\delta_i \in Z \cup (P \setminus J)$. If $m = 1$ then $\beta \in Z$ or $\beta \in P \setminus J$. If $\beta \in Z$ then $\beta \in \overline{J}$ because every element $\alpha \in \overline{J}$ is a sum of elements of $J$, hence $\langle \alpha, \gamma \rangle$ is a positive integer. By similar reasoning, if $\beta \in P \setminus J$ then $\beta \notin \overline{J}$. Let $m > 1$. We have $0 < \langle \beta, \delta \rangle = \sum \langle \beta, \delta_i \rangle$, hence $\langle \beta, \delta \rangle > 0$ for some $i$. We may assume $j = m$. Hence, $\beta - \delta_m$ is a root. Moreover, since $\beta - \delta_m = \delta_1 + \delta_2 + \cdots + \delta_{m-1}$, we see that $\beta - \delta_m \in Z \cup (P \setminus J)$.
induction, $\beta - \delta_m \in \mathcal{J}^c$. The case $m = 1$ gives $\delta_m \in \mathcal{J}^c$. Since $\mathcal{J}^c$ is closed, we have $\beta = (\beta - \delta_m) + \delta_m \in \mathcal{J}$, and the induction step is complete. 

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