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Orthogonal Polynomials on the Negative Multinomial Distribution

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Orthogonal polynomials on the multivariate negative binomial distribution,

$$(1+\theta)^{-\alpha-x}\left\{\prod_{j=0}^{p}\theta_j^{x_j}/x_j!\right\}\Gamma(\alpha+x)/\Gamma(\alpha),$$

where $\alpha > 0$, $\theta_j > 0$, $x = \Sigma x_j$, $\theta = \Sigma\theta_j$, $x_0, x_1, \ldots, x_p = 0, 1, \ldots$ are constructed and their properties studied.

INTRODUCTION

One generalization of the negative binomial to higher dimensions is given by the distribution

$$(1+\theta)^{-\alpha-x}\left(\prod_{j=0}^{p}\theta_j^{x_j}/x_j!\right)\Gamma(\alpha+x)/\Gamma(\alpha), \tag{1}$$

where $\alpha > 0$, $\theta_j > 0$, $x = \sum_0^p x_j$, $\theta = \sum_0^p \theta_j$, $x_0, x_1, \ldots, x_p = 0, 1, \ldots$.

The aim of this paper is to generalize the Meixner polynomials on the negative binomial distribution to a set of orthogonal polynomials on the distribution (1). If $\mathbf{k}$ is any vector the notation $k = \sum_i k_i$ will be used, and at times the parameters of a function will be suppressed to avoid cumbersome notation.

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Generating Function

Define a set of functions $\{m_{\mathbf{k}}(\mathbf{X})\}$ by

$$G(\mathbf{X}, \mathbf{w}) = \sum_{\mathbf{k} \geqslant 0} m_{\mathbf{k}}(\mathbf{X})\, w_0^{k_0} \cdots w_p^{k_p}/(k_0! \cdots k_p!), \tag{2}$$

$$= (1 - w_0)^{-X-\alpha} \prod_{i=0}^{p} \Big(1 - w_0(1+\theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j\Big)^{X_i},$$

where $\{v^{(j)}; j = 0, 1,..., p\}$ is a complete set of orthogonal functions on $\{\theta_i\}$ with $v^{(0)} \equiv 1$ and the summation is taken over $k_j = 0, 1,..., j = 0, 1,..., p$. The domain D of $\mathbf{w}$ is given by $\sum_0^p a_i^2 w_i^2 < 1$, where $a_0^2 = (1+\theta)/\theta$,

$$a_j^2 = \sum_{i=0}^{p} v_i^{(j)^2} \theta_i .$$

Let $b_i(\mathbf{w}) = 1 - w_0(1+\theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j$. Then

$$EG(\mathbf{X}, \mathbf{w})\, G(\mathbf{X}, \mathbf{v}) = EE \Big\{[(1-w_0)(1-v_0)]^{-\alpha-X} \prod_{i=0}^{p} [b_i(\mathbf{w})\, b_i(\mathbf{v})]^{X_i} \mid X\Big\}$$

$$= E[(1-w_0)(1-v_0)]^{-\alpha-X} \Big[\sum_{i=0}^{p} b_i(\mathbf{w})\, b_i(\mathbf{v})\, \theta_i/\theta\Big]^X$$

$$= \Big[(1+\theta)(1-w_0)(1-v_0) - \sum_{i=0}^{p} \theta_i b_i(\mathbf{w})\, b_i(\mathbf{v})\Big]^{-\alpha}$$

$$= \Big(1 - \sum_{j=0}^{p} a_j^2 w_j v_j\Big)^{-\alpha}.$$

Thus $Em_{\mathbf{h}}(\mathbf{X}) m_{\mathbf{k}}(\mathbf{X}) = \delta_{\mathbf{hk}} a_0^{2k_0} \cdots a_p^{2k_p} k_0! \cdots k_p!\, \Gamma(\alpha + k)/\Gamma(\alpha)$,

$$\delta_{\mathbf{hk}} = \begin{cases} 0 & \text{if } \mathbf{h} \neq \mathbf{k}, \\ 1 & \text{if } \mathbf{h} = \mathbf{k}, \end{cases}$$

and $M_{\mathbf{k}}(\mathbf{X}) = \{\Gamma(\alpha)/(\Gamma(\alpha + k)\, k_0! \cdots k_p!)\}^{1/2} a_0^{-k_0} \cdots a_p^{-k_p} m_{\mathbf{k}}(\mathbf{X})$ are the orthonormal functions.

Transform

Denote $Ef(\mathbf{X})s_0^{X_0} \cdots s_p^{X_p}$ by $f^*(\mathbf{s})$ for any function f for which $E\,|f(\mathbf{X})| < \infty$. It is straightforward to show

$$G^*(\mathbf{s}, \mathbf{w}) = \left\{(1 + \theta)(1 - w_0) - \sum_{i=0}^{p} \theta_i s_i \left(1 - w_0(1 + \theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j\right)\right\}^{-\alpha},$$

$$= \left\{(1 + \theta - S_0) - (1 + \theta)(\theta - S_0)\, w_0/\theta - \sum_{j=1}^{p} S_j w_j\right\}^{-\alpha},$$

where $S_j = \sum_{i=0}^{p} s_i \theta_i v_i^{(j)}$, $j = 0, 1,\ldots, p$. Thus

$$m_{\mathbf{k}}^*(\mathbf{s}) = a_0^{2k_0}(1 + \theta - S_0)^{-\alpha-k}\,(\theta - S_0)^{k_0} \left(\prod_{j=1}^{p} S_j^{k_j}\right) \Gamma(\alpha + k)/\Gamma(\alpha).$$

Completeness

Suppose $Ef(\mathbf{X})M_{\mathbf{k}}(\mathbf{X}) = 0$ for every index vector $\mathbf{k}$, then this is equivalent to

$$Ef(\mathbf{X})\, G(\mathbf{X}, \mathbf{w}) = 0 \qquad \text{for all} \quad \mathbf{w} \in D.$$

This means that

$$\sum_{\mathbf{x} \geqslant \mathbf{0}} f(\mathbf{x})\, \Gamma(\alpha + x) \prod_{i=0}^{p} (\phi_i^{x_i}/x_i!) = 0,$$

where $\phi_i = \theta_i\{1 - w_0(1 + \theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j\}/(1 - w_0)(1 + \theta)$, for $i = 0, 1,\ldots, p$, which is a one-to-one and onto transformation of $\mathbf{w}$ on account of the complete orthogonal functions $\{v^{(j)}\}$ on $\{\theta_i\}$. Hence $Ef(\mathbf{X})\, M_{\mathbf{k}}(\mathbf{X}) = 0$ for every $\mathbf{k} \geqslant \mathbf{0}$ is equivalent to $f(\mathbf{x}) \equiv 0$ for all $\mathbf{x}$.

Let $\mathbf{e}_h$ be the vector (δ_{jh}), $h = 0, 1,\ldots, p$. It will be shown that $\{m_{k\mathbf{e}_0 + r\mathbf{e}_j}(\mathbf{X});\ k = 0, 1,\ldots, r = 0, 1,\ldots\}$ is a complete orthogonal polynomial set in X_0 and X alone if and only if $v_1^{(j)} = v_2^{(j)} = \cdots = v_p^{(j)}$ and $v_0^{(j)} \neq v_1^{(j)}$.

To prove this, write $X_0 = Y_0$, $X = Y$, $X_i = Y_i$ for $i = 2, 3,\ldots, p$ in (2). Then, for any w_0 and w_j,

$$G(\mathbf{Y}, w_0, w_j) = \sum_{k,r} m_{k\mathbf{e}_0 + r\mathbf{e}_j}(\mathbf{X})\, w_0^{k} w_j^{r}/k!r!$$

$$= (1 - w_0)^{-Y-\alpha}\,(1 - w_0(1 + \theta)/\theta + v_0^{(j)} w_j)^{Y_0}$$

$$\cdot (1 - w_0(1 + \theta)/\theta + v_1^{(j)} w_j)^{Y-Y_0}$$

$$\cdot \prod_{i=2}^{p} \left[1 + \frac{(v_i^{(j)} - v_1^{(j)})\, w_j}{1 - w_0(1 + \theta)/\theta + v_1^{(j)} w_j}\right]^{Y_i}.$$

If $v_1^{(j)} = \cdots = v_p^{(j)}$ and $v_1^{(j)} \neq v_0^{(j)}$, then the result is obvious and conversely if $G(\mathbf{Y}, w_0, w_j)$ is a function of Y and Y_0 for any $Y_2, \ldots, Y_p, w_0$ and w_j; then this implies

$$\prod_{i=2}^{p} [1 + (v_i^{(j)} - v_1^{(j)})\, w_j/\{1 + w_0(1+\theta)/\theta + v_1^{(j)} w_j\}]^{Y_i} = 1,$$

which gives $v_i^{(j)} = v_1^{(j)}$ for $i = 2, 3, \ldots, p$ and $v_0^{(j)} \neq v_1^{(j)}$, because otherwise, it will be a function of Y only.

Polynomial Structure

Define the degree of a multidimensional polynomial

$$\sum_{\mathbf{a}} q(\mathbf{a})\, z_0^{a_0} \cdots z_p^{a_p}$$

to be the degree of the polynomial $\sum_{\mathbf{a}} q(\mathbf{a}) u^a$.

If $\chi_j = \sum_{i=0}^{p} v_i^{(j)} X_i$, $j = 0, 1, \ldots, p$, then any polynomial of degree δ in $\mathbf{X}$ is a polynomial of degree δ in $\boldsymbol{\chi}$, and vice versa.

$$\begin{aligned}
&E\chi_0^{\delta_0} \cdots \chi_p^{\delta_p} G(\mathbf{X}, \mathbf{w}) \\
&\quad = E \sum_{\beta \leqslant \delta} a_{\boldsymbol{\beta}} \mathbf{X}^{(\boldsymbol{\beta})} G(\mathbf{X}, \mathbf{w}) \\
&\quad = \sum_{\beta \leqslant \delta} a_{\boldsymbol{\beta}} (\alpha + \beta - 1)^{(\beta)} \prod_{i=0}^{p} \left[\theta_i \left(1 - w_0(1+\theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j\right)\right]^{\beta_i} \\
&\quad = (\alpha + \delta - 1)^{(\delta)} \prod_{l=0}^{p} \left\{\sum_{i=0}^{p} v_i^{(l)} \theta_i \left(1 - w_0(1+\theta)/\theta + \sum_{j=1}^{p} v_i^{(j)} w_j\right)\right\}^{\delta_l} + H(\mathbf{w}) \\
&\quad = (\alpha + \delta - 1)^{(\delta)}\, \theta^{\delta_0} (1 - w_0(1+\theta)/\theta)^{\delta_0} \prod_{l=1}^{p} a_l^{2\delta_l} w_l^{\delta_l} + H(\mathbf{w}),
\end{aligned}$$

where $\{\alpha_{\boldsymbol{\beta}}\}$ is a set of constants and $H(\mathbf{w})$ is a polynomial in $\mathbf{w}$ of degree at most $\delta - 1$. Thus $E\chi_0^{\delta_0} \cdots \chi_p^{\delta_p} M_{\mathbf{k}}(\mathbf{X}) = 0$ if either $k > \delta$, or $\mathbf{k} \neq \boldsymbol{\delta}$ with $k = \delta$, which implies that

$$M_{\mathbf{k}}(\mathbf{X}) = c_{\mathbf{k}} \chi_0^{k_0} \cdots \chi_p^{k_p} + Q(\mathbf{X}),$$

where $c_{\mathbf{k}}$ is a nonzero constant and $Q(\mathbf{X})$ is a polynomial of degree at most $k - 1$. The above representation means that $\{M_{\mathbf{k}}(\mathbf{X})\}$ can be constructed by applying the

Gram–Schmidt orthogonalization process to the sequence

$$1, \chi_0^{a_0}\chi_1^{a_1} \cdots \chi_p^{a_p}, \chi_0^{b_0}\chi_1^{b_1} \cdots \chi_p^{b_p}, \ldots,$$

where each of the products is distinct,

$$\sum a_i \leqslant \sum b_i \leqslant \cdots \text{ and the ordering when } \sum a_i = \sum b_i \text{ is arbitrary.}$$

Clearly, the $M_{\mathbf{k}}(\mathbf{X})$ are polynomials of maximum degree k and if $P_{\mathbf{h}}(\mathbf{X})$ is any polynomial of degree $h < k$, then $EM_{\mathbf{k}}(\mathbf{X})\, P_{\mathbf{h}}(\mathbf{X}) = 0$ and so any other set of orthogonal polynomials $\{N_{\mathbf{h}}(\mathbf{X})\}$ can be written as

$$N_{\mathbf{h}}(\mathbf{X}) = \sum_{k=h} (EN_{\mathbf{h}}(\mathbf{X})\, M_{\mathbf{k}}(\mathbf{X}))\, M_{\mathbf{k}}(\mathbf{X}).$$

Runge-Type Identity

Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent random vectors having the distribution (1) with parameters α_1 and α_2, then

$$G(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{w}; \alpha_1 + \alpha_2) = G(\mathbf{X}_1, \boldsymbol{\omega}; \alpha_1)\, G(\mathbf{X}_2, \mathbf{w}; \alpha_2).$$

Thus

$$m_{\mathbf{k}}(\mathbf{X}_1 + \mathbf{X}_2 ; \alpha_1 + \alpha_2) = \sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} \binom{k_0}{h_0} \cdots \binom{k_p}{h_p} m_{\mathbf{h}}(\mathbf{X}_1 ; \alpha_1)\, m_{\mathbf{l}}(\mathbf{X}_2 ; \alpha_2).$$

A Runge-type identity holds for a class of polynomials and the reader is referred to Eagleson [1] for background material.

Bilinear Sums

Define

$$g_h(\mathbf{x}, \mathbf{y}) = \sum_{k=h} M_{\mathbf{k}}(\mathbf{x})\, M_{\mathbf{k}}(\mathbf{y}).$$

The transform of $g_h(\mathbf{x}, \mathbf{y})$ is

$$\begin{aligned} g_h^*(\mathbf{s}, \mathbf{t}) &= [\Gamma(\alpha + h)/\Gamma(\alpha)\, h!](1 + \theta - S_0)^{-\alpha-h}\, (1 + \theta - T_0)^{-\alpha-h} \\ &\quad \cdot \left\{(\theta - S_0)(\theta - T_0)(1 + \theta)/\theta + \sum_{i=1}^{p} S_i T_i / a_i^2\right\}^h, \\ &= [\Gamma(\alpha + h)/\Gamma(\alpha) h!](1 + \theta - S_0)^{-\alpha}\, (1 + \theta - T_0)^{-\alpha} \\ &\quad \cdot \left\{1 - \left[(1 + \theta) - \sum_{i=0}^{p} \theta_i s_i t_i\right] \Big/ (1 + \theta - S_0)(1 + \theta - T_0)\right\}^h, \end{aligned}$$

where

$$S_j = \sum_{i=0}^{p} \theta_i s_i v_i^{(j)}, \qquad T_j = \sum_{i=0}^{p} \theta_i t_i v_i^{(j)}, \qquad j = 0, 1, 2,..., p.$$

Thus $g_h(\mathbf{x}, \mathbf{y})$ is invariant under any choice of $\{v^{(j)}\}$.

If $\{N_{\mathbf{k}}(\mathbf{X})\}$ is any other orthonormal polynomial set

$$g_h(\mathbf{X}, \mathbf{Y}) = \sum_{k=h} N_{\mathbf{k}}(\mathbf{X})\, N_{\mathbf{k}}(\mathbf{Y}),$$

and if $R(\mathbf{x})$ is a polynomial of degree n, then

$$R(\mathbf{x}) = \sum_{h\leqslant n} E g_h(\mathbf{x}, \mathbf{X})\, R(\mathbf{X}).$$

Poisson Limit

Suppose $\theta_i \to 0$, $\alpha \to \infty$ while $\alpha\theta_i \to \mu_i$, $0 < \mu_i < \infty$, $i = 0, 1,..., p$. It is easily shown that the distribution of $\mathbf{X}$ converges to that of $p + 1$ independent Poisson random variables with means μ_i, $i = 0, 1,..., p$. Further suppose $\alpha^{-1} v_i^{(j)} \to u_i^{(j)}$, $i = 0, 1,..., p$, $j = 1, 2,..., p$ and $\{u^{(j)}\}$ is complete on $\{\mu_i\}$; then

$$G(\mathbf{X}, \alpha^{-1}\mathbf{w}) \to e^{w_0} \prod_{i=0}^{p} \left(1 - w_0/\mu + \sum_{j=1}^{p} u_i^{(j)} w_j\right)^{X_i},$$

which generates a complete set of orthogonal functions on the limit distribution.

Normal Limit

If $\alpha \to \infty$, while $\boldsymbol{\theta}$ remains fixed, the distribution of $\mathbf{Y} = (\mathbf{X} - \alpha\boldsymbol{\theta})/\alpha^{1/2}$ converges to that of $p + 1$ normal random variables with variances

$$\theta_0(1 + \theta_0),..., \theta_p(1 + \theta_p)$$

and covariances $\theta_0\theta_1,..., \theta_{p-1}\theta_p$.

$$G(\alpha^{1/2}\mathbf{Y} + \alpha\boldsymbol{\theta}, \alpha^{-1/2}\mathbf{w}) \to \exp\left(\sum_{j=0}^{p} \eta_j w_j - \frac{1}{2}\sum_{j=0}^{p} a_j^2 w_j^2\right),$$

where $\eta_0 = -Y/\theta$, $\eta_j = -\sum_{i=0}^{p} Y_i v_i^{(j)}$, $j = 1, 2,..., p$.

The limiting generating function generates the product set of Hermite–Chebycheff polynomials on the independent normal variables $\eta_0, \eta_1,..., \eta_p$.

Gamma Limit

Suppose $\theta_i \to \infty$, $\theta_i/\theta \to \nu_i$, $i = 0, 1, \ldots, p$ and $\theta v^{(j)} \to 0$, $j = 1, 2, \ldots, p$ while α remains fixed.

The distribution of $\mathbf{X}/\theta$ converges to that of a random vector $\mathbf{Y}$; where $Y_i = \nu_i Z$ and Z has a gamma (α) distribution.

$$G(\theta\mathbf{Y}, \mathbf{w}) \to (1 - w_0)^{-\alpha} \exp\{-\nu Z w_0/(1 - w_0)\},$$

which generates the set of Laguerre polynomials on Z.

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