**S-matrices**

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**ABSTRACT**

A new class of so called S-matrices is introduced which allows investigating links between various known classes of matrices such as Vandermonde matrices, Hankel matrices, companion matrices, etc. For complex S-matrices, the problem of decomposition into a quasidirect sum (a sum for which the sum of the ranks of the summands equals the rank of the given matrix) of indecomposable complex S-matrices is completely solved, and the uniqueness of such a decomposition is proved.

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1. INTRODUCTION

We intend to investigate the class of so-called S-matrices, which comprises important classes of matrices such as Vandermonde matrices, extension of companion matrices, proper Hankel matrices, etc.

A complex matrix (in general over a field)

\[ A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ a_{10} & a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{kk} \end{pmatrix} \]

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with a finite number of columns is called an S-matrix if, for an indeterminate \( x \), the extended matrix

\[
A_x = \begin{pmatrix}
a_{00} & a_{01} & \cdots & a_{0k} & 1 \\
a_{10} & a_{11} & \cdots & a_{1k} & x \\
a_{20} & a_{21} & \cdots & a_{2k} & x^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots
\end{pmatrix}
\]

has the property that the greatest common divisor of all its subdeterminants of order \( r(A)+1 \), where \( r(A) \) is the rank of \( A \), is a nonzero polynomial of degree \( r(A) \). This (monic) polynomial will then be called S-polynomial of \( A \).

We shall call an S-matrix pure if its columns are linearly independent.

**Remark 1.1.** In order to form subdeterminants of order \( r(A)+1 \), there must be at least \( r(A)+1 \) rows in \( A \). Hence, any S-matrix has linearly dependent rows.

**Lemma 1.2.** An S-matrix has S-polynomial 1 iff it is a zero matrix.

**Proof.** This follows immediately from the definition.

**Example 1.3.** Any \( m \times n \) Vandermonde matrix, \( m > n \) — i.e., any matrix of the form

\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
t_1 & t_2 & \cdots & t_n \\
t_1^2 & t_2^2 & \cdots & t_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

with mutually distinct numbers \( t_1, \ldots, t_n \) and \( m \) rows—is a pure S-matrix. Its S-polynomial is \( \prod_{i=1}^{n}(x-t_i) \). The same is true if the number of rows is infinite.

This follows from Theorem 2.7 as a special case.

**Example 1.4.** The well-known companion matrix of a polynomial \( f(x) = x^n - a_1x^{n-1} - \cdots - a_n \) is the matrix

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1
\end{pmatrix}
\]
Its corresponding generalized companion matrix \([2]\) is the matrix

\[
C^\infty = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_n & a_{n-1} & \cdots & a_1 \\
a_n a_1 & a_n + a_1 a_{n-1} & \cdots & a_2 + a_1^2 \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

obtained in the following way: If the \(k\)th row of \(C^\infty\) is denoted as \(C_k\), then the first \(n\) rows \(C_1, \ldots, C_n\) are identical with the first \(n\) rows of the \(n \times n\) identity matrix, while for \(k > n + 1\),

\[
C_k = a_1 C_{k-1} + a_2 C_{k-2} + \cdots + a_n C_{k-n}.
\]  

We shall show in Theorem 2.5 that \(C^\infty\) is a pure S-matrix with S-polynomial \(f(x)\).

**Example 1.5.** Let \(t\) be a number. The \(m \times n\) \((m > n)\) matrix

\[
P_{mn}(t) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
t & 1 & 0 & \cdots & 0 \\
t^2 & 2t & 1 & \cdots & 0 \\
t^3 & 3t^2 & 3t & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix} = (p_{ij}),
\]

\[
p_{ij} = \binom{i}{j} t^{i-j}, \quad i = 0,1,\ldots,m-1, \quad j = 0,1,\ldots,n-1,
\]

is also a pure S-matrix. Its S-polynomial is \((x - t)^n\).

This again follows from Theorem 2.7 as a special case.

**Example 1.6.** Let \(A = (a_{i+k})\), \(i, k = 0,1,\ldots\), be an \(m \times n\) proper Hankel matrix with linearly dependent rows. Proper means that the upper left corner submatrix of \(A\) of order \(r(A)\), the rank of \(A\), is nonsingular. Then \(A\) is an S-matrix and its S-polynomial is identical with the \(R\)-polynomial in \([1]\).

We shall be interested in the properties of S-matrices, in particular in the so-called quasidirect decompositions of S-matrices into sums of matrices which
are again S-matrices. Here, a sum $B + C$ of matrices of the same size is called quasidirect if for the ranks, $r(B + C) = r(B) + r(C)$. This definition clearly extends to quasidirect sums of more than two matrices as well as to matrices which have, as in our case, a finite number of columns but maybe an infinite number of rows.

In Example 1.3, it is clear that $V$ is a quasidirect sum of the matrices $V_1, \ldots, V_n$ of the same size as $V$, where $V_i$ consists of the $i$th column of $V$, all the remaining entries being zero. The $S$-polynomial of $V_i$ is then $x - t_i$.

2. RESULTS

**Lemma 2.1.** Let $A$, $B$ be matrices with the same column space. Then $A$ is an S-matrix iff $B$ is. In such case, both matrices have the same $S$-polynomial.

**Proof.** Follows from the fact that the definition of the $S$-matrix and of the $S$-polynomial depend on the column-space only.

**Corollary 2.2.** If $A$ is an $S$-matrix with $m$ columns and $G$ is any $m \times n$ matrix with linearly independent rows, then $B = AG$ is again an $S$-matrix which has, in addition, the same $S$-polynomial as $A$.

We shall prove now:

**Lemma 2.3.** Let $A$ be an $S$-matrix with $n$ columns and rank $r$. Then the first $r$ rows of $A$ are linearly independent.

**Proof.** Suppose that the rank of the submatrix of $A$ consisting of the first $r$ rows of $A$ is $s < r$.

There exists a nonsingular matrix $G_1$ of order $n$ such that $A = AG_1$ has zeros in the first $n - r$ columns. By Corollary 2.2, $A$ is again an $S$-matrix. Since the submatrix $A_0$ of $A$ consisting of the first $r$ rows of $A$ has also rank $s$, there exists a nonsingular matrix $G_2$ of order $n$ such that $AG_2$ has in the first $n - r$ columns also zeros and $A_0G_2$ has zeros even in the first $n - s$ columns. Thus, $AG_1G_2 = \begin{pmatrix} 0 & 0 & Q \\ 0 & S_1 & S_2 \end{pmatrix}$, where $Q$ is an $r \times s$ matrix with rank $s$ and $S_1$ has $r - s$ linearly independent columns. Thus there exists a subset $J_1$ of $s + 1$ row indices such that the
corresponding submatrix of $Q$ has also rank $s$. Since $S_1$ has linearly independent columns, there exists a set $J_2$ of $r - s$ row indices such that the corresponding submatrix of $S_1$ is nonsingular. It follows that the submatrix of $A$ with the last $r + 1$ columns and the rows with the set of indices $J_1 \cup J_2$ has determinant which is a nonzero polynomial of degree less than $r$, a contradiction. 

Corollary 2.4. The rank of an S-matrix is equal to the number $r$ for which the first $r$ rows of $A$ are linearly independent whereas the first $r + 1$ rows are not.

Theorem 2.5. Let $A$ be an S-matrix with rank $r$, and $A_0$ its submatrix of the first $r$ rows. Then there exists a unique matrix $\hat{C}$ such that

$$A = \hat{C}A_0.$$

The matrix $\hat{C}$ is a pure S-matrix, equal to the leading submatrix, of the appropriate size, of the generalized companion matrix of the S-polynomial of $A$.

Proof. By Lemma 2.1 and Corollary 2.2, we can assume that $A$ with $n$ columns is pure, $r = n$, so that $A_0$ is a square nonsingular matrix. Thus the matrix $M = AA_0^{-1} = (m_{ik})$, $i = 0, 1, \ldots, k = 1, \ldots, n$, is a pure S-matrix whose first $n$ rows form the $n \times n$ identity matrix. Let

$$\varphi = x^n - \sum_{i=1}^{n} a_i x^{n-i}$$

be its S-polynomial, which means that the $(n + 1)$st row is a linear combination of the first $n$ rows with the coefficients $a_n, \ldots, a_1$. Denote by $\hat{C}$ the leading submatrix of the generalized companion matrix of $\varphi$ of the same size as $M$. We shall show that $M = \hat{C}$. Let us distinguish three cases:

Case 1: $a_1 = a_2 = \cdots = a_n = 0$ so that $\varphi = x^n$. Assume there is an entry $m_{ik} \neq 0$ for $i \geq n$ and $1 \leq k \leq n$. Then the determinant of $A_x$ in the first $n$ and the $(i + 1)$st row has the form $x^{i+1} - m_{ik} x^{k-1} + \cdots$ and is not divisible by $x^n$, a contradiction. Thus $M = \hat{C}$ in this case.

Case 2: $a_n \neq 0$. Denote, for a moment, by $M_t$, $t = 0, 1, \ldots$, that square submatrix of $M$ with $n$ rows which corresponds to row indices $t, t + 1, \ldots, n + t - 1$; and by $m_s$ the row vector of $M$ with the index $s$. Clearly $M_0 = I$ and $M_1$
is the usual companion matrix of the polynomial \( \varphi \). We shall show by induction with respect to \( t \) that \( M_t \) is nonsingular for all possible values of \( t = 0, 1, \ldots \) and that

\[
m_{n+t} = aM_t,
\]

where \( a = (a_n, a_{n-1}, \ldots, a_1) \). Clearly, these properties characterize, together with \( M_0 = I \), the matrix \( \hat{C} \).

For \( t = 0 \), the assertion is true. Suppose thus that \( t > 0 \) and that the assertion is true for \( t - 1 \). Consider the determinant

\[
\det \begin{pmatrix}
m_{t-1} & x^{t-1} \\
m_t & x^t \\
\vdots & \vdots \\
m_{n+t-1} & x^{n+t-1}
\end{pmatrix}
= x^{n+t-1} \det M_{t-1} + \cdots + (-1)^n x^{t-1} \det M_t.
\]

Since it is divisible by \( \varphi \) and \( a_n \neq 0 \), \( \det M_{t-1} \neq 0 \) implies \( \det M_t \neq 0 \). Now,

\[
\det \begin{pmatrix}
m_t & x^t \\
\vdots & \vdots \\
m_{n+t} & x^{n+t}
\end{pmatrix}
= x^t \det M_t \det \begin{pmatrix}
1 & X \\
M_t^{-1} & x^n
\end{pmatrix},
\]

where \( X = (1, \ldots, x^{n-1})^T \). Since \( \varphi \) and \( x^t \) are relatively prime, we have

\[
\varphi = \det \begin{pmatrix}
I & X \\
m_{n+t}M_t^{-1} & x^n
\end{pmatrix}
= x^n - m_{n+t}M_t^{-1}X.
\]

Therefore

\[
m_{n+t}M_t^{-1} = a,
\]

\[
aM_t = m_{n+t},
\]

and this together with (1) implies \( M = \hat{C} \).

Case 3: \( a_n = a_{n-1} = \cdots = a_{s+1} = 0, \ a_s \neq 0 \), where \( 1 \leq s \leq n-1 \). Then \( \varphi \) is divisible by \( x^{n-s} \) but not by \( x^{n-s+1} \). We shall show first that \( M \) has the
where \( \hat{M} \) is an S-matrix with the S-polynomial \( \hat{\phi} = x^i - a_1 x^{s-1} - \cdots - a_s \) and such that its leading submatrix with \( s \) rows is \( I_s \). Suppose that \( m_{ik} \neq 0 \) for some \( i \geq n - s \) and \( k \) satisfying \( 1 \leq k \leq n - s \). Then \( i = n \) and, choosing the submatrix \( M_1 \) in \( M_x \) with rows \( 0, \ldots, n - 1, i \), we obtain

\[
\det M_1 = \det \begin{pmatrix} I & X \\ m_i & x^i \end{pmatrix} = x^i - m_{ik} x^{k-1} - \cdots.
\]

However, this polynomial should be divisible by \( \phi \), and thus by \( x^{n-s} \). Consequently, \( k - 1 \geq n - s \), a contradiction. Therefore, \( \hat{M} \) has the form (3). The greatest common divisor of all determinants of order \( n + 1 \) in \( M_x \) being \( \phi \), it follows easily that the greatest common divisor of all determinants of order \( s + 1 \) in the analogous matrix \( \hat{M}_x \) is \( \hat{\phi} \). Thus \( \hat{M} \) is an S-matrix, and by case 2, \( \hat{M} \) is a section of the generalized companion matrix of \( \hat{\phi} \). It follows that then again \( M = \hat{C} \).

**Corollary 2.6.** If an S-matrix \( A \) and a pure S-matrix \( B \) have the same number of rows (finite or infinite) and the same S-polynomial, then there exists a matrix \( M \) with linearly independent rows such that

\[
A = BM.
\]  

**Proof.** By Theorem 2.5, \( A = CA_0 \), \( B = CB_0 \) with the same matrix \( C \), a nonsingular matrix \( B_0 \) and \( A_0 \) with linearly independent rows. Therefore, \( M = B_0^{-1} A_0 \) satisfies the condition (4). 

**Definition.** A generalized Vandermonde matrix is any matrix of the form

\[
V = \left( P_{m,n_1}(t_1), P_{m,n_2}(t_2), \ldots, P_{m,n_s}(t_s) \right)
\]

where \( t_1, \ldots, t_s \) are mutually distinct complex numbers, \( n_1, n_2, \ldots, n_s \) positive integers, and \( P_{m,n_k}(t_k) \) matrices of the form (2),

\[
m > n = \sum_{j=1}^{s} n_j.
\]
THEOREM 2.7. The matrix $V$ in (5) is a pure S-matrix. Its S-polynomial is

$$
\varphi = \prod_{i=1}^{s} (x - t_i)^{a_i}.
$$

Proof. Follows from the fact that each determinant of $V_i$ of order $n + 1$ is divisible by $(x - t_j)^n$ for each $j$ and hence by $\varphi$, whereas such a determinant of the matrix formed by the first $n + 1$ rows is equal to $\varphi$.

THEOREM 2.8. Let $A$ be an $m \times n$ S-matrix. Then the S-polynomial of $A$ is (6) iff there exists a matrix $M$ with $n$ linearly independent rows such that

$$
A = VM
$$

where $V$ has the form (5).

Proof. The fact that $A$ in (7) has S-polynomial $\varphi$ follows immediately from Corollary 2.2 and Theorem 2.7. The converse is also true by Corollary 2.6.

Let us turn now to quasidirect decompositions of S-matrices.

THEOREM 2.9. Let $B, C$ be S-matrices of the same dimensions such that the sum $B + C$ is quasidirect. Then $B + C$ is again an S-matrix, and its S-polynomial is the product of the S-polynomials of $B$ and $C$. In addition, these last two polynomials are relatively prime.

Proof. Let $B, C$ be $p \times q$. By Theorem 2.8,

$$
B = V_1 M_1,
$$

where $M_1$ is $r(B) \times q$ of rank $r(B)$, and

$$
C = V_2 M_2
$$

where $M_2$ is $r(C) \times q$ of rank $r(C)$, and $V_1, V_2$ are matrices of the form (5).

Consequently, we can write

$$
B + C = (V_1, V_2) \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}.
$$

Since $B + C$ is quasidirect, both factors on the right hand side have rank
This means, however, that $B + C$ has the form (7) and $(V_1, V_2)$ is again a generalized Vandermonde matrix with distinct numbers in $V_1$ and $V_2$. By Theorem 2.8, $B + C$ is an $S$-matrix whose $S$-polynomial is the product of the $S$-polynomials of $B$ and $C$, and these polynomials are relatively prime.

We shall be able to solve completely the problem of decomposition of $S$-matrices into quasidirect sum of $S$-matrices. We say that an $S$-matrix $A$ is $S$-indecomposable if no quasidirect decomposition $A = B + C$ exists where $B$ and $C$ are nonzero $S$-matrices. This notion depends, however, on the field, as the example of the matrix

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

with $S$-polynomial $x^2 + 1$ shows. For simplicity, we shall investigate the case of the complex field only.

Theorem 2.10. A complex $S$-matrix is $S$-indecomposable over the field of complex numbers iff its $S$-polynomial is a nonnegative power of a linear polynomial.

Every complex $S$-matrix $A$ is a quasidirect sum of $S$-indecomposable $S$-matrices, and this sum is unique up to the ordering of the summands. Each summand corresponds to one root of the $S$-polynomial of $A$.

Proof. The "if" part of the first assertion follows from Theorem 2.9 and Lemma 1.2. Let now $A$ be an $m \times n$ $S$-matrix whose $S$-polynomial $\varphi$ is not such a power. Then

$$
\varphi = \varphi_1\varphi_2\cdots\varphi_s, \quad \varphi_i = (x - t_i)^{n_i}, \quad n_i = 1, \quad i = 1, \ldots, s,
$$

$t_i$ mutually distinct, $s > 1$, $\sum n_i = n$. Let $V$ be the generalized Vandermonde matrix formed as in (5), let $V_i = P_{mn_i}(t_i)$. By Theorem 2.7, $V$ is a pure $S$-matrix with the $S$-polynomial $\varphi$. By Corollary 2.6,

$$
A = VM
$$

for some matrix $M$ with linearly independent rows. Let

$$
M = \begin{pmatrix}
M_1 \\
M_2 \\
\vdots \\
M_s
\end{pmatrix}
$$
be the partitioning of $M$ corresponding to the partitioning

$$V = (V_1, V_2, \ldots, V_s).$$

Then

$$A = \sum_{i=1}^{s} A_i,$$  \hspace{1cm} (9)

where

$$A_i = V_i M_i$$

are, by Theorem 2.8, S-matrices with S-polynomials $q_i$, $i = 1, \ldots, s$. These matrices are S-indecomposable. The ranks of $A_i$ being $n_i$ and the rank of $A$ being $n = \sum_{i=1}^{s} n_i$, (9) is quasidirect. This completes the proof of the first part and of the existence of a decomposition. To prove the uniqueness, assume that

$$A = \sum_{j=1}^{c} B_j$$

is also a quasidirect decomposition of $A$ into S-indecomposable S-matrices. By the previous part, each $B_j$ corresponds to one root of the S-polynomial of $A$. Therefore, $c = s$ and we can assume that $B_j$ corresponds to $t_j$. It follows that

$$B_j = V_j N_j, \quad j = 1, \ldots, s,$$

where $V_j = P_{mn_j}(t_j)$ as above. Thus,

$$A = V N, \quad N = \begin{pmatrix} N_1 \\ \vdots \\ N_s \end{pmatrix}.$$

Let $\hat{A}, \hat{V}$ be the submatrices of $A, V$ consisting of the first $n$ rows. Then $\hat{A} = \hat{V} M$ as well as $\hat{A} = \hat{V} N$. Since $\hat{V}$ is nonsingular, $M = N$. \hfill \blacksquare

**Remark 2.11.** A nonsingular matrix can also be written as a sum of S-matrices. It suffices to complete such a matrix by another row. The resulting matrix is an S-matrix and by Theorem 2.10 can be decomposed into a sum of S-indecomposable S-matrices. If there are at least two summands, we obtain, by leaving out the last rows in each summand, a quasidirect decomposition of
the original matrix into a sum of S-matrices. This decomposition is, of course, not unique if the order of the given matrix is greater than one.

We shall conclude with two corollaries of Theorem 2.10.

**Corollary 2.12.** Let $A$ be a proper Hankel matrix with linearly dependent rows. If $A = \sum_i A_i$ is a quasidirect decomposition of $A$ into a sum of S-matrices, then all matrices $A_i$ are again proper Hankel matrices.

**Proof.** We shall need a result of [1] which states that a proper complex Hankel matrix $A$ can be decomposed into a quasidirect sum of $H$-indecomposable proper Hankel matrices, each of the summands corresponding to one root of the $H$-polynomial of $A$. Since the $H$-polynomial of $A$ is identical with the $S$-polynomial of $A$, this decomposition is also the unique decomposition of $A$ into a sum of proper Hankel matrices with relatively prime $H$-polynomials, again a proper Hankel matrix.  

**Corollary 2.13.** Let $f(x)$, $g(x)$ be polynomials of degrees $m$, $n$ respectively. Let $\hat{C}_f, \hat{C}_g$ be the leading sections of the generalized companion matrices $C_{f}, C_{g}$ of the polynomials $f$, $g$, each having $m + n$ rows. Then $f$ and $g$ are relatively prime iff

$$\det(\hat{C}_f, \hat{C}_g) \neq 0.$$  

**Proof.** By Theorem 2.8, $\hat{C}_f = V_f M_f$, $\hat{C}_g = V_g M_g$, where $V_f, V_g$ are the generalized Vandermonde matrices with $m + n$ rows corresponding to $f$, $g$ respectively and $M_f, M_g$ are nonsingular matrices. Since

$$(\hat{C}_f, \hat{C}_g) = (V_f, V_g) \begin{pmatrix} M_f & 0 \\ 0 & M_g \end{pmatrix}$$

and $(V_f, V_g)$ is nonsingular iff $f$ and $g$ are relatively prime, the result follows.  

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