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A characterization of the external lines of a hyperoval cone in PG(3, q), q even

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ABSTRACT

In this article, the lines not meeting a hyperoval cone in PG(3, q), q even, are characterized by their intersection properties with points and planes.

1. Introduction

Let PG(3, q) be the projective space of dimension 3 and order q, where q is a prime power. A k-set L of lines is a set of k lines in PG(3, q). As usual, we call a star of lines the set of all lines through one point, the centre of the star. Let m_j denote non-negative integers, with $0 \le m_1 < m_2 < \cdots < m_s \le q^2+q+1$. A set L is said to be of *type* $(m_1, m_2, \ldots, m_s)$ with respect to stars (of lines), if any star contains either m_1, or m_2,..., or m_s lines of L, and all such stars do exist; see [2]. A j-secant star of L will be a star containing exactly j lines of L. Denote by t_j the number of m_j-secant stars. Then the following hold; see [5]:

$$\begin{cases} \sum_{j=1}^{s} t_j = \vartheta_3 \\ \sum_{j=1}^{s} m_j t_j = k\vartheta_1 \\ \sum_{j=1}^{s} m_j(m_j-1)t_j = k(k-1) - 2\tau, \end{cases} \tag{1.1}$$

τ being the number of unordered pairs of skew lines in L and $\vartheta_i = \frac{q^{i+1}-1}{q-1}$ the number of points of PG(i, q). Also the type of L with respect to ruled planes, i.e. planes considered as sets of their lines, can be defined and equations similar to (1.1) can be written; see [5]. An external plane is a plane such that no line of L belongs to it.

A plane hyperoval is a $(q+2)$-set of points in a plane π, no three of which are collinear. A hyperoval cone of PG(3, q), q even, consists of the points on the lines joining a plane oval to a point V, called the vertex, not belonging to π; see [4].

Quadrics in PG(3, q) are very interesting objects with many combinatorial properties. One is that lines can only meet a quadric in a few ways. So we may consider a family of lines that all meet a particular quadric in the same way. This family of lines has remarkable properties. An important question is whether we may use these properties to characterize them. The following result [1] enters into this scheme of things.

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Result (*[1] Barwick and Butler*)**.** Let L be a non-empty set of lines in $PG(3, q)$, q even, such that the following hold.

(I) Every point lies on 0 or $\frac{1}{2}q^2$ lines of L.
(II) Every plane contains 0, q^2, or $\frac{1}{2}q(q-1)$ lines of L.

Then L is the set of external lines to a hyperoval cone of $PG(3, q)$.

In this paper, we give a characterization of the set of external lines of a hyperoval cone of $PG(3, q)$, q even, as a set of type $(0, a, b)$ with respect to ruled planes and of type (m, n) with respect to stars of lines, which is the set of external lines to a hyperoval cone of $PG(3, q)$, q even. In particular, we prove the following.

Theorem. *In* $PG(3, q)$, *a* $\left(\frac{q^4}{2} - \frac{q^3}{2}\right)$*-set of lines, q even, having exactly* $\frac{q^4(q-2)(q-1)^2(q+1)}{8}$ *pairs of skew lines and* $\frac{(q+1)(q+2)}{2}$ *external planes, of type* $(0, a, b)$ *with respect to ruled planes and of type* (m, n) *with respect to stars of lines, is the set of external lines to a hyperoval cone.*

2. Proof of the theorem

Suppose that L is a k-set of lines in $PG(3, q)$ of type (m, n) with respect to stars of lines. According to (1.1), we get

$$\begin{cases} t_m + t_n = (q+1)(q^2+1) \\ mt_m + nt_n = k(q+1) \\ m(m-1)t_m + n(n-1)t_n = k(k-1) - 2\tau. \end{cases} \tag{2.1}$$

Thus, a two-character set with respect to stars of lines depends on four parameters, k, τ, m, and n, and a complete classification seems to be extremely difficult; see [3,6]. Therefore, in order to give a characterization, we fix two of these parameters, k and τ.

For $k = \left(\frac{q^4}{2} - \frac{q^3}{2}\right)$ and $\tau = \frac{q^4(q-2)(q-1)^2(q+1)}{8}$ system (2.1) becomes

$$\begin{cases} t_m + t_n = (q+1)(q^2+1) \\ mt_m + nt_n = \left(\dfrac{q^4}{2} - \dfrac{q^3}{2}\right)(q+1) \\ m(m-1)t_m + n(n-1)t_n = \dfrac{q^3}{2}(q-1)\left(\dfrac{q^2}{2} - 1\right)(q+1). \end{cases} \tag{2.2}$$

From the first two equations of (2.2), we get

$$\begin{cases} t_m = \dfrac{(q+1)\left[q^3(q-1) - 2n(q^2+1)\right]}{2(m-n)} \\ t_n = \dfrac{(q+1)\left[2m(q^2+1) - q^3(q-1)\right]}{2(m-n)}. \end{cases} \tag{2.3}$$

Since $t_n > 0$, by the second equation of (2.3), we get

$$m < \frac{q^2}{2} - \frac{q}{2} - \frac{1}{2} + \frac{q+1}{2q^2+2}.$$

Therefore, $0 \le m < \frac{q^2}{2} - \frac{q}{2} + 1$.

If we solve system (2.2), choosing m as a parameter and t_m, t_n, and n as variables, and examining the cases $m = 0$ and $m = 1$ separately, we get the following. For $m = 0$, we have $n = \frac{q^2}{2}$; q being even, we have that n is an integer, and hence this is an acceptable solution. For $m = 1$, we have $n = \frac{q^3(q-1)(q^2-2)}{2(q^4-q^3-2q^2-2)} = \frac{q^2}{2} + \frac{2q^2(q+1)}{2(q^4-q^3-2q^2-2)}$, which is not a non-negative integer.

For $m \neq 0$ and $m \neq 1$, we get

$$n = \frac{q^3(q-1)(2m-q^2)}{2\left[2m(q^2+1) - q^3(q-1)\right]}. \tag{2.4}$$

Since $1 < m < \frac{q^2}{2} - \frac{q}{2} + 1$, we can suppose that $m = m(q)$, (i.e. m is a polynomial function of q); hence

$$m = \alpha q^2 + \beta q + \gamma.$$

By substituting m into (2.4), we get $n = \frac{(2\alpha-1)q^6+(2\beta-2\alpha+1)q^5+(2\gamma-2\beta)q^4-2\gamma q^3}{2(2\alpha-1)q^4+2(2\beta+1)q^3+4(\alpha+\gamma)q^2+4\beta q+4\gamma}$.

Putting $N(q) = (2\alpha - 1)q^6 + (2\beta - 2\alpha + 1)\, q^5 + (2\gamma - 2\beta)q^4 - 2\gamma q^3$ and $D(q) = 2(2\alpha - 1)q^4 + 2\,(2\beta + 1)\, q^3 + 4\,(\alpha + \gamma)\, q^2 + 4\beta q + 4\gamma$, $n = \frac{N(q)}{D(q)}$.

As n is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero for any q.

The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$R(q) = \left\{ \frac{2\left[\alpha\,(2\gamma+1) - 2\beta^2 - 2\beta - \gamma - 1\right]}{(2\alpha-1)^2} + 2\alpha + 2 \right\} q^3 - \left\{ \frac{2\left[\alpha + \beta\,(2\gamma+1) + \gamma\right]}{(2\alpha-1)^2} - 2\alpha - 2\beta \right\} q^2 + \left\{ \frac{4\left[2\alpha^2\,(\beta+\gamma) - \alpha\,(2\beta+\gamma) - \beta^2\right]}{(2\alpha-1)^2} \right\} q + \left\{ \frac{4\gamma\left(2\alpha^2 - 2\alpha - \beta\right)}{(2\alpha-1)^2} \right\}.$$

If $\alpha \neq \frac{1}{2}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q.

Thus we obtain a system of four equations and three variables (α, β, γ), which gives the following solution: $\alpha = 0$, $\beta = 0$, $\gamma = 0$; we exclude this solution because it gives $m = 0$, and we have just studied it.

If $\alpha = \frac{1}{2}$, then $m = \frac{1}{2}q^2 + \beta q + \gamma$; hence $n = \frac{\beta q^5 + (\gamma-\beta)q^4 - \gamma q^3}{(2\beta+1)q^3 + (2\gamma+1)q^2 + 2\beta q + 2\gamma}$.

Putting $N(q) = \beta q^5 + (\gamma - \beta)\, q^4 - \gamma q^3$ and $D(q) = (2\beta + 1)\, q^3 + (2\gamma + 1)\, q^2 + 2\beta q + 2\gamma$, $n = \frac{N(q)}{D(q)}$.

As n is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero for any q.

The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$R(q) = \left\{ \frac{2\left[4\beta^2 + 8\beta^3 + 2\beta^2\,(1-3\gamma) - \beta\,(4\gamma+1) + \gamma\,(\gamma+1)\,(2\gamma+1)\right]}{(2\beta+1)^3} \right\} q^2 + \left\{ \frac{2\left[4\beta^4 + 4\beta^3\gamma + 2\beta^2\,(3\gamma-1) + 4\beta\gamma - \gamma^2\right]}{(2\beta+1)^3} \right\} q + \left\{ \frac{4\gamma\left[2\beta^3 - \beta + \gamma\,(\gamma+1)\right]}{(2\beta+1)^3} \right\} q.$$

If $\beta \neq -\frac{1}{2}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q.

Thus we obtain a system of three equations and two variables (β, γ). The only acceptable solution is $\beta = 0$, $\gamma = 0$, for which $m = \frac{1}{2}q^2$ and $n = 0$; a contradiction, because $m < n$.

If $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, $m = \frac{1}{2}q^2 - \frac{1}{2}q + \gamma$; hence $n = \frac{-q^5 + (2\gamma+1)q^4 - 2\gamma q^3}{2(2\gamma+1)q^2 - 2q + 4\gamma}$.

Putting $N(q) = -q^5 + (2\gamma + 1)\, q^4 - 2\gamma q^3$ and $D(q) = 2\,(2\gamma + 1)\, q^2 - 2q + 4\gamma$, $n = \frac{N(q)}{D(q)}$.

As n is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero for any q.

The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$R(q) = \left\{ \frac{4\gamma\left(8\gamma^4 - 12\gamma^2 - 4\gamma + 1\right)}{(2\gamma+1)^4} \right\} q + \frac{8\gamma^2\left(4\gamma^3 + 8\gamma^2 + 2\gamma - 1\right)}{(2\gamma+1)^4}.$$

If $\gamma \neq -\frac{1}{2}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q.

Thus we obtain a system of two equations and one variable (γ). The only acceptable solution is $\gamma = 0$, for which $n = -\frac{1}{2}q^3$, a contradiction.

If $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ and $\gamma = -\frac{1}{2}$, it follows that $m = \frac{1}{2}q^2 - \frac{1}{2}q - \frac{1}{2}$, and it is not an integer; therefore, $m = 0$ and $n = \frac{q^2}{2}$. Thus L is a $\left(\frac{q^4}{2} - \frac{q^3}{2}\right)$-set of type $\left(0, \frac{q^2}{2}\right)$ with respect to stars of lines.

Now suppose that L is a $\left(\frac{q^4}{2} - \frac{q^3}{2}\right)$-set of type $(0, a, b)$ in PG(3, q) with respect to ruled planes. According to (1.1), we get

$$\begin{cases} t_0 + t_a + t_b = \left(q^3 + q^2 + q + 1\right) \\ at_a + bt_b = \left(\dfrac{q^4}{2} - \dfrac{q^3}{2}\right)(q+1) \\ a\,(a-1)\,t_a + b\,(b-1)\,t_b = \dfrac{q^3}{2}(q-1)\left(\dfrac{q^2}{2} - 1\right)(q+1)\,, \end{cases} \tag{2.5}$$

with $t_0 = \frac{(q+1)(q+2)}{2}$. From the first two equations of (2.5), we get

$$t_a = \frac{q(q+1)\left[q^2(q-1) - b(2q-1)\right]}{2\,(a-b)},$$

$$t_b = \frac{q(q+1)\left[a(2q-1) - q^2(q-1)\right]}{2\,(a-b)}.$$

Since $t_b > 0$ and $a < b$, we have

$$a < \frac{q^2(q-1)}{2q-1} \quad \text{and hence a} \quad < \frac{q^2}{2} - \frac{q}{4} - \frac{1}{8} - \frac{1}{8\,(2q-1)}.$$

We can suppose that $a < \frac{q^2}{2} - \frac{q}{4} - \frac{1}{8}$.

If we solve system (2.5), choosing a as a parameter and t_a, t_b, and b as variables and excluding the cases $a = 0$ because $a > 0$ and $a = 1$ because in this case b is not a positive integer, we get

$$b = \frac{q^2(q-1)\left(2a-q^2\right)}{2\left[a(2q-1)-q^2(q-1)\right]}. \tag{2.6}$$

Since $a < \frac{q^2}{2} - \frac{q}{4} - \frac{1}{8}$, we can suppose that $a = a(q)$, (i.e. a is a polynomial function of q); hence $a = \alpha q^2 + \beta q + \gamma$. By substituting a into (2.6), we get

$$b = \frac{(2\alpha-1)q^5 + (2\beta-2\alpha+1)\,q^4 + (2\gamma-2\beta)q^3 - 2\gamma q^2}{2\,(2\alpha-1)\,q^3 + 2\,(2\beta-\alpha+1)\,q^2 + 2\,(2\gamma-\beta)\,q - 2\gamma}.$$

Putting $N(q) = (2\alpha-1)q^5 + (2\beta-2\alpha+1)\,q^4 + (2\gamma-2\beta)q^3 - 2\gamma q^2$ and $D(q) = 2(2\alpha-1)q^3 + 2\,(2\beta-\alpha+1)\,q^2 + 2\,(2\gamma-\beta)\,q - 2\gamma$, we have $b = \frac{N(q)}{D(q)}$.

As b is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero. The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$\begin{aligned} R(q) = &\left\{\frac{\alpha\,(8\gamma+1) - 8\beta^2 - 4\beta - 4\gamma - 1}{4(2\alpha-1)^2} - \frac{\alpha}{4} + \frac{1}{4}\right\} q^2 \\ &- \left\{\frac{\alpha^2\beta + \alpha(\alpha-\beta) - \beta(\beta-2\gamma)}{(2\alpha-1)^2}\right\} q - \frac{\gamma\left(\alpha^2-\alpha-\beta\right)}{(2\alpha-1)^2}. \end{aligned}$$

If $\alpha \neq \frac{1}{2}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q. Thus we obtain a system of three equations and three variables (α, β, γ), which gives the following solutions:

$\alpha = 0,\ \beta = 0,\ \gamma = 0$, a contradiction;
$\alpha = 1,\ \beta = 0,\ \gamma = 0$, for which we have $a = q^2$ and $b = \frac{1}{2}q(q-1)$; a contradiction, because $a < b$.

If $\alpha = \frac{1}{2}$, we get $b = \frac{2\beta q^4 + 2(\gamma-\beta)q^3 - 2\gamma q^2}{(4\beta+1)q^2 + 2(2\gamma-\beta)q - 2\gamma}$.

As b is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero. The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$R(q) = \left\{\frac{-4\left[4\beta^4 + 2\beta^3 + 2\beta^2\gamma + 2\beta\gamma - \gamma^2\,(8\gamma+3)\right]}{(4\beta+1)^3}\right\} q - \frac{4\gamma\left[4\beta^3 + 2\beta^2 + \gamma\,(4\gamma+1)\right]}{(4\beta+1)^3}.$$

If $\beta \neq -\frac{1}{4}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q. Thus we obtain a system of two equations and two variables (β, γ), which gives the following acceptable solutions:

$\beta = 0,\ \gamma = 0$, and hence we get $b = 0$, a contradiction;
$\beta = -\frac{1}{2}$, $\gamma = 0$, for which we get $a = \frac{1}{2}q^2 - \frac{1}{2}q$ and $b = q^2$, an acceptable solution.

If $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{4}$, then $a = \frac{1}{2}q^2 - \frac{1}{4}q + \gamma$; hence $b = \frac{-q^4 + (4\gamma+1)q^3 - 4\gamma q^2}{(8\gamma+1)q - 4\gamma}$.

Putting $N(q) = -q^4 + (4\gamma+1)\,q^3 - 4\gamma q^2$ and $D(q) = (8\gamma+1)\,q - 4\gamma$, $b = \frac{N(q)}{D(q)}$.

As b is an integer, the remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ must be zero for any q.

The remainder $R(q)$ of the division by $\frac{N(q)}{D(q)}$ is

$$R(q) = -\frac{512\gamma^4\,(4\gamma+1)}{(8\gamma+1)^4}.$$

If $\gamma \neq -\frac{1}{8}$ and $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q. Thus we have the following solutions:

$$\gamma = 0 \quad \text{or} \quad \gamma = -\frac{1}{4};$$

for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{4}$ and $\gamma = 0$, we get $b = q^2\,(1-q) < 0$, a contradiction;

for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{4}$ and $\gamma = -\frac{1}{4}$, we get $b = \frac{q^2}{1-q} < 0$, a contradiction.

Thus the last case is $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{4}$ and $\gamma = -\frac{1}{8}$, for which we get $b = q^2\,(1-q)\,(2q+1) < 0$, a contradiction.

Finally, we claim that $a = \frac{1}{2}q^2 - \frac{1}{2}q$ and $b = q^2$. So L is a $\left(\frac{q^4}{2} - \frac{q^3}{2}\right)$-set of type $\left(0, \frac{1}{2}q^2 - \frac{1}{2}q, q^2\right)$ with respect to ruled planes in $PG(3, q)$, q even.

Then according to Result (I), L is the set of external lines to a hyperoval cone of $PG(3, q)$, q even.

Thus, the theorem is completely proved.

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