J. Math. Anal. Appl. 387 (2012) 419–432

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On quadratic integral equations in Orlicz spaces

Mieczysław Cichoń [a,*], Mohamed M.A. Metwali [a,b]

[a] Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
[b] Department of Mathematics, Faculty of Sciences, Damanhour University, Egypt

ARTICLE INFO

Article history:
Received 30 June 2011
Available online 10 September 2011
Submitted by D. O'Regan

Keywords:
Integral equation
Monotonic solutions
Orlicz spaces
Superposition operators

ABSTRACT

In this paper we study the quadratic integral equation of the form

$$x(t) = g(t) + \lambda x(t) \int_a^b K(t,s) f\big(s, x(s)\big)\, ds.$$

Several existence theorems for a.e. monotonic solutions in Orlicz spaces are proved for strongly nonlinear functions f. The presented method of the proof can be easily extended to different classes of solutions.

1. Introduction

The paper is devoted to study the following quadratic integral equation

$$x(t) = g(t) + \lambda x(t) \int_a^b K(t,s) f\big(s, x(s)\big)\, ds. \tag{1}$$

We will deal with problems in which either the growth of the function f or the kernel K is not polynomial. This is motivated, for instance, by some mathematical models in physics. An interesting discussion about such a kind of problems can be found in [27] or [49]. The considered thermodynamical problem leads to the integral equation $x(t) + \int_I k(t,s) \exp x(s)\, ds = 0$ and thus the integral equations with exponential nonlinearities turn out important from an application point of view. Let us also note, that such a kind of problems can be applied for integral equations associated (making use of the Green kernel) for the operator $-\Delta u + \exp u$ on a bounded regular subsets of $\mathbb{R}^2$ (see [20]) and that the solutions in Orlicz spaces are also sometimes studied in partial differential equations ([19], for instance).

We are interested in a different class of solutions than in the earlier papers. Such a kind of integral equations was investigated in spaces of continuous or integrable functions. Similar problems are also important when L^p-solutions are checked. The currently considered case is less restrictive and includes a large class of real problems. Whenever one has to deal with some problems involving strong nonlinearities (of exponential growth, for instance), it is a useful device to look for solutions not in Lebesgue spaces, but in Orlicz spaces.

In the literature, mostly solutions of integral equations are sought in $C[0,1]$ and $L^p[0,1]$ with $p > 1$. The results obtained for $L^p[0,1]$ invariably assume a polynomial growth (in x) on the nonlinearity $f(t,x)$. On the other hand, seeking solutions in other Orlicz spaces will lead to restrictions that are not of polynomial type, and hence will allow us to consider new classes of equations. All very basic types of integral or differential equations were satisfactory examined (cf. [36,40,41,43,46]). Some

* Corresponding author.
E-mail addresses: mcichon@amu.edu.pl (M. Cichoń), m.metwali@yahoo.com (M.M.A. Metwali).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.013

additional properties of solutions (in a simplest case of the Δ_2-condition) are also investigated (constant-sign solutions, for instance [1,2]). An interesting discussion about advantages of integral equations in Orlicz spaces can be found in [37, Section 40] (see also [5]).

Nevertheless, for the quadratic integral equations the operators generated by the right-hand side of the equation are more complicated and were not investigated in this case. Let us note, that in this case the methods based on properties of some Banach algebras are usually applied (cf. [8,17,38,39]). This approach seems to be strictly related with continuous or Hölder continuous solutions (the product is an inner operation in the Banach algebra of continuous functions and at the same time is an operator used in the integral equation). Let us note, that this method is dependent on some properties of $C(0,1)$ and cannot be easily applied to different classes of spaces.

We propose to use a different factorization and we will assume the Hammerstein integral operator which is multiplied by the function $x(\cdot)$ has values in the space conjugated to the space of solutions. Such an idea was used by Brézis and Browder for Hammerstein integral equations [21] by considering conjugated Lebesgue L^p spaces. We extend this procedure for Orlicz (or: ideal) spaces and for a triple of spaces (two for Hammerstein operator and one more for a multiplication operator). This allows us to prove the existence theorems under much more general conditions than previously considered ones. Our method leads to extensions for both types of results mentioned above.

We concentrate on the property of monotonicity of solutions for Eq. (1). This notion is broaden to some function spaces and the basic properties of families of such functions are investigated.

Especially, the quadratic integral equation of Chandrasekhar type

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)x(s)\,ds$$

can be very often encountered in many applications (cf. [7,8,13,15,22,26]).

The theorems proved by us extend, in particular, that presented in [4,7,16,14] considered in the space $C(I)$ or in Banach algebras (cf. [17]). In the context of non-quadratic integral equations in Orlicz spaces, which are also cover by our theorems, let us mention the papers [1,2,40,41,43,46].

Our theorems allow to consider the cases of integral equations when the kernel function K is more singular than in previously considered cases. Moreover, we are able to consider strongly nonlinear functions f. Both extensions seem to be important from the applications point of view (cf. [19,20,27,49], for instance). Let us note, that our results are motivated by the paper of Cheng and Kozak [27]. We generalize some of their assumptions and we consider more complicated quadratic integral equations.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers. In the paper we will denote by I a finite interval $[a,b]\subset\mathbb{R}$.

Assume that $(E,\|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B_r(x)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta,r)$. When necessary we will also indicate the space by using the notation $B_r(E)$. If X is a subset of E, then $\bar{X}$ and *conv* X denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols $k\cdot X$ and $X+Y$.

Let M and N be complementary N-functions i.e. $N(x)=\sup_{y\geqslant 0}(xy-M(x))$, where $N:\mathbb{R}\to[0,+\infty)$ is continuous, even and convex with $\lim_{x\to 0}\frac{N(x)}{x}=0$, $\lim_{x\to\infty}\frac{N(x)}{x}=\infty$ and $N(x)>0$ if $x>0$ ($N(u)=0 \iff u=0$). The Orlicz class, denoted by $\mathcal{O}_P$, consists of measurable functions $x:I\to\mathbb{R}$ for which

$$\rho(x;M)=\int_I M\bigl(x(t)\bigr)\,dt<\infty.$$

We shall denote by $L_M(I)$ the *Orlicz space* of all measurable functions $x:I\to\mathbb{R}$ for which

$$\|x\|_M=\inf_{\lambda>0}\left\{\int_I M\left(\frac{x(s)}{\lambda}\right)ds\leqslant 1\right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M\subseteq L_M\subseteq\mathcal{O}_M$. The inclusion $L_M\subset L_P$ holds if, and only if, there exist positive constants u_0 and a such that $P(u)\leqslant aM(u)$ for $u\geqslant u_0$.

An important property of E_M spaces lies in the fact that this is a class of functions from L_M having absolutely continuous norms.

Moreover, we have $E_M=L_M=\mathcal{O}_M$ if M satisfies the Δ_2-condition, i.e.

$$(\Delta_2)\quad \text{there exist } \omega, t_0\geqslant 0 \text{ such that for } t\geqslant t_0,\ \text{we have } M(2t)\leqslant\omega M(t).$$

Let us observe, that an N-function complementary to $M(u)=\exp u^2-1$ satisfies this condition, while the function $M(u)=\exp|u|-|u|-1$ does not.

An N-function M is said to satisfy Δ'-condition if there exist $K, t_0 \geqslant 0$ such that for $t, s \geqslant t_0$, we have $M(ts) \leqslant KM(t)M(s)$.

If the N-function M satisfies the Δ'-condition, then it also satisfies Δ_2-condition. Typical examples: $M_1(u) = \frac{|u|^\alpha}{\alpha}$ for $\alpha > 1$, $M_2(u) = (1+|u|)\ln(1+|u|) - |u|$ or $M_3(u) = |u|^\alpha(|\ln|u|| + 1)$ for $\alpha > 3 + \frac{\sqrt{5}}{2}$.

The last class of N-functions, interesting for us, consists of functions which increase more rapidly than power functions. An N-function M is said to satisfy Δ_3-condition if there exist K, $t_0 \geqslant 0$ such that for $t \geqslant t_0$, we have $tM(t) \leqslant M(Kt)$.

Sometimes, we will use more general concept of function spaces i.e. ideal spaces. A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x: I \to U$ (U is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y: I \to U$ the relation $|y(s)| \leqslant |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \leqslant \|x\|$. If X is also complete, it is called an ideal space (see [48]). The class of Orlicz spaces stands for an important example of ideal spaces.

We will use the technique of factorization for some operators acting on Orlicz spaces through another Orlicz spaces. We can mention, that by using in this place different ideal spaces it is possible to obtain some extensions of our results and then we try to facilitate this approach. To stress the connection of our results with the growth condition of f we restrict ourselves to the case of Orlicz spaces.

3. Nonlinear operators

One of the most important operator studied in nonlinear functional analysis is the so-called superposition operator [6]. Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F(x)(t) = f\big(t, x(t)\big), \quad t \in I.$$

The operator F in such a way is called the superposition operator generated by the function f. We will be interested in the case when F acts between some Orlicz spaces.

A full discussion about necessary and sufficient conditions for continuity and boundedness of such a type of operators can be found in [6]. The following properties will be useful:

Lemma 3.1. *Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then the superposition operator F transforms measurable functions into measurable functions.*

Lemma 3.2. *(See [37, Lemma 17.5] in S and [42] in L_M.) Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F maps a sequence of functions convergent in measure into a sequences of functions convergent in measure.*

Lemma 3.3. *(See [36, Theorem 17.5].) Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then*

$$M_2\big(f(s,x)\big) \leqslant a(s) + bM_1(x),$$

where $b \geqslant 0$ and $a \in L^1(I)$, if and only if the superposition operator F acts from $L_{M_1}(I)$ to $L_{M_2}(I)$.

In Orlicz spaces there is no automatic continuity of superposition operators like in L^p spaces, but the following lemma is useful (remember, that the Orlicz space L_M is ideal and if M satisfies Δ_2-condition it is also regular cf. [5, Theorem 1]):

Lemma 3.4. *(See [48, Theorem 5.2.1].) Let f be a Carathéodory function, X an ideal space, and W a regular ideal space. Then the superposition operator $F: X \to W$ is continuous.*

Let us note, that in the case of functions of the form $f(t,x) = g(t)h(x)$, the superposition operator F is continuous from the space of continuous functions $C(I)$ into $L_M(I)$ even when M does not satisfy Δ_2-condition [5]. Since $E_M(I)$ is a regular part of an Orlicz space $L_M(I)$ (cf. [47, p. 72]), in the context of Orlicz spaces, we will use the following (see also Lemma 3.3):

Lemma 3.5. *Let f be a Carathéodory function. If the superposition operator F acts from $L_{M_1}(I)$ into $E_{M_2}(I)$, then it is continuous.*

The problem of boundedness of such a type of operators in different classes of Orlicz spaces will be described in the proofs of our main results.

Two more operators will play an important role it this paper, namely the linear integral operator

$$H(x) = \lambda \int_a^b K(t,s)x(s)\,ds$$

and the multiplication operator. The first one is well known and all necessary results concerning the properties of such a kind of operators in Orlicz spaces can be found in [36], so here we omit the details and important results will be pointed out in the proofs of our main results.

Now, we need to describe the second one. By $U(x)(t)$ we will denote the operator of the form:

$$U(x)(t) = x(t) \cdot A(x)(t),$$

where $A = H \circ F$ is a Hammerstein operator.

Generally speaking, the product of two functions $x, y \in L_M(I)$ is not in $L_M(I)$. However, if x and y belong to some particular Orlicz spaces, then the product $x \cdot y$ belongs to a third Orlicz space. Let us note, that one can find two functions belonging to Orlicz spaces: $u \in L_U(I)$ and $v \in L_V$ such that the product uv does not belong to any Orlicz space (this product is not integrable). Nevertheless, we have:

Lemma 3.6. *(See [36, Lemma 13.5].) Let M_1, M_2 and Φ be arbitrary N-functions. The following conditions are equivalent*:

1. *For every functions $u \in L_{M_1}(I)$ and $w \in L_{M_2}$, $u \cdot w \in L_\Phi(I)$.*
2. *There exists a constant $k > 0$ such that $\|uw\|_\Phi \leqslant k\|u\|_{M_1}\|w\|_{M_2}$.*
3. *There exist numbers $k_0 > 0$, $u_0 \geqslant 0$ such that for all $u_1, u_2 \geqslant u_0$*

$$\Phi\left(\frac{u_1u_2}{k_0}\right) \leqslant M_1(u_1) + M_2(u_2).$$

Some particular cases when the assumptions of the above lemma are satisfied are presented in [36]. In an interesting case of N-functions satisfying the Δ'-condition we have:

Proposition 3.1. *(Cf. [36, Theorem 15.1, Theorem 15.2, Lemma 15.4].) Assume, that M satisfies the Δ'-condition. Then there exists a constant $k_1 > 0$ such that*

$$\|uv\|_M \leqslant k_1\|u\|_M\|v\|_M \tag{2}$$

for all $u, v \in L_M(I)$.

An interesting discussion about necessary and sufficient conditions for product operators can be found in [36].

4. Monotone functions

Let $S = S(I)$ denote the set of measurable (in Lebesgue sense) functions on I and let meas stand for the Lebesgue measure in $\mathbb{R}$. Identifying the functions which are equal almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0}\bigl[a + \operatorname{meas}\bigl\{s\colon \bigl|x(s) - y(s)\bigr| \geqslant a\bigr\}\bigr]$$

becomes a complete space. Moreover, the space S with the topology convergence in measure on I is a metric space, because the convergence in measure is equivalent to convergence with respect to d (cf. Proposition 2.14 in [47]).

The compactness in such spaces we will be called a "compactness in measure" and such sets have important properties when considered as subsets of some Orlicz spaces (ideal spaces). Let us recall, in metric spaces the set U_0 is compact if and only if each sequence from U_0 has a subsequence that converges in U_0 (i.e. sequentially compact).

In this paper, we need to investigate some properties of sets and operators in such a class of spaces instead of the space S. Some of them are obvious, the rest will be proved.

We are interested in finding of almost everywhere monotonic solutions for our problem. We will need to specify this notion in considered solution spaces.

Let X be a bounded subset of measurable functions. Assume that there is a family of subsets $(\Omega_c)_{0\leqslant c\leqslant b-a}$ of the interval I such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geqslant x(t_2)$ $(t_1 \in \Omega_c,\ t_2 \notin \Omega_c)$.

It is clear, that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c) \setminus Z$, where Z is a set with measure zero, this family contains nonincreasing functions (possibly except for a set Z). We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose a measurable and nonincreasing *function* y and all functions equal a.e. to y satisfy the above condition. This means that such a notion can be also be considered in the space S. Thus we can write, that *elements* from $L_M(I)$ belong to this class of functions. Further, let Q_r stand for the subset of the ball B_r consisting of all functions which are a.e. nonincreasing on I. Functions a.e. nondecreasing are defined by similar way.

It is known, that such a family constitute a set which is compact in measure in S. We are interested, if the set is still compact in measure as a subset of subspaces of S. In general, it is not true, but for the case of Orlicz spaces, we have the following:

Lemma 4.1. *Let X be a bounded subset of $L_M(I)$ consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I. Then X is compact in measure in $L_M(I)$.*

Proof. Let $r > 0$ be such that $X \subset B_r \subset L_M(I)$. It is known (cf. [37,9]), that X is compact in measure as a subset of S. By taking an arbitrary sequence (x_n) in X we obtain that there exists a subsequence (x_{n_k}) convergent in measure to some $x \in S$. Since Orlicz spaces are perfect (cf. [48]), the balls in $L_M(I)$ are closed in the topology of convergence in measure. Thus $x \in B_r \subset L_M(I)$ and then $x \in X$. $\square$

Remark 4.1. The above lemma remains true for subsets of arbitrary perfect ideal spaces [48].

We have also an important

Lemma 4.2. *(See Lemma 4.2 in [11].) Suppose the function $t \to f(t,x)$ is a.e. nondecreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \to f(t,x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in I$. Then the superposition operator F generated by f transforms functions being a.e. nondecreasing on I into functions having the same property.*

We will use the fact, that the superposition operator takes the bounded sets compact in measure into the sets with the same property. Namely, we have

Proposition 4.1. *Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and the function $t \to f(t,x)$ is a.e. nondecreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \to f(t,x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in I$. Assume, that $F: L_M(I) \to E_M(I)$. Then $F(V)$ is compact in measure for arbitrary bounded and compact in measure subset V of $L_M(I)$.*

Proof. Let V be a bounded and compact in measure subset of $L_M(I)$. By our assumption $F(V) \subset E_M(I)$. As a subset of S the set $F(V)$ is compact in measure (cf. [9]). Since the topology of convergence in measure is metrizable, the compactness of the set is equivalent with the sequential compactness. By taking an arbitrary sequence $(y_n) \subset F(V)$ we get a sequence (x_n) in V such that $y_n = F(x_n)$. Since $(x_n) \subset V$, as follows from Lemma 3.2 F transforms this sequence into the sequence convergent in measure. Thus (y_n) is compact in measure, so is $F(V)$. $\square$

5. Measures of noncompactness

Now we present the concept of a regular measure of noncompactness: we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Definition 5.1. (See [12].) A mapping $\mu: \mathcal{M}_E \to [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(conv\, X) = \mu(X)$.
(iv) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.
(v) $\mu(X+Y) \leqslant \mu(X) + \mu(Y)$.
(vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1,2,3,\ldots,$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following:

Definition 5.2. (See [12].) Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as

$$\beta_H(X) = \inf\{r > 0\colon \text{ there exists a finite subset } Y \text{ of } E \text{ such that } x \subset Y + B_r\}.$$

For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X in $L_M(I)$ (cf. Definition 3.9 in [47] or [32,33]):

$$c(X) = \lim_{\varepsilon\to 0} \sup_{mes\, D \leqslant \varepsilon} \sup_{x\in X} \|x \cdot \chi_D\|_{L_M(I)},$$

where χ_D denotes the characteristic function of D.

Then we have the following theorem, which clarifies the connections between different coefficients in Orlicz spaces. Since Orlicz spaces $L_M(I)$ are regular, when M satisfies Δ_2-condition, then Theorem 1 in [33] reads as follows:

Proposition 5.1. *Let X be a nonempty, bounded and compact in measure subset of an ideal regular space Y. Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in $L_M(I)$ iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms cf. [3]).

An importance of such a kind of functions can be clarified by using the contraction property with respect to this measure instead of compactness in the Schauder fixed point theorem. Namely, we have a theorem [12]:

Theorem 5.1. *Let Q be a nonempty, bounded, closed and convex subset of E and let $V : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0, 1)$ such that*

$$\mu\bigl(V(X)\bigr) \leqslant k\mu(X),$$

for any nonempty subset X of E. Then V has at least one fixed point in the set Q.

6. Main results

Denote by B the operator associated with the right-hand side of Eq. (1) i.e. $B(x) = g + U(x)$, where $U(x)(t) = \lambda x(t)\int_a^b K(t,s) f(s,x(s))\,ds$.

We will try to choose the domains of operators defined above in such a way to obtain the existence of solutions in a desired space (here: an Orlicz space). We stress on conditions allowing us to consider strongly nonlinear operators.

We need to distinguish two different cases. We will assume in the paper that this space L_φ is a Banach–Orlicz algebra – cf. Proposition 3.1. This allows us to obtain more general growth conditions on f (cf. [41,43,46] for non-quadratic equations).

6.1. The case of Δ'-condition

Let us recall, that for φ satisfying the Δ'-condition there exists a constant k_1 such that the inequality (2) holds true (cf. Proposition 3.1).

Recall, that in this case $k_1 = u_0^2[1 + K + 2(b-a) + \varphi(u_0^2)\cdot(b-a)^2]$ [37, p. 145], where K is a constant taken from the definition of the Δ'-condition.

Theorem 6.1. *Assume, that M and N are complementary N-functions and that*:

(C1) *$g \in E_\varphi(I)$ is nondecreasing a.e. on I,*
(C2) *$f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $f(t,x)$ is assumed to be nondecreasing with respect to both variables t and x separately,*
(C3) *$|f(t,x)| \leqslant b(t) + G(|x|)$ for $t \in I$ and $x \in \mathbb{R}$, where $b \in E_N(I)$ and G is nonnegative, nondecreasing, continuous function defined on $\mathbb{R}^+$,*
(C4) *let N and φ satisfy the Δ'-condition and suppose that there exist $\omega, \gamma, u_0 \geqslant 0$ for which*

$$N\bigl(\omega\bigl(G(u)\bigr)\bigr) \leqslant \gamma\varphi(u) \leqslant \gamma M(u) \quad \textit{for } u \geqslant u_0,$$

(K1) *$s \to K(t,s) \in E_M(I)$ for a.e. $t \in I$,*
(K2) *$K \in E_M(I^2)$ and $t \to K(t,s) \in E_\varphi(I)$ for a.e. $s \in I$, $\|K\|_M < \frac{1}{2k_1\cdot|\lambda|\cdot G(1)}$,*
(K3) *$\int_a^b K(t_1,s)\,ds \geqslant \int_a^b K(t_2,s)\,ds$ for $t_1, t_2 \in [a,b]$ with $t_1 < t_2$.*

Then there exists a number $\rho > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all g with $\|g\|_\varphi < 1$ there exists a solution $x \in E_\varphi(I)$ of (1) *which is a.e. nondecreasing on I.*

Proof. We need to divide the proof into a few steps.

I. The operator B is well defined from $L_\varphi(I)$ into itself and continuous on a domain depending on the considered case.
II. We will construct an invariant ball B_r for B in $L_\varphi(I)$.
III. We construct a subset Q_r of this ball which contains a.e. nondecreasing functions and we investigate the properties Q_r.
IV. We check the continuity and monotonicity properties of B in Q_r, so $U : Q_r \to Q_r$.
V. We prove that B is a contraction with respect to a measure of noncompactness.
VI. We use the Darbo fixed point theorem to find a solution in Q_r.

I. First of all observe that under the assumptions (C2) and (C3) by Lemma 3.3 the superposition operator F acts from $L_\varphi(I)$ to $L_N(I)$.

In this case we will prove, that U is a continuous mapping from the unit ball in $E_\varphi(I)$ into the space $E_\varphi(I)$ (note: $L_\varphi(I)$ is not sufficient for our proof). Let us recall, that $x \in E_\varphi(I)$ iff for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\chi_T\|_\varphi < \varepsilon$ for every measurable subset T of I with measure smaller that δ (i.e. x has absolutely continuous norm).

Since N is an N-function satisfying Δ'-condition and by (C3), we are able to use [36, Lemma 19.1]. From this there exists a constant C (not dependent on the kernel) such that for any measurable subset T of I and $x \in L_\varphi(I)$, $\|x\|_\varphi \leqslant 1$ we have

$$\bigl\|A(x)\chi_T\bigr\|_\varphi \leqslant C\|K\chi_{T\times I}\|_M. \tag{3}$$

First, let us observe that in view of Lemma 3.1, it is sufficient to check the desired property for the operator A.

Moreover, by (C3) and (C4) there exist $\omega, \gamma, u_0 > 0$, s.t. (cf. [36, p. 196])

$$\begin{aligned}\bigl\|G\bigl(\bigl|x(\cdot)\bigr|\bigr)\bigr\|_N &= \frac{1}{\omega}\bigl\|\omega G\bigl(\bigl|x(\cdot)\bigr|\bigr)\bigr\|_N \\ &\leqslant \frac{1}{\omega}\inf_{r>0}\biggl\{\int N\bigl(\omega G\bigl(\bigl|x(t)\bigr|\bigr)/r\bigr)\,dt \leqslant 1\biggr\} \\ &\leqslant \frac{1}{\omega}\Biggl(1+\int_a^b N\bigl(\omega G\bigl(\bigl|x(t)\bigr|\bigr)\bigr)\,dt\Biggr) \\ &\leqslant \frac{1}{\omega}\Biggl(1+N\bigl(\omega G(u_0)\bigr)\cdot(b-a)+\gamma\int_a^b \varphi\bigl(\bigl|x(t)\bigr|\bigr)\,dt\Biggr),\end{aligned}$$

whenever $x \in L_\varphi(I)$ with $\|x\|_\varphi \leqslant 1$.

Now, by the Hölder inequality and the assumption (C2) we get

$$\bigl|K(t,s)f\bigl(s,x(s)\bigr)\bigr| \leqslant \bigl\|K(t,s)\bigr\|\cdot\bigl|f\bigl(s,x(s)\bigr)\bigr| \leqslant \bigl\|K(t,s)\bigr\|\cdot\bigl|\bigl(b(s)+G\bigl(\bigl|x(s)\bigr|\bigr)\bigr)\bigr|$$

for $t, s \in I$. Put $k(t) = 2\|K(t,\cdot)\|_M$ for $t \in I$. As $K \in E_M(I^2)$ this function is integrable on I. By the assumptions (K1) and (K2) about the kernel K of the operator H (cf. [46]) we obtain that

$$\bigl\|A(x)(t)\bigr\| \leqslant k(t)\cdot\bigl(\|b\|_N + \bigl\|G\bigl(\bigl|x(\cdot)\bigr|\bigr)\bigr\|_N\bigr) \quad \text{for a.e. } t \in I.$$

Whence for arbitrary measurable subset T of I and $x \in E_\varphi(I)$

$$\bigl\|A(x)\chi_T\bigr\|_\varphi \leqslant \|k\chi_T\|_\varphi\cdot\bigl(\|b\|_N + \bigl\|G\bigl(\bigl|x(\cdot)\bigr|\bigr)\bigr\|_N\bigr).$$

Finally if t is such that $K(t,\cdot) \in E_M(I)$ and $x \in E_\varphi(I)$ we have

$$\int_T \bigl\|K(t,s)f\bigl(s,x(s)\bigr)\bigr\|\,ds \leqslant 2\bigl\|K(t,\cdot)\chi_T\bigr\|_M\cdot\bigl(\|b\|_N + \bigl\|G\bigl(\bigl|x(\cdot)\bigr|\bigr)\bigr\|_N\bigr) \quad \text{for a.e. } t \in I.$$

From this it follows that A maps $B_1(E_\varphi(I))$ into $E_\varphi(I)$.

We are in a position to prove the continuity of A as a mapping from the unit ball $B_1(E_\varphi(I))$ into the space $E_\varphi(I)$. Let $x_n, x_0 \in B_1(E_\varphi(I))$ be such that $\|x_n - x_0\|_\varphi \to 0$ as n tends to ∞. Suppose, contrary to our claim, that A is not continuous and the $\|A(x_n) - A(x_0)\|_\varphi$ does not converge to zero. Then there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$\bigl\|A(x_{n_k}) - A(x_0)\bigr\|_\varphi > \varepsilon \quad \text{for } k = 1, 2, \ldots \tag{4}$$

and the subsequence is a.e. convergent to x_0. Since (x_n) is a subset of the ball the sequence $(\int_a^b \varphi(|x_n(t)|)\,dt)$ is bounded. As the space $E_\varphi(I)$ is regular the balls are norm-closed in L_1 so the sequence $(\int_a^b |x_n(t)|\,dt)$ is also bounded.

Thus

$$\begin{aligned}\int_T \bigl\|K(t,s)f\bigl(s,x_n(s)\bigr)\bigr\|\,ds &\leqslant 2\bigl\|K(t,\cdot)\chi_T\bigr\|_M\cdot\bigl(\|b\|_N + \bigl\|G\bigl(\bigl|x_n(\cdot)\bigr|\bigr)\bigr\|_N\bigr) \\ &\leqslant 2\bigl\|K(t,\cdot)\chi_T\bigr\|_M\cdot\Biggl(\|b\|_N + \frac{1}{\omega}\Biggl[1+N\bigl(\omega G(u_0)\bigr)\cdot(b-a)+\gamma\int_a^b \varphi\bigl(\bigl|x_n(t)\bigr|\bigr)\,dt\Biggr]\Biggr)\end{aligned}$$

and then the sequence $(\|K(t,s)f(s,x_n(s))\|)$ is equiintegrable on I for a.e. $t \in I$. By the continuity of $f(t,\cdot)$ we get $\lim_{k\to\infty} K(t,s)f(s,x_{n_k}(s)) = K(t,s)f(s,x_0(s))$ for a.e. $s \in I$. Now, applying the Vitali convergence theorem we obtain that

$$\lim_{k\to\infty} A(x_{n_k})(t) = A(x_0)(t) \quad \text{for a.e. } t \in I.$$

But Eq. (3) implies that $A(x_{n_k})$ is a subset of $E_\varphi(I)$ and then $\lim_{k\to\infty} A(x_{n_k})(t) = A(x_0)(t)$ which contradicts the inequality (4). Since A is continuous between indicated spaces, by Lemma 3.6 the operator U has the same property and then U is a continuous mapping from $B_1(E_\varphi(I))$ into the space $E_\varphi(I)$ by assumption (C1) B maps $B_1(E_\varphi(I))$ into $E_\varphi(I)$ continuously.

II. We will prove the boundedness of the operator U, namely we will construct the invariant ball for this operator. By B we will denote the right-hand side of our integral equation i.e. $B = g + U$.

Put $r = 1$ and let

$$\rho = \frac{1 - \|g\|_\varphi}{2k_1 \cdot C \cdot \|K\|_M}.$$

Let x be an arbitrary element from $B_1(E_\varphi(I))$. Then by using the above consideration, the assumption (C3), the formula (3) and Proposition 3.1 for sufficiently small λ (i.e. $|\lambda| < \rho$) we obtain

$$\begin{aligned}
\|B(x)\|_\varphi &\leqslant \|g\|_\varphi + \|Ux\|_\varphi \\
&= \|g\|_\varphi + \|x \cdot A(x)\|_\varphi \\
&\leqslant \|g\|_\varphi + k_1\|x\|_\varphi \cdot \|A(x)\|_\varphi \\
&= \|g\|_\varphi + k_1|\lambda|\|x\|_\varphi \cdot \left\| \int_a^b K(\cdot,s) f\big(s,x(s)\big)\,ds \right\|_\varphi \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot \|x\|_\varphi \cdot \|K\|_M \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot r \cdot \|K\|_M \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot \rho \cdot C \cdot \|K\|_M \leqslant r
\end{aligned}$$

whenever $\|x\|_\varphi \leqslant 1$.

Then we have $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$. Moreover, B is continuous on $B_r(E_\varphi(I))$ (see part I of the proof).

III. Let Q_r stand for the subset of $B_r(E_\varphi(I))$ consisting of all functions which are a.e. nondecreasing on I. Similarly as claimed in [10] this set is nonempty, bounded (by r), convex (direct calculation from the definition) and closed in $L_\varphi(I)$.

To prove the last property, let (y_n) be a sequence of elements in Q_r convergent in $L_\varphi(I)$ to y. Then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem for Orlicz spaces and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on I (cf. [41]). Moreover, y is still nondecreasing a.e. on I which means that $y \in Q_r$ and so the set Q_r is closed. Now, in view of Lemma 4.1 the set Q_r is compact in measure.

IV. Now, we will show, that B preserves the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nondecreasing on I and consequently $F(x)$ is also of the same type in virtue of the assumption (C2) and Theorem 4.2. Further, $A(x) = HF(x)$ is a.e. nondecreasing on I thanks for the assumption (C4). Since the pointwise product of a.e. monotone functions is still of the same type, the operator U is a.e. nondecreasing on I.

Moreover, the assumptions (C1) and (C2) permit us to deduce that $Bx(t) = g(t) + U(x)(t)$ is also a.e. nondecreasing on I. This fact, together with the assertion that $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$ gives us that B is also a self-mapping of the set Q_r. From the above considerations it follows that H maps continuously Q_r into Q_r.

V. We will prove that B is a contraction with respect to a measure of strong noncompactness. Assume that X is a nonempty subset of Q_r and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\operatorname{meas} D \leqslant \varepsilon$ we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\leqslant \|g\chi_D\|_\varphi + \|U(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \|x \cdot A(x)\chi_D\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + k_1\|x\chi_D\|_\varphi \cdot \|A(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot \left\| \int_D K(\cdot,s) f\big(s,x(s)\big)\,ds \right\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot \left\| \int_D \big|K(\cdot,s)\big| \big(b(s) + G\big(\big|x(s)\big|\big)\big)\,ds \right\|_\varphi
\end{aligned}$$

$$\leqslant \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot 2\|K\|_M \big\| \big[b\chi_D + G(1)\big] \big\|_N$$
$$\leqslant \|g\chi_D\|_\varphi + 2k_1 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot \|K\|_M \big[\|b\chi_D\|_N + G(1)\big].$$

Hence, taking into account that $g \in E_\varphi$, $b \in E_N$

$$\lim_{\varepsilon\to 0}\Big\{\sup_{mes\, D\leqslant\varepsilon}\Big[\sup_{x\in X}\big\{\|g\chi_D\|_\varphi = 0\big\}\Big]\Big\} \quad \text{and} \quad \lim_{\varepsilon\to 0}\Big\{\sup_{mes\, D\leqslant\varepsilon}\Big[\sup_{x\in X}\big\{\|b\chi_D\|_N = 0\big\}\Big]\Big\}.$$

By definition of $c(x)$,

$$c\big(B(X)\big) \leqslant 2k_1 \cdot |\lambda| \cdot \|K\|_M \cdot G(1) \cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space E_φ, we can use Proposition 5.1 and get

$$\beta_H\big(B(X)\big) \leqslant 2k_1 \cdot |\lambda| \cdot \|K\|_M \cdot G(1) \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and the inequality from the assumption (K2) allow us to apply the Darbo fixed point theorem (see [12]), which completes the proof. □

6.2. The case of Δ_3-condition

Let us consider the case of N-functions with growth essentially more rapid than a polynomial. In fact, we will consider functions satisfying Δ_3-condition. This is very large and important class, especially from an application point of view (cf. [19,20,44,49]). An extensive description of this class can be found in [44, Section 2.5]. Recall, that an N-function M determines the properties of the Orlicz space $L_M(I)$ and then the less restrictive rate of the growth of this function implies the “worser” properties of the space. By ϑ we will denote the norm of the identity operator from $L_\varphi(I)$ into $L^1(I)$ i.e. $\sup\{\|x\|_1\colon x \in B_1(L_\varphi(I))\}$. For the discussion about the existence of φ which satisfies our conditions see [37, p. 61].

Theorem 6.2. *Assume, that M and N are complementary N-functions and that* (C1), (C2), (C3), (K1) *and* (K3) *hold true. Moreover, put the following assumptions*:

(C5) (1) *N satisfies the Δ_3-condition,*
(2) *$K \in E_M(I^2)$ and $t \to K(t,s) \in E_\varphi(I)$ for a.e. $s \in I$,*
(3) *there exist $\beta, u_0 > 0$ such that*

$$G(u) \leqslant \beta\frac{M(u)}{u}, \quad \textit{for } u \geqslant u_0,$$

(K4) *φ is an N-function satisfying the condition Δ' and such that*

$$\iint_{I^2} \varphi\big(M\big(\big|K(t,s)\big|\big)\big)\,dt\,ds < \infty$$

and

$$2k_1 \cdot \big(2 + (b-a)\big(1+\varphi(1)\big)\big) \cdot |\lambda| \cdot \|K\|_{\varphi\circ M} \cdot G(r_0) < 1,$$

where

$$r_0 = \frac{1}{\vartheta}\left[\frac{\omega}{2|\lambda| \cdot k_1 \cdot (2+(b-a)(1+\varphi(1))) \cdot \|K\|_{\varphi\circ M}} - \|b\|_N\right].$$

Then there exist a number $\rho > 0$ and a number $\varpi > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all $g \in E_\varphi(I)$ with $\|g\|_\varphi < \varpi$ there exists a solution $x \in E_\varphi(I)$ of (1) *which is a.e. nondecreasing on I.*

Proof. We will indicate only the points of the proof if they differ from the previous proof.

I. In this case the operator B can be considered as continuous when acting on the whole $E_\varphi(I)$.

By [37, Lemma 15.1 and Theorem 19.2] and the assumption (K4):

$$\big\|A(x)\chi_T\big\|_\varphi \leqslant 2 \cdot \big(2 + (b-a)\big(1+\varphi(1)\big)\big) \cdot \|K \cdot \chi_{T\times I}\|_{\varphi\circ M}\big(\|b\|_N + \big\|G\big(\big|x(\cdot)\big|\big)\big\|_N\big) \tag{5}$$

for arbitrary $x \in L_\varphi(I)$ and arbitrary measurable subset T of I.

Let us note, that the assumption (C5)(3) implies that there exist constants $\omega, u_0 > 0$ and $\eta > 1$ such that $N(\omega G(u)) \leqslant \eta u$ for $u \geqslant u_0$. Thus for $x \in L_\varphi(I)$

$$\|G(|x(\cdot)|)\|_N \leqslant \frac{1}{\omega}\left(1 + \int_I N(\omega G(|x(s)|))\,ds\right) \leqslant \frac{1}{\omega}\left(1 + \eta u_0(b-a) + \eta \int_I |x(s)|\,ds\right).$$

The remaining estimations can be derived as in the first main theorem and then we obtain, that $A : E_\varphi(I) \to E_\varphi(I)$.

II. Put

$$\rho = \frac{1}{2 \cdot k_1 \cdot (2 + (b-a)(1+\varphi(1))) \cdot \|K\|_{\varphi\circ M} \cdot [\|b\|_N + \frac{1}{\omega}(1 + \eta u_0(b-a))]}.$$

Fix λ with $|\lambda| < \rho$.

Choose a positive number r in such a way that

$$\|g\|_\varphi + 2|\lambda| \cdot k_1 \cdot (2 + (b-a)(1+\varphi(1))) \cdot \|K\|_{\varphi\circ M} \cdot r \cdot \left(\|b\|_N + \frac{1}{\omega}(1 + \eta u_0(b-a) + \eta\vartheta r)\right) \leqslant r. \tag{6}$$

As a domain for the operator B we will consider the ball $B_r(E_\varphi(I))$.

Let us remark, that the above inequality is of the form $a + (b + vr)cr \leqslant r$ with $a, b, c, v > 0$. Then $vc > 0$ and if we assume that $bc - 1 < 0$ and that the discriminant is positive, then Viète's formulas imply that the quadratic equation has two positive solutions $r_1 < r_2$ for sufficiently small λ. By the definition of ρ it is clear, that our assumptions guarantee the above requirements, so there exists a positive number r satisfying this inequality.

Put $C = (2 + (b-a)(1+\varphi(1)))$. Let us note, in view of the above considerations, that the assumption about the discriminant which implies the existence of solutions for the above problem is of the form:

$$\left[\|b\|_N + \frac{1}{\omega}(\eta u_0(b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot C \cdot \|K\|_{\varphi\circ M}}\right]^2 2|\lambda| \cdot k_1 \cdot C \cdot \|K\|_{\varphi\circ M} > \frac{4\|g\|_\varphi \eta\vartheta}{\omega}$$

i.e.

$$\varpi = \left[\|b\|_N + \frac{1}{\omega}(\eta u_0(b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot C \cdot \|K\|_{\varphi\circ M}}\right]^2 \frac{|\lambda| \cdot k_1 \cdot C \cdot \omega\|K\|_{\varphi\circ M}}{2\eta\vartheta}.$$

For $x \in B_r(E_\varphi(I))$ we have the following estimation:

$$\begin{aligned}
\|B(x)\|_\varphi &\leqslant \|g\|_\varphi + \|Ux\|_\varphi \\
&= \|g\|_\varphi + \|x \cdot A(x)\|_\varphi \\
&\leqslant \|g\|_\varphi + k_1\|x\|_\varphi \cdot \|A(x)\|_\varphi \\
&= \|g\|_\varphi + k_1|\lambda|\|x\|_\varphi \cdot \left\|\int_a^b K(\cdot, s) f(s, x(s))\,ds\right\|_\varphi \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot C \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi\circ M}\left[\|b\|_N + \frac{1}{\omega}\left(1 + N(\omega G(u_0)) \cdot (b-a) + \eta \int_I |x(s)|\,ds\right)\right] \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot C \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi\circ M}\left[\|b\|_N + \frac{1}{\omega}(1 + N(\omega G(u_0)) \cdot (b-a) + \eta\|x\|_1)\right] \\
&\leqslant \|g\|_\varphi + 2k_1 \cdot C \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi\circ M}\left[\|b\|_N + \frac{1}{\omega}(1 + N(\omega G(u_0)) \cdot (b-a) + \eta\vartheta\|x\|_\varphi)\right] \\
&= \|g\|_\varphi + 2rk_1 \cdot C \cdot |\lambda| \cdot \|K\|_{\varphi\circ M}\left(\|b\|_N + \frac{1}{\omega}(1 + \eta u_0(b-a) + \eta\vartheta r)\right) \leqslant r.
\end{aligned}$$

Then $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$.

Note, that parts III and IV of the previous proof are similar to those from the first theorem, so we omit the details.

V. We will prove that B is a contraction with respect to a measure of strong noncompactness. Assume that X is a nonempty subset of Q_r and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\operatorname{meas} D \leqslant \varepsilon$ we obtain

$$\begin{aligned}
\left\|B(x)\cdot\chi_D\right\|_\varphi &\leqslant \|g\chi_D\|_\varphi + \left\|U(x)\cdot\chi_D\right\|_\varphi \\
&= \|g\chi_D\|_\varphi + \left\|x\cdot A(x)\chi_D\right\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + k_1\|x\chi_D\|_\varphi\cdot\left\|A(x)\cdot\chi_D\right\|_\varphi \\
&= \|g\chi_D\|_\varphi + k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\left\|\int_D K(\cdot,s)f\big(s,x(s)\big)\,ds\right\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\left\|\int_D \big|K(\cdot,s)\big|\big(b(s)+G\big(\big|x(s)\big|\big)\big)\,ds\right\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\left(\left\|\int_D \big|K(\cdot,s)\big|b(s)\,ds\right\|_\varphi + \left\|\int_D \big|K(\cdot,s)\big|G\big(\big|x(s)\big|\big)\,ds\right\|_\varphi\right) \\
&\leqslant \|g\chi_D\|_\varphi + 2\cdot C\cdot k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\vartheta\cdot\|K\|_{\varphi\circ M}\|b\chi_D\|_N \\
&\quad + 2\cdot C\cdot k_1\|x\chi_D\|_\varphi\cdot\left\|\int_D \big|K(\cdot,s)\big|G\big(\big|x(s)\big|\big)\,ds\right\|_\varphi \\
&\leqslant \|g\chi_D\|_\varphi + 2\cdot C\cdot k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\|K\|_{\varphi\circ M}\big[\|b\chi_D\|_N + G(r)\big] \\
&\leqslant \|g\chi_D\|_\varphi + 2\cdot C\cdot k_1\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\|K\|_{\varphi\circ M}\big[\|b\chi_D\|_N + G(r_0)\big],
\end{aligned}$$

where

$$r_0 = \frac{1}{\vartheta}\left[\frac{\omega}{2|\lambda|\cdot k_1\cdot(2+(b-a)(1+\varphi(1)))\cdot\|K\|_{\varphi\circ M}} - \|b\|_N\right].$$

Let us note, that r_0 is an upper bound for solutions of (6).

Since $g\in E_\varphi$, $b\in E_N$,

$$\lim_{\varepsilon\to 0}\left\{\sup_{mes\,D\leqslant\varepsilon}\left[\sup_{x\in X}\big\{\|g\chi_D\|_\varphi = 0\big\}\right]\right\} \quad\text{and}\quad \lim_{\varepsilon\to 0}\left\{\sup_{mes\,D\leqslant\varepsilon}\left[\sup_{x\in X}\big\{\|b\chi_D\|_N = 0\big\}\right]\right\}.$$

Thus by definition of $c(x)$,

$$c\big(B(X)\big)\leqslant 2\cdot k_1\cdot C\cdot|\lambda|\|K\|_{\varphi\circ M}\cdot G(r_0)\cdot c(X).$$

Since $X\subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space E_φ, we can use Proposition 5.1 and get

$$\beta_H\big(B(X)\big)\leqslant 2\cdot k_1\cdot C\cdot|\lambda|\|K\|_{\varphi\circ M}\cdot G(r_0)\cdot\beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and the assumption (K4) allow us to apply the Darbo fixed point theorem (see [12]), which completes the proof. □

Remark 6.1. We need to stress on some aspects of our results. First of all we can observe, that our solutions are not necessarily continuous as in previously investigated cases. In particular, we need not to assume, that the Hammerstein operator transforms the space $C(I)$ into itself. For the examples and conditions related to Hammerstein operators in Orlicz spaces we refer the readers to [44, Chapter VI.6.1, Corollary 6 and Example 7]. Our solutions belong to the space $L_\varphi(I)$, where φ satisfies Δ'-condition (no restriction in our Theorem 6.2 and appropriately chosen for Theorem 6.1, cf. (C4)). In both cases this is more general assumption than in earlier papers and at the same time we have more information about the solution set.

We allow the kernel to be more "singular". An interesting pair of conjugated N-functions for which the results apply is the following one: $N(u)=\exp|u|-|u|-1$ satisfying Δ_3-condition and $M(u)=(1+|u|)\ln(1+|u|)-|u|$ satisfying Δ'-condition.

Example 6.1. To stress some advantages of our results let us recall that the obtained results, in particular, extend the results for the quadratic equations of the type

$$x(t)=g(t)+\lambda x(t)\int_0^1 K(t,s)\big|x(s)\big|\,ds$$

or

$$x(t) = g(t) + \lambda x(t) \int_0^1 K(t,s)\bigl(\log\bigl(1 + |x(s)|\bigr)\bigr)\,ds$$

with the kernel of the type $K(t,s) = \exp(s-t)$ (cf. [36, Theorem 19.2]). Here $G(u) = u$ or $G(u) = \log(1+|u|)$, respectively. In the last equation the function $N(u) = (1+|u|)\ln(1+|u|) - |u|$ satisfies the Δ'-condition and then $M(u) = e^{|u|} - |u| - 1$. Whence $K \in E_M(I^2)$. We put $\varphi = N$ and take an arbitrary $g \in E_\varphi$ with suitable set of properties.

Example 6.2. In the case $g(t) = 1$ and $f(t,x) = (\log(1+|x(s)|))$, Eq. (1) takes the form

$$x(t) = 1 + \lambda x(t) \int_0^1 \frac{t}{t+s}\psi(s)\bigl(\log\bigl(1 + |x(s)|\bigr)\bigr)\,ds. \tag{7}$$

Eq. (7) is the quadratic integral equation of generalized Chandrasekhar type which is considered in many papers and monographs (cf. [7,15,26,35] for instance). It arose originally in connection with scattering through a homogeneous semi-infinite plane atmosphere (see [22,26,30]).

In this case we have $K(t,s) = \frac{t}{t+s}\psi(s)$ and then for some sufficiently good functions ψ our result applies ($\psi(s) = (1/2)\cdot e^{-s}$, for instance).

Let us recall that the quadratic equations have numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [7,8,15,17,22,24–26,28,30,35]. In order to apply earlier results we have to impose an additional condition that the so-called "characteristic" function ψ is continuous (cf. [22, Theorem 3.2]) or even Hölder continuous [8]. This function is immediately related to the angular pattern for single scattering and then our results allow to consider some peculiar states of the atmosphere. In astrophysical applications of the Chandrasekhar equation the only restriction, that $\int_0^1 \psi(s)\,ds \leqslant 1/2$ is treated as necessary (cf. [22, Chapter VIII; Corollary 2, p. 187] or [34]).

An interesting discussion about this condition and the applicability of such equations can be found in [22]. Recall that to ensure the existence of solutions normally one assumes that $\psi(t)$ is an even polynomial (as in the book of Chandrasekhar [26, Chapter 5]) or continuous [22]. The using of different solution spaces in our paper allow to remove this restriction and then we give a partial answer to the problem from [22]. The continuity assumption for ψ implies the continuity of solutions for the considered equation (cf. [22]) and then seems to be too restrictive even from the theoretical point of view. For example, the following quadratic Volterra equation $x(t) = g(t) + \lambda x(t)\int_0^t \frac{t}{t+s}\exp(-s)(\log(1+|x(s)|^\alpha))\,ds$ can be reformulated in the form considered in our paper:

$$x(t) = g(t) + \lambda x(t) \int_0^1 \frac{t}{t+s}\exp(-s)\cdot\chi_{[0,t]}\cdot\bigl(\log\bigl(1 + |x(s)|^\alpha\bigr)\bigr)\,ds$$

with discontinuous function ψ. Our result is also new for *any* problem with the function g being not continuous, but belonging to the space L_φ!

Why the Orlicz space approach? When we are looking for discontinuous solutions (not belonging to the space of continuous functions $C(I)$) we are unable to use the Banach algebra approach. For the quadratic integral equations we need to check the solution space S for which the pointwise multiplication of two elements from S is still in S and some estimations are known. Thus the natural choice is the Orlicz space L_φ for φ satisfying the Δ'-condition.

If an N-function φ is of the form $\varphi(u) = |u|^\alpha(\ln(|u|+1))$ for $\alpha > 3 + \frac{\sqrt{5}}{2}$, then it satisfies the Δ'-condition. Under the standard conditions our theorems allow to find solutions for the considered quadratic integral equation not only in the space of continuous functions or in L^p spaces (cf. [21]), but also in the space L_φ (for the Chandrasekhar equation, in particular). This shows one of the directions of applications for our results.

Let us note, that interesting discussion about the kernels of Hammerstein operators in different Orlicz spaces can be found in [36, Chapter II, Section 16; Chapter IV, Section 19]. Let us note, that some of the examples from this book can be also modified in such a way to be proper for our quadratic integral equation.

Let us conclude this part of the paper by presenting an illustrative example motivated by [36, p. 206]:

$$x(t) = \alpha\cdot g(t) + x(t)\cdot\int_0^1 K(t,s)\bigl(b(s) + \ln\bigl(1 + |x(s)|\bigr)\bigr)\,ds,$$

where $\alpha > 0$, $b \in E_N(I)$ and g is discontinuous a.e. decreasing function:

$$g(t) = \begin{cases} 0, & t \text{ is rational}, \\ 1-t, & t \text{ is irrational} \end{cases}$$

and

$$K(t,s) = \begin{cases} t \cdot \ln(\frac{t+s}{t}), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Since $\iint_{I^2} \exp K(t,s)\,dt\,ds < \infty$, $K \in E_M(I^2)$ for $M(u) = e^{|u|} - |u| - 1$ (cf. [36, p. 160]).

More examples of equations can be found in recent papers of Banaś and co-authors [14,15,18] or in [29,39] and for monotonic solutions in [23].

Another problem solved in the paper is related with the monotonicity properties of solutions. To do it we extend the notion of "monotonicity almost everywhere" for Orlicz spaces. This is important property of solutions considered in recent papers (see [14,16,31], for example) and we extend this aspect of earlier papers too.

The last aspect of applicability of our results deals with the technique of Orlicz spaces for partial differential equations and then for integral equations. This context we require to consider more singular equations than in a classical case. Motivated by previously considered equations (see [19,20,27,49] or [41,46]) we extend this method to the case of quadratic integral equations. It should be noted that our method of the proof can be also adapted to classical equations considered in [19,27,36,49]. For more information we refer the readers to Chapter IX "Nonlinear PDEs and Orlicz spaces" in [45].

Remark 6.2. From the above proofs it is clear, that we are able to generalize our main theorems by assuming, that instead of our assumptions specified for F and K we can directly assume, that the operator U acts between $L_M(I)$ and $L_M(I)$ and that it has the desired properties. It is more general assumption, but not so descriptive from the application point of view.

Let X, Y be ideal spaces. A superposition operator $F : X \to Y$ is called improving if it takes bounded subsets of X into the subsets of Y with equiabsolutely continuous norms. The applications of such operators are based on the observation that large classes of linear integral operators

$$Hy(t) = \lambda \int_D k(s,t)y(t)\,dt,$$

although not being compact, map sets with equiabsolutely continuous norms into precompact sets, and thus one may apply the classical Schauder fixed point principle to obtain solutions of the nonlinear Hammerstein integral equation.

If we assume in our main theorem, that F is improving and H is a regular operator (cf. [37]), then A will be completely continuous. Thus such a kind of assumptions can be treated as an extension for our conditions. Let us note, that for operators from Lebesgue spaces L_p into L_r (i.e. Orlicz spaces with $p(x) = x^p$ and $r(x) = x^r$, respectively), the characterization of improving operators is known [50]: a superposition operator $F : L_p \to L_r$ is improving if and only if there exists a continuous and even function M satisfying $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$ and such that $G(x)(t) = M(f(t,x(t)))$ is also an operator from L_p into L_r (for an appropriate growth condition of f see [50]).

Remark 6.3. Till now, we are interested in finding monotonic solutions of our problem. Assume, that we have the decomposition of the interval I into the disjoint subsets T_1 and T_2 with $T_1 \cup T_2 = I$, such that $f(\cdot,x)$ is a.e. nondecreasing on T_1 and a.e. nonincreasing on T_2 and $f(t,\cdot)$ is a.e. nondecreasing for any t. By an appropriate change of the monotonicity assumptions we are able to prove the existence of solutions belonging to the class of functions described above (similarly like in [11]). In such a case the considered operators should preserve this property.

Acknowledgment

We are grateful to the referee for his/her helpful suggestions and comments on this paper which greatly improved this paper.

References

[1] R.P. Agarwal, D. O'Regan, P. Wong, Constant-sign solutions of a system of Volterra integral equations in Orlicz spaces, J. Integral Equations Appl. 20 (2008) 337–378.
[2] R.P. Agarwal, D. O'Regan, P. Wong, Solutions of a system of integral equations in Orlicz spaces, J. Integral Equations Appl. 21 (2009) 469–498.
[3] J. Alexopoulos, De La Vallée Poussin's theorem and weakly compact sets in Orlicz spaces, Quaest. Math. 17 (1994) 231–248.
[4] G. Anichini, G. Conti, Existence of solutions of some quadratic integral equations, Opuscula Math. 28 (2008) 433–440.
[5] J. Appell, The importance of being Orlicz, Banach Center Publ. 64 (2004) 21–28.
[6] J. Appell, P.P. Zabrejko, Nonlinear Superposition Operators, Cambridge University Press, Cambridge, 1990.
[7] I.K. Argyros, Quadratic equations and applications to Chandrasekhar and related equations, Bull. Aust. Math. Soc. 32 (1985) 275–292.
[8] I.K. Argyros, On a class of quadratic integral equations with perturbations, Funct. Approx. Comment. Math. 20 (1992) 51–63.
[9] J. Banaś, On the superposition operator and integrable solutions of some functional equations, Nonlinear Anal. 12 (1988) 777–784.
[10] J. Banaś, Integrable solutions of Hammerstein and Urysohn integral equations, J. Aust. Math. Soc. 46 (1989) 61–68.
[11] J. Banaś, Applications of measures of weak noncompactness and some classes of operators in the theory of functional equations in the Lebesgue space, Nonlinear Anal. 30 (1997) 3283–3293.
[12] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Math., vol. 60, M. Dekker, New York/Basel, 1980.
[13] J. Banaś, B. Rzepka, On existence and asymptotic stability of a nonlinear integral equation, J. Math. Anal. Appl. 284 (2003) 165–173.
[14] J. Banaś, B. Rzepka, Nondecreasing solutions of a quadratic singular Volterra integral equation, Math. Comput. Modelling 49 (2009) 488–496.

[15] J. Banaś, M. Lecko, W.G. El-Sayed, Existence theorems of some quadratic integral equations, J. Math. Anal. Appl. 222 (1998) 276–285.
[16] J. Banaś, A. Martinon, Monotonic solutions of a quadratic integral equation of Volterra type, Comput. Math. Appl. 47 (2004) 271–279.
[17] J. Banaś, K. Sadarangani, Solutions of some functional-integral equations in Banach algebras, Math. Comput. Modelling 38 (2003) 245–250.
[18] J. Banaś, T. Zając, Solvability of a functional integral equation of fractional order in the class of functions having limits at infinity, Nonlinear Anal. 71 (2009) 5491–5500.
[19] A. Benkirane, A. Elmahi, An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces, Nonlinear Anal. 36 (1999) 11–24.
[20] J. Berger, J. Robert, Strongly nonlinear equations of Hammerstein type, J. Lond. Math. Soc. 15 (1977) 277–287.
[21] H. Brézis, F. Browder, Existence theorems for nonlinear integral equations of Hammerstein type, Bull. Amer. Math. Soc. 81 (1975) 73–78.
[22] J. Caballero, A.B. Mingarelli, K. Sadarangani, Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer, Electron. J. Differential Equations 57 (2006) 1–11.
[23] J. Caballero, D. O'Regan, K. Sadarangani, On monotonic solutions of some integral equations, Arch. Math. (Brno) 41 (2005) 325–338.
[24] B. Cahlon, M. Eskin, Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation, J. Math. Anal. Appl. 83 (1981) 159–171.
[25] S. Chandrasekhar, The transfer of radiation in stellar atmospheres, Bull. Amer. Math. Soc. 53 (1947) 641–711.
[26] S. Chandrasekhar, Radiative Transfer, Dover Publications, New York, 1960.
[27] I.-Y.S. Cheng, J.J. Kozak, Application of the theory of Orlicz spaces to statistical mechanics. I. Integral equations, J. Math. Phys. 13 (1972) 51–58.
[28] M.M. Crum, On an integral equation of Chandrasekhar, Q. J. Math. 18 (1947) 244–252.
[29] M.A. Darwish, S.K. Ntouyas, Monotonic solutions of a perturbed quadratic fractional integral equation, Nonlinear Anal. 71 (2009) 5513–5521.
[30] H. Domke, An equivalence theorem for Chandrasekhar's H-functions and its application for accelerating convergence, J. Quant. Spectrosc. Radiat. Transfer 39 (1988) 283–286.
[31] W.G. El-Sayed, B. Rzepka, Nondecreasing solutions of a quadratic integral equation of Urysohn type, Comput. Math. Appl. 51 (2006) 1065–1074.
[32] N. Erzakova, On measures of non-compactness in regular spaces, Z. Anal. Anwend. 15 (1996) 299–307.
[33] N. Erzakova, Compactness in measure and measure of noncompactness, Sib. Math. J. 38 (1997) 926–928.
[34] Ch. Fox, A solution of Chandrasekhar's integral equation, Trans. Amer. Math. Soc. 99 (1961) 285–291.
[35] S. Hu, M. Khavani, W. Zhuang, Integral equations arising in the kinetic theory of gases, Appl. Anal. 34 (1989) 261–266.
[36] M.A. Krasnosel'skii, Yu. Rutitskii, Convex Functions and Orlicz Spaces, Noordhoff, Gröningen, 1961.
[37] M.A. Krasnosel'skii, P.P. Zabreiko, J.I. Pustyl'nik, P.E. Sobolevskii, Integral Operators in Spaces of Summable Functions, Noordhoff, Leyden, 1976.
[38] K. Maleknejad, R. Mollapourasl, K. Nouri, Study on existence of solutions for some nonlinear functional-integral equations, Nonlinear Anal. 69 (2008) 2582–2588.
[39] K. Maleknejad, K. Nouri, R. Mollapourasl, Existence of solutions for some nonlinear integral equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 2559–2564.
[40] D. O'Regan, Solutions in Orlicz spaces to Urysohn integral equations, Proc. R. Ir. Acad., Sect. A 96 (1996) 67–78.
[41] W. Orlicz, S. Szufla, On some classes of nonlinear Volterra integral equations in Banach spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (1982) 239–250.
[42] R. Płuciennik, The superposition operator in Musielak–Orlicz spaces of vector-valued functions, in: Proceedings of the 14th Winter School on Abstract Analysis, Srní, 1986, Rend. Circ. Mat. Palermo (2) Suppl. 14 (1987) 411–417.
[43] R. Płuciennik, S. Szufla, Nonlinear Volterra integral equations in Orlicz spaces, Demonstratio Math. 17 (1984) 515–532.
[44] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.
[45] M.M. Rao, Z.D. Ren, Applications of Orlicz Spaces, Marcel Dekker, New York, 2002.
[46] A. Sołtysiak, S. Szufla, Existence theorems for L_φ-solutions of the Hammerstein integral equation in Banach spaces, Comment. Math. (Prace Mat.) 30 (1990) 177–190.
[47] M. Väth, Volterra and Integral Equations of Vector Functions, Marcel Dekker, New York/Basel, 2000.
[48] M. Väth, Ideal Spaces, Lecture Notes in Math., vol. 1664, Springer, Berlin/Heidelberg, 1997.
[49] J.D. Weeks, S.A. Rice, J.J. Kozak, Analytic approach to the theory of phase transitions, J. Chem. Phys. 52 (1970) 2416–2426.
[50] P.P. Zabreiko, E.I. Pustyl'nik, On continuity and complete continuity of nonlinear integral operators in L_p spaces, Uspekhi Mat. Nauk 19 (116) (1964) 204–205.