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M-groups of Fitting length three

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1. Introduction

A character of a finite group G is monomial if it is induced from a linear character of a subgroup of G. A group G is an M-group if all its complex irreducible characters (the set $\mathrm{Irr}(G)$) are monomial.

In [2], Dornhoff asked if normal subgroups of M-groups were again M-groups. In [1], Dade (and independently Van der Wall [9]) gave an example of an M-group which has a normal subgroup which was not an M-group. Dade's example depended very strongly on the use of the prime 2. Therefore, the question of whether normal subgroups of odd M-groups were again M-groups was left open.

It is called that a group G has a Sylow tower if G has a normal series of Hall subgroups $G_i \lhd G$ such that $G_0 = 1$, $G_n = G$, and $|G_i : G_{i-1}|$ is a power of a prime, for each $i = 1, \dots, n$. In [3], Gunter showed that normal subgroups of M-groups with Sylow tower were M-groups. Furthermore, in [8], Parks showed that if G is an odd M-group and $N \lhd G$ with N nilpotent and G/N supersolvable, then every normal subgroup of G is an M-group. After I submitted this paper for publication, I was informed that M. Loukaki recently proved that normal subgroups of odd order monomial $\{p, q\}$-groups are monomial.

Now, in this paper the following is shown.

Theorem. *Let p be a prime and let G be an M-group. Assume that there exist normal subgroups K and L of G such that*

(i) *$|G/K|$ divides p;*
(ii) *K/L is nilpotent p'-group;*
(iii) *L is abelian p-group.*

Then every normal subgroup N of G is an M-group.

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All of Park's, Gunter's, Loukaki's, and our result apply to families of groups where the result does not hold "one character at a time." (By Berger's example in [5].)

If ψ is a character of G, we denote the set of its irreducible constituents by $\mathrm{Irr}(G|\psi)$. Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. We write $I_G(\theta)$ to denote the inertia group $\{g \in G \mid \theta^g = \theta\}$.

Let a group A act on a group G. We say that A-invariant character $\chi \in \mathrm{Irr}(G)$ is A-primitive if it is not induced from any A-invariant character of any A-invariant proper subgroup. Furthermore, we say that χ is A-monomial if it is induced from an A-invariant linear character of an A-invariant subgroup.

2. Preliminaries

In this section we shall give some lemmas which will be used to prove the theorem.

Lemma 1. *Let a group A act on a group G. Let N be a normal A-invariant subgroup of G and assume that G/N is nilpotent and N has abelian Sylow subgroups. Suppose also that $(|A|, |G/N|) = 1$. If $\chi \in \mathrm{Irr}(G)$ is A-invariant, then the degrees of any two A-primitive characters that induce χ coincide.*

This is essentially Theorem C of [7]. There it is required that $(|A|, |G|) = 1$, but the same proof works in our case.

Lemma 2 [3]. *Let G be an M-group with a Sylow tower. Then every normal subgroup of G is again an M-group.*

Lemma 3 [8, p. 939, Lemma 3.4]. *Let $A \lhd G$ and suppose that $\chi \in \mathrm{Irr}(G)$. Let $\varphi \in \mathrm{Irr}(A|\chi_A)$. Let $\chi_1 \in \mathrm{Irr}(I_G(\varphi))$ be such that $(\chi_1)_N$ is a multiple of φ, and such that $(\chi_1)^G = \chi$. Suppose that χ is monomial and that φ is linear. Then χ_1 is monomial.*

Lemma 4 (see the proof of [8, p. 940, Step 1]). *Let $N \lhd G$ and suppose that $\chi \in \mathrm{Irr}(G)$. Let $\zeta \in \mathrm{Irr}(N|\chi_N)$. Let $A \subseteq N$, $A \lhd G$, and $A' = 1$. Choose $\varphi \in \mathrm{Irr}(A|\zeta_A)$. Let $\chi_1 \in \mathrm{Irr}(I_G(\varphi))$, $\zeta_1 \in \mathrm{Irr}(I_N((\varphi))$ be such that $(\chi_1)_A$, $(\zeta_1)_A$ are multiples of φ, and such that $\chi_1^G = \chi$, $\zeta_1^G = \zeta$, respectively. Then $\zeta_1 \mid (\chi_1)_{I_N(\varphi)}$.*

Lemma 5 [10, Lemma 2.1]. *Assume $N \lhd G$, $H \subseteq G$, $NH = G$, and $N \cap H = M$. Assume that $\phi \in \mathrm{Irr}(N)$ is invariant in G and $\phi_M \in \mathrm{Irr}(M)$. Then $\chi \leftrightarrow \chi_H$ defines a one-to-one correspondence between $\mathrm{Irr}(G|\phi^G)$ and $\mathrm{Irr}(H|(\phi_M)^H)$.*

Lemma 6 [6, p. 30, Corollary 1.4]. *Assume that every characteristic abelian subgroup of G is cyclic. Let $p_1, \ldots, p_r$ be the distinct prime divisors of $|F|$ for $F = F(G)$ and let $Z \subseteq Z(F)$ with $|Z| = p_1 \ldots p_r$. Then there exist $E, T \subseteq G$ such that*

(i) *$F = ET$, $Z = E \cap T$, and $T = C_F(E)$.*
(ii) *The Sylow subgroups of E are extra-special or cyclic of prime order.*
(iii) *If every characteristic abelian subgroup of G is in $Z(F)$, then $T = Z(F)$.*

3. Proof of theorem

In this section we shall prove the theorem stated in the Introduction. Let G be a minimal counterexample and $N \lhd G$ a non-M-group with $|G:N|$ as small as possible. Then N is a maximal normal subgroup of G. Therefore $|G:N|$ is a prime number. Since K/L is nilpotent and L is abelian, every subgroup of K is an M-group (by Theorem 6.23 of [4]) and, in particular, $G = NK$. Put $R = K \cap N$. We have that $|G:K| = |N:R| = p$.

Let $\zeta \in \mathrm{Irr}(N)$ be a non-monomial character. Since G is an M-group, by Mackey's theorem we deduce that ζ does not extend to G. $\chi = \zeta^G \in \mathrm{Irr}(G)$.

If p divides $\zeta(1)$ then, since R has a normal abelian Sylow p-subgroup, we deduce that ζ is induced from some character of R and, since R is an M-group, it follows that ζ is monomial. We conclude that ζ has p'-degree.

Let $U \subseteq N$ and $\alpha \in \mathrm{Irr}(U)$ be a primitive character such that $\alpha^N = \zeta$. Since ζ has p'-degree, there exists $P \in \mathrm{Syl}_p(N)$ such that $P \subseteq U$. Note that $N = PR$ and $G = PK$.

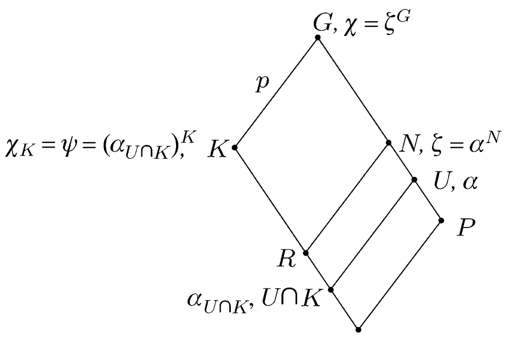
Case 1. $|G/N| \neq p$.

In this case, p does not divide $\chi(1)$ and $\chi_K \in \mathrm{Irr}(K)$. Hence, $\psi = (\zeta_R)^K = \chi_K$ is irreducible and P-invariant. Furthermore, $\psi = ((\alpha^N)_R)^K = ((\alpha_{U\cap R})^R)^K = (\alpha_{U\cap R})^K$. In particular, $\alpha_{U\cap K}$ is irreducible and P-invariant.

Suppose that $\alpha_{U\cap K}$ is not P-primitive. Then $\alpha_{U\cap K} = \beta^{U\cap K}$ for some P-invariant character $\beta \in \mathrm{Irr}(H)$, where H is a P-invariant proper subgroup of $U \cap K$. Since α has p'-degree, $p \nmid |U \cap K : H|$, and so H contains a Sylow p-subgroup of $U \cap K$. Thus $|PH : H| = p$. Since β is P-invariant, $\beta^{PH} = \beta_1 + \cdots + \beta_p$, where each β_i, $1 \leqslant i \leqslant p$, is an irreducible character of PH. Hence $(\alpha_{U\cap K})^U = \beta^U = (\beta^{PH})^U = \beta_1^U + \cdots + \beta_p^U$. On the other hand, $(\alpha_{U\cap K})^U = \alpha_1 + \cdots + \alpha_p$, where each α_i, $1 \leqslant i \leqslant p$, is an irreducible character of U. Since $\alpha = \alpha_i$ for some i, $\alpha = \beta_j^U$ for some j. Since PH is a proper subgroup of U, this contradicts that α is primitive. Thus $\alpha_{U\cap K}$ is P-primitive.

If ψ is P-monomial, then $\alpha(1) = 1$ by Lemma 1. Then $\zeta = \alpha^N$ is monomial, which is a contradiction.

On the other hand, since χ is monomial, $\chi = \lambda^G$ for some linear character λ of a subgroup T. Since χ has p'-degree, we may assume that $T \supseteq P$. Then $\psi = \chi_K = (\lambda^G)_K = (\lambda_{T\cap K})^K$. Since $T \cap K$ is P-invariant subgroup of K and $\lambda_{T\cap K}$ is P-invariant, ψ is P-monomial, which is a contradiction.



Case 2. $|G/N| = p$.

Step 1. $O_p(N) \subsetneq L$.

Proof. If $O_p(N) \nsubseteq L$, then G has a normal Sylow p-subgroup. Hence G has a Sylow tower, and so N is an M-group by Lemma 2, which is a contradiction. Thus $O_p(N) \subseteq L$. If $O_p(N) = L$, then $N = K$. Since K is an M-group, and so is N, which is a contradiction. □

Let H be a Hall p'-subgroup of G, and set $L_1 = L \cap N (= O_p(N))$.

Step 2. $C_{L_1}(H) \neq 1$.

Proof. Since $H \subseteq N$, $[H, L] \subseteq N \cap L = L_1$, and hence $V = C_L(H)L_1$.

If $C_{L_1}(H) = 1$, then $L = C_L(H) \times L_1$ and $|C_L(H)| = p$. By the Frattini argument, $G = LN_G(H)$, and so $C_L(H) \lhd G$. Then $G/C_L(H) \simeq N$. Since G is an M-group, $G/C_L(H)$ is also an M-group, and so is N, which is a contradiction. □

Let $\varphi_1 \in \mathrm{Irr}(L_1)$ be a linear character with $\varphi_1 | \zeta_{L_1}$. We set $K_1 = \mathrm{Ker}\,\varphi_1$.

Step 3. $K_1 \nsupseteq C_{L_1}(H)$. *In particular,* $\varphi_1 \neq 1$.

Proof. Suppose that $K_1 \supseteq C_{L_1}(H)$. We set $\overline{G} = G/C_{L_1}(H)$ since $C_{L_1}(H) \lhd G$. Then φ_1 is regarded as a character of $\overline{L}_1 \subseteq \overline{G}$, and so ζ is also a character of $\overline{N}$. Since $|\overline{G}| < |G|$, $\overline{N}$ is an M-group by induction. Then ζ is monomial, which is a contradiction. Thus $K_1 \nsupseteq C_{L_1}(H)$. In particular, $\varphi_1 \neq 1$. □

Step 4. $K_1 \nsupseteq [H, L_1]$.

Proof. If $[H, L_1] = 1$, then $H \lhd N$, and so $H \lhd G$. Then G has a Sylow tower, and hence N is an M-group by Lemma 2, which is a contradiction.

If $K_1 \supseteq [H, L_1] \neq 1$, then we have a contradiction by an argument similar to that above. □

Next we set $N_1 = I_N(\varphi_1)$. By Clifford's theorem, there exists a $\zeta_1 \in \mathrm{Irr}(N_1)$ such that $\varphi_1 | (\zeta_1)_{N_1}$ and $\zeta_1^N = \zeta$. ζ is not monomial and neither is ζ_1.

Set $G_1 = I_G(\varphi_1)$. Since $\varphi_1 \mid \chi_{L_1}$, there exists a $\chi_1 \in \mathrm{Irr}(G_1)$ such that $\varphi_1 \mid (\chi_1)_{L_1}$ and $\chi_1^G = \chi$ by Clifford's theorem. By Lemmas 3 and 4, χ_1 is monomial and $\zeta_1 \mid (\chi_1)_{N_1}$.

Setting $\overline{G}_1 = G_1/K_1$ gives $\overline{G}_1 \rhd \overline{N}_1$. If $\overline{N}_1$ is nilpotent, then $\overline{N}_1$ is an M-group, and so ζ_1 is monomial, which is a contradiction. In particular, N_1 has a Hall p'-subgroup $H_1 \neq 1$. Since $[\overline{H}_1, \overline{L}_1] = 1$ and $\overline{L}_1\overline{H}_1 \lhd \overline{N}_1$, $\overline{H}_1$ char $\overline{N}_1 \lhd \overline{G}_1$, and hence $\overline{H}_1 \lhd \overline{G}_1$. Therefore there exists a subgroup L_2 of G_1 such that $L_1 \subsetneq L_2 \subseteq N_1$ and $\overline{L}_2$ is an abelian normal subgroup of $\overline{G}_1$. Let $\varphi_2 \in \mathrm{Irr}(\overline{L}_2)$, with $\varphi_2 \mid (\zeta_1)_{\overline{L}_2}$. Then $1 \neq \varphi_2$ is a linear character. Let $\overline{K}_2 = \mathrm{Ker}\,\varphi_2$ with $K_1 \subseteq K_2$. Then $1 \neq L_2/K_2$ is cyclic. We set $G_2 = I_{G_1}(\varphi_2)$ and $N_2 = I_{N_1}(\varphi_2)$. By Clifford's theorem, there exists a $\chi_2 \in \mathrm{Irr}(G_2)$ such that $\varphi_2 \mid (\chi_2)_{L_2}$ and

${\chi_2}^{G_1} = \chi_1$. Similarly, there exists a $\zeta_2 \in \mathrm{Irr}(N_2)$ such that $\varphi_2 \mid (\zeta_2)_{L_2}$ and ${\zeta_2}^{N_1} = \zeta_1$. Then ζ_1 is not monomial and so is ζ_2. Furthermore, $\zeta_2 \mid (\chi_2)_{N_2}$ by Lemma 4.

Now we set $\overline{G}_2 = G_2/K_2$.

Step 5. *There exists a subgroup L_3 such that $L_2 \subsetneq L_3 \subseteq N_2$ and $\overline{L}_3$ is an abelian normal subgroup of $\overline{G}_2$.*

Proof. Suppose false. Set $R_1 = C_{L_1}(H) \times ([L_1, H] \cap K_1)$. Since L_1/K_1 is cyclic, so is $[L_1, H]/([L_1, H] \cap K_1)$. Hence L_1/R_1 is cyclic. Let H_1 be a Hall p'-subgroup of N_1. By the Frattini argument, $N = L_1 N_N(H)$, and hence $H_1^v \subseteq H$ for some $v \in L_1$. Thus we may assume that $H_1 \subseteq H$.

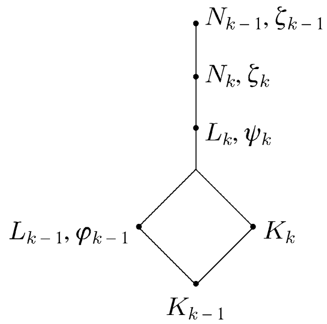
Since $[L_1, H_1] \subseteq K_1$ and $[L_1, H_1] \subseteq [L_1, H]$, $[L_1, H_1] \subseteq K_1 \cap [L_1, H] \subseteq R_1$. Hence $H_1 \subseteq \{x \in N \mid [x, L_1] \subseteq R_1\} = I_N(\mu_1)$, where $\mu_1 \in \mathrm{Irr}(L_1/R_1)$ is faithful. Let S be a Hall p'-subgroup of $I_N(\mu_1)$ with $H_1 \subseteq S$. By an argument similar to that above, we may assume that $S \subseteq H$. Since $[S, L_1] \subseteq R_1$ and $[S, L_1] \subseteq [H, L_1]$, $[S, L_1] \subseteq [H, L_1] \cap R_1 = [H, L_1] \cap (C_{L_1}(H) \times ([H, L_1] \cap K_1)) = [H, L_1] \cap K_1 \subseteq K_1$. Hence $S \subseteq \{x \in N \mid [x, L_1] \subseteq K_1\} = I_N(\varphi_1) = N_1$. This implies that $S = H_1$.

Since $N = L_1 N_N(H)$ and $L_1 \subseteq N_1$, $N_1 = L_1 N_{N_1}(H)$. If $N_1 = L_1 H_1$, then $\overline{N}_1 = \overline{L}_1 \times \overline{H}_1$ is nilpotent, and so $\overline{N}_1$ is an M-group. This contradicts the fact that $\zeta_1 \in \mathrm{Irr}(\overline{N}_1)$ is not monomial. Hence $H_1 \subsetneq N_{N_1}(H)$. Let $a \in N_{N_1}(H)$ be a p-element with $a \notin L_1 H_1$. Since $[a, [H, L_1] \cap K_1] \subseteq [H, L_1] \cap K_1 \subseteq R_1$ and $[a, C_{L_1}(H)] \subseteq C_{L_1}(H) \subseteq R_1$, $[a, L_1] \subseteq R_1$. Hence $a \in I_N(\mu_1)$. By condition (i) of the theorem, $N_1 = L_1 H_1 \langle a \rangle$, and so $N_1 = I_N(\mu_1)$.

Since $\overline{L}_2 \subseteq \overline{L}_1 \times \overline{H}_1$, $\overline{L}_2 = \overline{L}_1 \times (\overline{L}_2 \cap \overline{H}_1)$. Hence $\varphi_2 = \varphi_1 \lambda$, for some $\lambda \in \mathrm{Irr}(L_2 \cap H_1)$ (λ is regarded as a character of $\overline{L_2 \cap H_1}$). Setting $\mu_2 = \mu_1 \lambda$ gives $\mu_2 \in \mathrm{Irr}(L_2/R_1)$. Then $I_{N_1}(\mu_2) = I_{N_1}(\lambda) = I_{N_1}(\varphi_2) = N_2$.

If $\overline{N}_2$ is nilpotent, then $\overline{N}_2$ is an M-group. This contradicts the fact that ζ_2 is not monomial. Let $F = F(\overline{N}_2)$ and let H_2 be a Hall p'-subgroup of N_2. By conditions (i), (ii) of the theorem, $|\overline{N}_2/F| = p$ and $F = \overline{H}_2 \overline{L}_2$. Hence $\overline{N}_2 = F\langle \bar{x} \rangle$ for some p-element $x \in N_2$ with $\bar{x}^p \in F$. Since every characteristic abelian subgroup of $\overline{N}_2$ is cyclic, there exist $E, T \lhd \overline{N}_2$ which satisfy the conditions (i)–(iii) of Lemma 6. By Lemma 6(iii), $T = Z(F)$. If E is abelian, then $F = Z(F)$ and $\overline{N}_2 = C_{\overline{N}_2}(\overline{L}_1) = C_{\overline{N}_2}(F) \subseteq F$. Hence $\overline{N}_2$ is nilpotent, which is a contradiction. Thus E is non-abelian. By Lemma 6(ii), $F = (\overline{E}_1 \times \cdots \times \overline{E}_r)\overline{L}_2$, where $H_1 \cap K_2 \subseteq E_i \subseteq H_2$, $1 \leqslant i \leqslant r$, and each $\overline{E}_i$ is an extra-special group of order $q_i^{2n_i+1}$ for a prime q_i, and an integer n_i. Let $E_0 = E_1 \cdots E_r$.

Set $R_2 = \mathrm{Ker}\,\mu_2$ and $\widetilde{N}_2 = N_2/R_2$. Then $R_2 = R_1(\mathrm{Ker}\,\lambda) = R_1(H_1 \cap K_2)$, and so $\widetilde{H}_2 = H_2/H_1 \cap K_2 \simeq \overline{H}_2$. Hence $F(\widetilde{N}_2) = (\widetilde{E}_1 \times \cdots \times \widetilde{E}_r)\widetilde{L}_2 = \widetilde{E}_0 \widetilde{L}_2$ such that $[\widetilde{E}_0, \widetilde{L}_2] = 1$, $\widetilde{E}_0 \cap \widetilde{L}_2 = Z(\widetilde{E}_0)$. Setting $U = \widetilde{E}_0\langle \tilde{x} \rangle$, $\widetilde{N}_2 = U\widetilde{L}_2$ ensures $\widetilde{L}_2 \lhd \widetilde{N}_2$ and $\widetilde{L}_2 \cap U = Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle$. Then there exist $\psi_i \in \mathrm{Irr}(\widetilde{E}_i)$, $1 \leqslant i \leqslant r$, such that $\psi_i(1) = q_i^{n_i}$, $\psi_0 \in \mathrm{Irr}(\langle \tilde{x}^p \rangle)$, and $(\mu_2)_{Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle} \mid \psi_1 \cdots \psi_r {\psi_0}_{Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle}$. Let $\psi \in \mathrm{Irr}(U)$ with $\psi_1 \cdots \psi_r \psi_0 \mid \psi_{\widetilde{E}_0 \times \langle \tilde{x}^p \rangle}$. Since $\psi_1 \cdots \psi_r \psi_0$ is U-invariant and $|U/\widetilde{E}_0 \times \langle \tilde{x}^p \rangle| = p$, $(\psi)_{\widetilde{E}_0 \times \langle \tilde{x}^p \rangle} = \psi_1 \cdots \psi_r \psi_0$. Thus there exists a $\psi \in \mathrm{Irr}(U)$ such that $(\mu_2)_{Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle} \mid \psi_{Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle}$ and $\psi(1) = q_1^{n_1} \cdots q_r^{n_r}$.



Now $\widetilde{N}_2 = \widetilde{L}_2 U$, $\widetilde{L}_2 \cap U = Z(\widetilde{E}_0) \times \langle \tilde{x}^p \rangle$, and $\mu_2 \in \mathrm{Irr}(\widetilde{L}_2)$ is invariant in $\widetilde{N}_2$. By Lemma 5, there exists a $\psi^* \in \mathrm{Irr}(\widetilde{N}_2)$ such that $(\psi^*)_U = \psi$ and $\mu_2 \mid (\psi^*)_{\widetilde{L}_2}$. Since $I_{N_1}(\mu_2) = N_2$, $(\psi^*)^{N_1} \in \mathrm{Irr}(N_1/R_1)$ with $\mu_2 \mid ((\psi^*)^{N_1})_{L_2/R_1}$ by Clifford's theorem. Similarly, we have that $((\psi^*)^{N_1})^N = (\psi^*)^N \in \mathrm{Irr}(N)$ with $\mu_1 \mid ((\psi^*)^N)_{L_1}$. Since $\operatorname{Ker}\mu_1 \supseteq R_1 \supseteq C_{L_1}(H) \neq 1$ and $C_{L_1}(H) \lhd G$, $(\psi^*)^N$ is regarded as a character of $N/C_{L_1}(H)$. By the minimality of $|G|$, $N/C_{L_1}(H)$ is an M-group since $N/C_{L_1}(H) \lhd G/C_{L_1}(H)$. Hence $(\psi^*)^N$ is monomial and so $(\psi^*)^{N_1}$ and ψ^* are monomial by Lemma 3. Thus $\psi^* = \lambda^{\widetilde{N}_2}$ for some linear character λ of $\widetilde{A}$, where $R_2 \subseteq A$ is a subgroup of N_2.

If $\widetilde{E}_i \cap \widetilde{A}$ is non-abelian, then $(\widetilde{E}_i \cap \widetilde{A})' = Z(\widetilde{E}_i)$. Then $Z(\widetilde{E}_i) \subseteq \operatorname{Ker}\lambda$. Hence $Z(\widetilde{E}_i) \subseteq \operatorname{Ker}\psi^*$, which is a contradiction. Thus $\widetilde{E}_i \cap \widetilde{A}$ is abelian. Since $|\widetilde{N}_2 : \widetilde{A}| = \psi^*(1) = q_1^{n_1} \cdots q_r^{n_r}$, $p \nmid |\widetilde{N}_2 : \widetilde{A}|$. Hence $\widetilde{N}_2 = F(\widetilde{N}_2)\widetilde{A}$, and so $(\widetilde{E_i \cap A})Z(\widetilde{E}_i) = (\widetilde{E}_i \cap \widetilde{A})Z(\widetilde{E}_i) \lhd \widetilde{N}_2$. Then, in $\overline{N}_2 = N_2/K_2$, $(\overline{E_i \cap A})Z(\overline{E}_i) \lhd \overline{N}_2$. By Step 1, there exists an element $1 \neq y \in L$ with $y \notin L_1$. Then $G_2 = N_2\langle y \rangle$. Since $[\bar{y}, \overline{H}_2] = 1$ in $\overline{G}_2 = G_2/K_2$, $(\overline{E_i \cap A})Z(\overline{E}_i) \lhd \overline{G}_2$. Hence $\overline{E_i \cap A} \subseteq Z(\overline{E}_i)$. This means that $\widetilde{E_i \cap A} \subseteq Z(\widetilde{E}_i)$. Thus $|\widetilde{E}_i : \widetilde{E}_i \cap \widetilde{A}| \geqslant |\widetilde{E}_i : Z(\widetilde{E}_i)| = q_i^{2n_i}$. On the other hand, since $|\widetilde{E}_i : \widetilde{E}_i \cap \widetilde{A}| \mid \lambda^{\widetilde{N}_2} = \psi^*(1)$, $|\widetilde{E}_i : \widetilde{E}_i \cap \widetilde{A}| \leqslant q_i^{n_i}$, and hence $q_i^{2n_i} \leqslant q_i^{n_i}$, which is a contradiction. □

Step 6. *A contradiction.*

Proof. Let $\varphi_3 \in \mathrm{Irr}(\overline{L}_3)$ with $\varphi_3 \mid (\zeta_2)_{\overline{L}_3}$. Then $1 \neq \varphi_3$ is a linear character. Let $\overline{K}_3 = \operatorname{Ker}\varphi_3$ with $K_2 \subseteq K_3$. Then $1 \neq L_3/K_3$ is cyclic. Furthermore, there exists a $\zeta_3 \in \mathrm{Irr}(N_3)$ such that $\varphi_3 \mid (\zeta_3)_{L_3}$ and $(\zeta_3)^{N_2} = \zeta_2$, where $N_3 = I_{N_2}(\varphi_3)$. Then ζ_3 is not monomial.

Repeating this argument, there exist $K_k, L_k, N_k, \zeta_k \in \mathrm{Irr}(N_k/K_k)$ $(k = 4, \ldots)$ such that $L_{k-1} \subsetneq L_k \subseteq N_k \subseteq N_{k-1}$, L_k/K_k is cyclic, and ζ_k is not monomial. Since $L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots \subseteq N_3 \subseteq N_2 \subseteq N_1$, there exists an integer n with $L_n = N_n$. Since $N_n/K_n = L_n/K_n$ is cyclic and $\zeta_n \in \mathrm{Irr}(N_n/K_n)$, ζ_n is monomial, which is a contradiction, and this completes the proof of the theorem.

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