A representation of mixed derivatives with an application to the edge-of-the-wedge theorem

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ABSTRACT

The authors prove a lemma which expresses the mixed derivatives of a function in $\mathbb{R}^n$ in terms of its directional derivatives of the same order in an angle. The lemma is used to derive an edge-of-the-wedge theorem for $\mathbb{C}^n$ with an explicit domain of analytic continuation. Other applications will be given in subsequent papers.

1. INTRODUCTION

In this paper we present a convenient condition for real-analyticity of continuous functions in $\mathbb{R}^n$ (section 4). The result is used to prove a form of the edge-of-the-wedge theorem which includes a description of a minimal domain of analytic continuation (section 5). Our method of proof is related to that of F.E. Browder [3], but somewhat simpler. In a subsequent article [9] one of us will prove a refinement of Helgason's support theorem for Radon transforms on $\mathbb{R}^n$ [5]. A later paper [10] will explore the relation between 1-dimensional analyticity of a function in $\mathbb{C}^n$ on a family of complex lines through the origin and n-dimensional holomorphy of the function on a neighborhood of the origin, cf. Forelli's paper [4].

Although the above results may seem unrelated, our proofs are all based on the following lemma, by which mixed derivatives can be estimated in terms of directional derivatives in an angle.

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MAIN LEMMA. *For every open subset Ω of the unit sphere S^{n-1} in $\mathbb{R}^n$, there exist a constant $B=B_\Omega$ and a family of integrable functions $\{g_\alpha\}$ with the following properties:*

$$(1.1)\qquad \int_\Omega (\omega_1 D_1+\ldots+\omega_n D_n)^{|\alpha|} g_\alpha(\omega)d\sigma=\frac{|\alpha|!}{\alpha!}D^\alpha$$

for all n-tuples $\alpha=(\alpha_1,\ldots,\alpha_n)$ of non-negative integers;

$$(1.2)\qquad \int_\Omega |g_\alpha(\omega)|d\sigma\le (B+\varepsilon)^{|\alpha|}$$

for every $\varepsilon>0$ and all α of sufficiently large height $|\alpha|$.

Here we have used the standard notations

$$D_j=\partial/\partial x_j,\ D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n},$$

$$\alpha!=\alpha_1!\ldots\alpha_n!,\ |\alpha|=\alpha_1+\ldots+\alpha_n$$

while $d\sigma$ denotes the area-element of S^{n-1}.

The main lemma will be derived from the following special case for $n=2$:

LEMMA 1. *For every subinterval $(0,\lambda)$ of $(0,2\pi)$ there exist a constant $B=B_\lambda$ and a family of integrable functions $\{g_{pq}\}$ with the following properties:*

$$(1.1')\qquad \int_0^\lambda \left\{(\cos\theta)\frac{\partial}{\partial x_1}+(\sin\theta)\frac{\partial}{\partial x_2}\right\}^{p+q} g_{pq}(\theta)d\theta=\binom{p+q}{p}\frac{\partial^{p+q}}{\partial x_1^p\partial x_2^q}$$

for all non-negative integers p, q;

$$(1.2')\qquad \int_0^\lambda |g_{pq}(\theta)|d\theta\le(B+\varepsilon)^{p+q}$$

for every $\varepsilon>0$ and all p, q with sufficiently large sum $p+q$.

It is easy to see that the constants B must be ≥ 1, cf. (2.1) below. The minimal constants B_λ will form a decreasing function of λ. We prove lemma 1 with

$$(1.3)\qquad B_\lambda=\frac{3+2\sqrt{2}}{\sin\lambda}\ \text{for}\ 0<\lambda\le\pi/4,\ B_\lambda=B_{\pi/4}\ \text{for}\ \lambda>\pi/4.$$

Under translation of the interval $(0,\lambda)$, the constant B is at most doubled (lemma 2). Our constants B are not best possible; it would be of interest for the applications to have minimal values. We do have a sharp result for the case where Ω is the whole sphere (or a hemisphere): in that case B can be taken equal to 1 if we suppress the factor $|\alpha|!/\alpha!$ on the right-hand side of (1.1), see [10].

2. PROOF OF LEMMA 1

By the binomial theorem, equation (1.1′) is equivalent to the set of conditions

$$(2.1)\qquad \int_0^\lambda (\cos\theta)^{p+q-k}(\sin\theta)^k g_{pq}(\theta)d\theta=\delta_{kq},\ k=0,\ldots,p+q.$$

Taking $g_{pq}(\theta)=0$ for $\theta>\pi/4$ if necessary, we may assume that $\lambda\leq\pi/4$ so that

$$\tau=\tan\lambda\leq 1.$$

We now substitute $\theta=\arctan\tau s$. Then (2.1) takes the form

$$\tau^{k+1}\int_0^1 s^k\frac{g_{pq}(\arctan\tau s)}{(1+\tau^2s^2)^{\frac{1}{2}(p+q)+1}}\,ds=\delta_{kq},\ k=0,\ldots,p+q.$$

Since the right-hand side is zero for $k\neq q$, we may replace τ^{k+1} by τ^{q+1}. Setting

$$(2.2)\qquad h_{pq}(s)=\frac{\tau^{q+1}g_{pq}(\arctan\tau s)}{(1+\tau^2s^2)^{\frac{1}{2}(p+q)+1}},$$

our system of equations reduces to

$$(2.3)\qquad \int_0^1 s^k h_{pq}(s)ds=\delta_{kq},\ k=0,\ldots,p+q.$$

It is convenient to introduce the linear span S_q of the powers

$$s^0,\ldots,s^{q-1},s^{q+1},\ldots,s^{p+q}\text{ in }L^2(0,1).$$

Equation (2.3) requires that $h_{pq}\perp S_q$ and furthermore that $\int_0^1 s^q h_{pq}=1$. We take for h_{pq} the unique element of $L^2(0,1)$ that satisfies these conditions and has minimal L^2 norm. Then h_{pq} must be a constant c times the difference between s^q and its orthogonal projection Q on S_q (hence h_{pq} is a polynomial of degree $\leq p+q$). The condition

$$1=\int_0^1 s^q h_{pq}=\int_0^1 (s^q-Q)c(s^q-Q)$$

gives the value of c and hence

$$(2.4)\qquad \|h_{pq}\|_2=\{\text{distance }(s^q,S_q)\}^{-1}.$$

The above distance may be estimated rather accurately with the aid of functional analysis and Laplace integrals. However, this L^2 distance has been computed exactly by Müntz [6] and Szász [8]. Their formula gives

$$\text{distance }(s^q,S_q)=(2q+1)^{-\frac{1}{2}}\prod_{k=0,\,k\neq q}^{p+q}\frac{|k-q|}{k+q+1}$$
$$=(2q+1)^{\frac{1}{2}}\frac{q!p!q!}{(p+2q+1)!}.$$

For fixed $p+q=m$ the reciprocal $\|h_{pq}\|_2$ of the distance is maximal if $2q^2\approx m^2$, so that by a short calculation

$$(2.5)\qquad \|h_{pq}\|_2\leq C^m\text{ for any }C>3+2\sqrt{2},$$

provided $m=p+q>m_0$.

Defining g_{pq} according to (2.2), we will have (2.1) and hence (1.1′). Finally, Schwarz's inequality gives

$$
(2.6)\qquad \begin{cases} \int_0^\lambda |g_{pq}(\theta)|d\theta = \int_0^1 |g_{pq}(\arctan \tau s)| \dfrac{\tau ds}{1+\tau^2 s^2} \\ \le \|h_{pq}\|_2 \tau^{-q} \{ \int_0^1 (1+\tau^2 s^2)^{p+q} ds \}^{\frac{1}{2}} \\ \le \|h_{pq}\|_2 (1+1/\tau^2)^{\frac{1}{2}(p+q)} \le (C/\sin\lambda)^{p+q}, \end{cases}
$$

provided $p+q$ is sufficiently large, cf. (2.5). Formula (2.6) completes the proof of lemma 1. It gives (1.2′) with $B = B_\lambda$ as in (1.3); in the case $\lambda > \pi/4$ we may of course take $B_\lambda = B_{\pi/4}$.

3. DERIVATION OF THE MAIN LEMMA

We first investigate what happens in lemma 1 under rotation.

LEMMA 2. *For every interval $(a, a+\lambda)$ there exist a constant $\tilde{B} = \tilde{B}_{a\lambda}$ and a family of integrable functions $\{\tilde{g}_{pq}\}$ with the following properties:*

$$
(3.1)\qquad \int_a^{a+\lambda} \{(\cos\theta)D_1 + (\sin\theta)D_2\}^{p+q} \tilde{g}_{pq}(\theta)d\theta = \binom{p+q}{p} D_1^p D_2^q
$$

for all non-negative integers p, q;

$$
(3.2)\qquad \int_a^{a+\lambda} |\tilde{g}_{pq}(\theta)|d\theta \le (\tilde{B}+\varepsilon)^{p+q}
$$

for every $\varepsilon > 0$ and all p, q with sufficiently large sum $p+q$.

In fact, we may take $\tilde{B} = 2B$ where B is as in lemma 1.

PROOF. Substituting

$$
D_1 = \tilde{D}_1 \cos a - \tilde{D}_2 \sin a, \; D_2 = \tilde{D}_1 \sin a + \tilde{D}_2 \cos a
$$

and setting $\theta - a = t$, equation (3.1) becomes

$$
(3.3)\qquad \begin{cases} \int_0^\lambda (\tilde{D}_1 \cos t + \tilde{D}_2 \sin t)^{p+q} \tilde{g}_{pq}(a+t)dt = \\ \binom{p+q}{p} \sum_{\substack{0\le j\le p \\ 0\le k\le q}} \binom{p}{j}\binom{q}{k} (\cos a)^{j+q-k}(-1)^{p-j}(\sin a)^{p-j+k} \tilde{D}_1^{j+k} \tilde{D}_2^{p+q-j-k} \end{cases}
$$

Now, using functions g as in lemma 1,

$$
\binom{p+q}{j+k} \tilde{D}_1^{j+k} \tilde{D}_2^{p+q-j-k} = \int_0^\lambda (\tilde{D}_1 \cos t + \tilde{D}_2 \sin t)^{p+q} g_{j+k,p+q-j-k}(t)dt.
$$

It follows that we may satisfy (3.1) by defining

$$\tilde{g}_{pq}(a+t)=$$

$$\binom{p+q}{p} \sum_{j,k} \frac{\binom{p}{j}\binom{q}{k}}{\binom{p+q}{j+k}} (\cos a)^{j+q-k}(-1)^{p-j}(\sin a)^{p-j+k} g_{j+k,p+q-j-k}(t).$$

With this choice of $\tilde{g}$, the inequalities (1.2′) give

$$\begin{aligned}\|\tilde{g}_{pq}\|_1 &\leq \sum_{j,k} \frac{(j+k)!}{j!k!} \frac{(p+q-j-k)!}{(p-j)!(q-k)!} \|g_{j+k,p+q-j-k}\|_1 \\ &\leq \sum_{j,k} 2^{j+k} 2^{p+q-j-k}(B+\varepsilon)^{p+q} \leq C^{p+q}\end{aligned}$$

for any constant $C>2(B+\varepsilon)$ and all large $p+q$. □

PROOF OF THE MAIN LEMMA. The proof is by induction with respect to the dimension. For $n=2$ the result follows from lemma 2, because Ω must then contain an interval $(a, a+\lambda)$ and we can take $g_\alpha = \tilde{g}_{pq}$ on that interval and $g_\alpha = 0$ outside. We thus take $n \geq 3$ and indicate the step from $n-1$ to n. Accordingly, let Ω be an open set in S^{n-1}. We represent the points ω of S^{n-1} as follows:

$$\omega_1 = \cos\theta,\ (\omega_2, \dots, \omega_n) = (\sin\theta)\omega'$$

where $0 \leq \theta \leq \pi$ and $\omega' \in S^{n-2}$. The area element of S^{n-1} then becomes

$$d\sigma = d\theta\ (\sin\theta)^{n-2} d\sigma', \tag{3.4}$$

where $d\sigma'$ is the area element on S^{n-2}.

Next choose open subsets Ω_1 and Ω' of $(0,\pi)$ and S^{n-2} such that the points ω corresponding to $\Omega_1 \times \Omega'$ form an open subset Ω_0 of Ω. The constants and functions which the main lemma (for dimensions $\leq n-1$) associates with Ω_1 and Ω' will be denoted by $\tilde{B}$, $\tilde{g}$ and B^*, g^*, respectively. In order to establish (1.1), we will use functions $g_\alpha(\omega)$ that are equal to zero outside Ω_0. Representing the vector $D=(D_1, \dots, D_n)$ as (D_1, D'), the inner product $\omega \cdot D$ becomes

$$\omega \cdot D = (\cos\theta)D_1 + (\sin\theta)\omega' \cdot D'.$$

The desired formula (1.1) may thus be written in the equivalent form

$$\left\{\begin{aligned} &\int_{\Omega_1 \times \Omega'} \{(\cos\theta)D_1 + (\sin\theta)\omega' \cdot D'\}^{\alpha_1 + |\alpha'|} g_{\alpha_1,\alpha'}(\omega)(\sin\theta)^{n-2} d\theta d\sigma' \\ &= \frac{|\alpha|!}{\alpha_1!|\alpha'|!} D_1^{\alpha_1} \frac{|\alpha'|!}{\alpha'!} (D')^{\alpha'}. \end{aligned}\right. \tag{3.5}$$

Now by the result for $n=2$, cf. lemma 2,

$$(3.6)\qquad \begin{cases} I(\omega') \stackrel{\text{def}}{=} \int_{\Omega_1} \{(\cos\theta)D_1+(\sin\theta)\omega'\cdot D'\}^{\alpha_1+|\alpha'|}\tilde{g}_{\alpha_1|\alpha'|}(\theta)d\theta \\ \quad = \dfrac{|\alpha|!}{\alpha_1!|\alpha'|!} D_1^{\alpha_1}(\omega'\cdot D')^{|\alpha'|}. \end{cases}$$

Multiplying (3.6) by $g^*_{\alpha'}(\omega')$ and integrating over Ω', we obtain

$$(3.7)\qquad \int_{\Omega'} I(\omega')g^*_{\alpha'}(\omega')d\sigma' = \frac{|\alpha|!}{\alpha_1!|\alpha'|!} D_1^{\alpha_1} \int_{\Omega'} (\omega'\cdot D')^{|\alpha'|}g^*_{\alpha'}(\omega')d\sigma'.$$

Thus by the main lemma for dimension $n-1$ and our choice of functions g^*, the right-hand side of (3.7) is precisely equal to the right-hand side of (3.5).

The conclusion is that we may satisfy (3.5) or (1.1) by defining

$$(3.8)\qquad g_\alpha(\omega)(\sin\theta)^{n-2} = \tilde{g}_{\alpha_1|\alpha'|}(\theta)g^*_{\alpha'}(\omega') \text{ for } \omega\in\Omega_0.$$

For the norm of $g_\alpha(\omega)$ we then obtain, cf. (3.4),

$$\int_{\Omega} |g_\alpha(\omega)|d\sigma = \int_{\Omega_0} = \int_{\Omega_1} |\tilde{g}_{\alpha_1|\alpha'|}(\theta)|d\theta \int_{\Omega'} |g^*_{\alpha'}(\omega')|d\sigma'.$$

By the induction hypothesis, the right-hand side is bounded by

$$(\tilde{B}+\varepsilon)^{|\alpha|}(B^*+\varepsilon)^{|\alpha'|}$$

provided $|\alpha'|$ is large. Since $B^*\geq 1$, the weaker inequality

$$\|g_\alpha\|_1 \leq \{(\tilde{B}+\varepsilon)(B^*+\varepsilon)\}^{|\alpha|}$$

will hold for all α of large height, even when $|\alpha'|$ is small. Suitably adjusting ε, it follows that we have (1.2) with $B=B_\Omega=\tilde{B}B^*$.

4. A SUFFICIENT CONDITION FOR REAL-ANALYTICITY OF CONTINUOUS FUNCTIONS

We start with a result for C^∞ functions.

PROPOSITION 1. *Let f be a C^∞ function on a domain D in $\mathbb{R}^n$ such that, for the fixed angle spanned by a given open subset Ω of S^{n-1} and at each point of D, all directional derivatives of order k are bounded by $C^k k!$ for all large k. Then f is real-analytic on D and f has a holomorphic extension to the neighborhood*

$$(4.1)\qquad U=D+\Delta(0,1/BC)=\{z\in\mathbb{C}^n,\ z=a+b,\ a\in D,\ |b_j|<1/BC\}$$

of D in $\mathbb{C}^n$, with $B=B_\Omega$ as in the main lemma.

PROOF. We estimate the mixed derivatives of f at $a\in D$ with the aid of the main lemma:

$$(4.2)\qquad \frac{|\alpha|!}{\alpha!}|D^\alpha f(a)| = \left|\int_\Omega (\omega\cdot D)^{|\alpha|}f(a)g_\alpha(\omega)d\sigma\right| \leq C^{|\alpha|}(B+\delta)^{|\alpha|}|\alpha|!$$

for every $\delta>0$ and all α of large height. It follows that the Taylor series

$$(4.3) \qquad \sum_{\alpha\geq 0} \frac{1}{\alpha!} D^{\alpha}f(a)(x-a)^{\alpha}$$

for $f(x)$ around the point a converges throughout the open set $\square(a)$ given by $|x_j-a_j|<1/BC$, $j=1,\ldots,n$. The series will converge to f on $\square(a)\cap D$, because the difference between $f(x)$ and the partial sums of the series will tend to zero there. Thus f is real-analytic on D.

The complexified series (4.3), obtained by replacing x with z, will converge throughout the polydisc $\Delta(a,1/BC)$, hence f has a holomorphic extension to the set U of (4.1).

Now we are ready for

THEOREM 1. *Let f be a continuous function on a domain D in $\mathbb{R}^n$ such that, for the fixed angle spanned by a given open subset Ω of S^{n-1} and at each point of D, all directional derivatives of order k exist and, for large k, are bounded by $C^k k!$. Then f is real-analytic on D and f has a holomorphic extension to the neighborhood U of D in $\mathbb{C}^n$ described in* (4.1).

PROOF. Let $\{\varphi_\varepsilon\}$ be a C^∞ approximate identity relative to convolution on $\mathbb{R}^n$ such that the support of $\varphi_\varepsilon\geq 0$ belongs to the ball $|x|\leq\varepsilon$ and $\int\varphi_\varepsilon=1$. It is enough to prove the desired result for every relatively compact subdomain D' of D, because the union of the corresponding neighborhoods

$$U'=D'+\Delta(0,1/BC)$$

will be U.

For given D' we take ε less than the distance between D' and the boundary of D and we form the convolutions

$$f_\varepsilon=f*\varphi_\varepsilon$$

on D'. Observe that $f_\varepsilon\to f$ on D' as $\varepsilon\to 0$. The functions f_ε will satisfy the conditions of proposition 1 for D'. Indeed, they are of class C^∞ and for large k, the directional derivatives $D^k_\omega f_\varepsilon$ in the given angle will satisfy the inequality

$$|D^k_\omega f_\varepsilon(a)|=|(D^k_\omega f*\varphi_\varepsilon)(a)|\leq C^k k!\int\varphi_\varepsilon=C^k k!$$

at each point $a\in D'$.

It thus follows from proposition 1 that the functions f_ε have a holomorphic extension to U'. Moreover, by (4.2) applied to f_ε, the extended functions f_ε form a bounded family on every relatively compact subdomain of U'. In other words, they form a normal family on U'. Any limit function of this family will be equal to f on D' and holomorphic on U'. We conclude that f is real-analytic on D' and has a holomorphic extension to U'.

REMARK 1. Observe that in theorem 1, it would be enough to require that f be a distribution on D for which the distributional directional derivatives satisfy the given conditions. On the other hand, one needs more than just real-analyticity of f on every line, cf. the example $x_1x_2/(x_1^2+x_2^2)$ for $\mathbb{R}^2$.

REMARK 2. There are related results on the real-analyticity of separately analytic functions, cf. F.E. Browder [2], Bochnak and Siciak [1]. Their work shows that the conditions for real-analyticity in theorem 1 are much more stringent than necessary. However, the point is that theorem 1 is easy to prove and just right for certain applications, among them the one below.

5. THE EDGE-OF-THE-WEDGE THEOREM

In this section we obtain a form of the edge-of-the-wedge theorem, in which we explicitly indicate a minimal set of analytic continuation. Our proof of the theorem is related to one by Browder, cf. [3]. The main difference with Browder's proof is that he made use of his theorem on real-analyticity of functions that are separately analytic, cf. [2], whereas we use the simpler theorem 1. For general information on the edge-of-the-wedge theorem one may consult Rudin [7].

In the following D is an arbitrary domain in $\mathbb{R}^n$, where we identify $\mathbb{R}^n$ with $\mathbb{R}^n+i0$ in $\mathbb{C}^n$. We let V denote a truncated open cone in (another) $\mathbb{R}^n$:

(5.1) $\quad V=\{t\Omega, \Omega\subset S^{n-1} \text{ open}, 0<t<R\}.$

With D and V we associate the following two open sets in $\mathbb{C}^n$:

(5.2) $\quad W^+=D+iV,\ W^-=D-iV.$

Although it is not necessary, it will be assumed that Ω is connected, so that W^+ and W^- are domains in $\mathbb{C}^n$. The sets W^+ and W^- need not intersect; they are "wedges", with common "edge" D. We finally introduce the basic set

(5.3) $\quad W=W^+\cup D\cup W^-.$

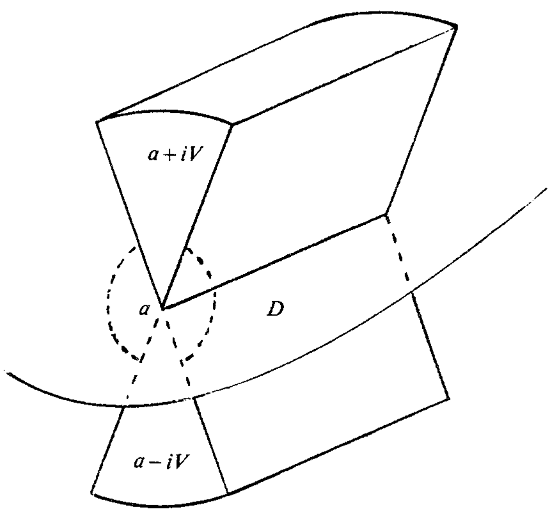

Fig. 1.

It is not easy to draw a picture even for $n=2$, but a little reflection will show that with V as in fig. 1, the set W does not contain a $\mathbb{C}^n$ neighborhood of any point $a \in D$. Indeed, the points $a+iy$ with $y \neq 0$ not in V or $-V$ are outside W. On the other hand, W does contain the intersection of a $\mathbb{C}^n$ neighborhood of a with the family of complex lines $\{z=a+s\omega,\ s \in \mathbb{C}\}$ where ω runs over Ω.

THEOREM 2. *Let D, V, W^+, W^- and W be as above. Then there exists an open neighborhood X of W in $\mathbb{C}^n$ such that every continuous function f on W which is holomorphic on W^+ and on W^- has a holomorphic extension to X. A minimal domain of analytic continuation is given by the union of W and the polydiscs $\Delta(a, r_a)$, where a runs over D and*

$$r_a = \frac{1}{B} \min\{\operatorname{dist}(a, \partial D), R\},$$

with $B = B_\Omega$ as in the main lemma.

PROOF. Let D_0 be an arbitrary relatively compact subdomain of D. We choose

$$0 < \varrho < \min\{\operatorname{dist}(D_0, \partial D), R\}$$

and define

$$V_0 = \{t\Omega, 0 < t < \varrho\}.$$

It will be sufficient to prove that every function f as in the theorem has an analytic continuation to the open set

$$U_0 = D_0 + \Delta(0, \varrho/B).$$

Indeed, the union of these sets is a neighborhood of D in $\mathbb{C}^n$ which contains the polydiscs $\Delta(a, r_a)$ (take small neighborhoods D_0 of a in $\mathbb{R}^n$ and take ϱ close to its upper bound).

With D_0 and Ω we associate the family of discs

$$\Delta_{a\omega}(\varrho) = \{z \in \mathbb{C}^n : z = a + s\omega,\ s \in \mathbb{C},\ |s| < \varrho\}$$

where a runs over D_0 and ω over Ω, while ϱ remains fixed. All these discs belong to the compact subset of W given by

$$\bar{D}_\varrho \pm i\bar{V}_0, \tag{5.4}$$

where D_ϱ stands for the ϱ-neighborhood of D_0 in $\mathbb{R}^n$.

Now let f be as in the theorem. We consider its restrictions to the discs $\Delta_{a\omega}(\varrho)$, writing

$$f_{a\omega}(s) = f(a + s\omega), \quad |s| < \varrho.$$

These functions will be continuous for $|s| < \varrho$ and analytic off the real axis. Hence by application of Morera's theorem, they are analytic for $|s| < \varrho$. Since f is continuous on the compact set (5.4), $|f|$ is bounded by a constant M there.

Thus by the Cauchy-inequalities for the derivatives of $f_{a\omega}$ at the point $s=0$,

$$\left|D_\omega^k f(a)\right| = \left|\frac{\partial^k}{\partial s^k} f_{a\omega}(0)\right| \le k!M/\varrho^k$$

for all $a \in D_0$ and all $\omega \in \Omega$.

It now follows from theorem 1 that the restriction of f to D_0 has a holomorphic extension f_0 to the open set $U_0 = D_0 + \Delta(0, \varrho/B)$ in $\mathbb{C}^n$. However, does this extension coincide with f on $U_0 \cap W$? The answer is affirmative for $U_0 \cap W^+$ because f_0 coincides with f on the discs $\Delta_{a\omega}(\varrho/B)$ (at the centers, the derivatives are the same), and the union of these discs contains an open subset of W^+. Similarly $f_0 = f$ on $U_0 \cap W^-$; by continuity, $f_0 = f$ also on $U_0 \cap D$.

We conclude that all functions f in the theorem can be extended analytically to an open set X in $\mathbb{C}^n$ which contains the union of W and the polydiscs $\Delta(a, r_a)$.

REMARK 3. A sharp estimate for the constant B_Ω in the main lemma would give information about the size of the common domain of analytic continuation X. The precise shape of X is unknown.

REMARK 4. From theorem 2 one usually derives various other forms of the edge-of-the-wedge theorem. In one strong version, the boundary values on the edge are only assumed to exist in distribution sense. Another version is a theorem on analytic continuation by reflection: Every holomorphic function f on W^+ whose imaginary part tends to zero as $y = \operatorname{Im} z \to 0$ in V has a holomorphic extension to X; on W^-, the extension is given by the complex conjugate of $f(\bar{z})$. Cf. [7].

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