ON A GENERALISATION OF LEGENDRE'S ASSOCIATED DIFFERENTIAL EQUATION. II

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1. In order to obtain a second solution of the differential equation (see [1]):

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

we choose another closed path C for the integrand

$$\frac{(t-1)^{k-\frac{m-n}{2}}(t+1)^{k+\frac{m-n}{2}}}{(t-z)^{k+\frac{m+n}{2}+1}},\ \text{or } \psi(t),\ \text{say},$$

namely that, where the variable t, starting from a point A, which may for simplicity be taken on the segment joining -1 and 1, describes a positive turn about -1, then a positive turn about 1, followed by a negative turn about -1, and finally by a negative turn about 1. The point z is not encircled by this path. (See fig. 1).

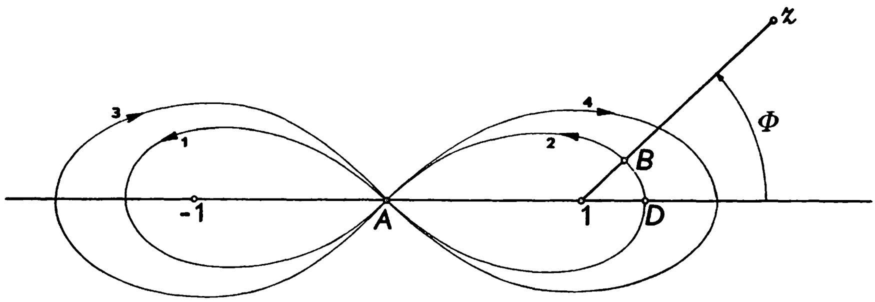

Fig. 1

Now for all values of k, m and n, except those for which $k+\frac{m+n}{2}$ is a negative integer, and those for which $k-\frac{m-n}{2}$ is an integer, and those for which $k+\frac{m-n}{2}$ is an integer, we define, using Pochhammer's notation:

$$(1)\quad \begin{cases} Q_k^{m,n}(z)= \\ =\dfrac{e^{-\pi i(2k-1)}}{\sin\pi\left(k-\frac{m-n}{2}\right)\sin\pi\left(k+\frac{m-n}{2}\right)}\cdot\dfrac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}\cdot\dfrac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+3}}\cdot \\ \cdot\displaystyle\int_C^{(-1+,1+,-1-,1-)}\psi(t)\,dt \end{cases}$$

The phase of $t-z$ will be measured so that the phase at B is $-(\pi-\Phi)$, where Φ is the angle (between $-\pi$ and π) that the vector from B to z makes with the positive direction of the real axis. The phases of $t-1$, $t+1$ will be taken to be zero, 2π respectively, when t passes through D (located on turn 2) where $t-1$ and $t+1$ are real and positive. Thus the initial phases of $t-1$, $t+1$ at A, are $-\pi$ and zero respectively. Furthermore we give arg $(z-1)$ and arg $(z+1)$ their principal values, after making a cross-cut from 1 to $-\infty$ along the real axis in the z-plane.

Remark. Assume that $m=n$. Then (1) becomes

$$\frac{e^{-\pi i(2k-1)}}{\sin^2 \pi k} \cdot \frac{\Gamma(k+m+1)}{\Gamma(k+1)} \cdot \frac{(z^2-1)^{m/2}}{2^{k+3}} \int_C^{(-1+,1+,-1-,1-)} \frac{(t^2-1)^k\, dt}{(t-z)^{k+m+1}}. \tag{2}$$

Now the integral in (2) is equal to

$$(1-e^{2\pi i k}) \int_C^{(-1+,1-)} \frac{(t^2-1)^k\, dt}{(t-z)^{k+m+1}},$$

so that (2) can be written in the form:

$$\frac{e^{-\pi i(k-1)}}{i \sin \pi k} \cdot \frac{\Gamma(k+m+1)}{\Gamma(k+1)} \cdot \frac{(z^2-1)^{m/2}}{2^{k+2}} \int_C^{(-1+,1-)} \frac{(t^2-1)^k\, dt}{(t-z)^{k+m+1}},$$

which is Hobson's definition of $Q_k^m(z)$. See [2], p. 195.

2. Let $2k$ *be an integer*. If we denote the values of the integral in (1) along the paths 1 and 2 by P and Q respectively, then the integral

$$\int_C^{(-1+,1+,-1-,1-)} \psi(t)dt$$

is equal to

$$P+Q-Pe^{2\pi i\left(k-\frac{m-n}{2}\right)}-Qe^{-2\pi i\left(k+\frac{m-n}{2}\right)}=$$
$$=-2\,i\,e^{\pi i\left(k-\frac{m-n}{2}\right)} \sin \pi\left(k-\frac{m-n}{2}\right) \cdot (P+Q).$$

Hence, in this case, (1) can be transformed into

$$\left\{\begin{aligned} & Q_k^{m,n}(z)= \\ & =\frac{i\, e^{\pi i\left(k-\frac{m-n}{2}+1\right)}}{\sin \pi\left(k-\frac{m-n}{2}\right)} \cdot \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)} \cdot \frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+2}} \cdot \int_C^{(-1+,1+)} \psi(t)dt. \end{aligned}\right. \tag{3}$$

3. Now assume that $|z-1|>2$, and that $k+\dfrac{m+n}{2}$ and $2k+1$ are not negative integers, and $k-\dfrac{m-n}{2}$ and $k+\dfrac{m-n}{2}$ are not integers. Making in

(1) the substitution $t-1=-2u$, we find for the integral in (1) the expression:

$$(4)\qquad e^{-\pi i\left(k-\frac{m-n}{2}+1\right)}\,2^{2k+1}\int\limits_{C'}^{(1+,0+,1-,0-)}\frac{u^{k-\frac{m-n}{2}}(1-u)^{k+\frac{m-n}{2}}}{(-2u+1-z)^{k+\frac{m+n}{2}+1}}\,du.$$

Now $-2u+1-z=(z-1+2u)\,e^{-i\pi}=(z-1)\left(1-\frac{2u}{1-z}\right)e^{-i\pi}$, as follows from the choice of the phase of $t-z$ in the preceding section. The phase of $1-\frac{2u}{1-z}$ is between $-\pi$ and π for all points u of the path; furthermore the path of integration C' is placed such that throughout $|2u|<|1-z|$. The expression (4) can be written as

$$(5)\qquad \begin{cases} e^{\pi i m}\cdot\dfrac{2^{2k+1}}{(z-1)^{k+\frac{m+n}{2}+1}}\displaystyle\sum_{r=0}^{\infty}(-1)^r\left(\frac{2}{1-z}\right)^r\cdot \\ \cdot\dfrac{\Gamma\left(-k-\frac{m+n}{2}\right)}{\Gamma(r+1)\,\Gamma\left(-k-\frac{m+n}{2}-r\right)}\displaystyle\int\limits_{C'}^{(1+,0+,1-,0-)}u^{k-\frac{m-n}{2}+r}(1-u)^{k+\frac{m-n}{2}}du. \end{cases}$$

The initial phases of u and $1-u$ at the point corresponding to A are zero. The integral in (5) is equal to

$$(6)\qquad \frac{-4e^{\pi i(2k+r)}\sin\pi\left(k-\frac{m-n}{2}+r\right)\sin\pi\left(k+\frac{m-n}{2}\right)\Gamma\left(k-\frac{m-n}{2}+r+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)}{\Gamma(2k+r+2)}.$$

Furthermore

$$(7)\qquad \begin{cases} \displaystyle\sum_{r=0}^{\infty}\frac{\Gamma\left(k-\frac{m-n}{2}+1+r\right)\Gamma\left(k+\frac{m+n}{2}+1+r\right)}{\Gamma(r+1)\,\Gamma(2k+r+2)}\left(\frac{2}{1-z}\right)^r= \\ =\dfrac{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma(2k+2)}\,F\left\{k-\frac{m-n}{2}+1,k+\frac{m+n}{2}+1;2k+2;\frac{2}{1-z}\right\}. \end{cases}$$

From (1), (4), (5), (6) and (7) we have under the conditions concerning k, m and n, and for $|z-1|>2$ the relation:

$$(8)\qquad \begin{cases} Q_k^{m,n}(z)=\dfrac{e^{\pi i m}\,2^{k-\frac{m-n}{2}}(z-1)^{-k-n/2-1}(z+1)^{n/2}\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)}{\Gamma(2k+2)}. \\ \qquad F\left\{k-\frac{m-n}{2}+1,\ k+\frac{m+n}{2}+1;\ 2k+2;\frac{2}{1-z}\right\}. \end{cases}$$

For $m=n$ this formula is the same as Hobson's expression for the second associated Legendre function $Q_k^m(z)$. See [2], p. 202 (27).

4. In order to get an expression similar to (8) for the case that

$2k+2=0, -1, -2, \ldots$ under the additional restrictions: $k+\frac{m+n}{2}$ is not a negative integer, $k-\frac{m-n}{2}$ is not an integer, we transform

$$\frac{1}{\Gamma(2k+2)}\cdot F\left\{k-\frac{m-n}{2}+1, k+\frac{m+n}{2}+1; 2k+2; \frac{1}{1-z}\right\}$$

into

$$\frac{1}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)}\sum_{r=-2k-1}^{\infty}\frac{\Gamma\left(k-\frac{m-n}{2}+1+r\right)\Gamma\left(k+\frac{m+n}{2}+1+r\right)}{\Gamma(r+1)\,\Gamma(2k+2+r)}\left(\frac{2}{1-z}\right)^r=$$

$$=\frac{1}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)}\sum_{s=0}^{\infty}\frac{\Gamma\left(-k-\frac{m-n}{2}+s\right)\Gamma\left(-k+\frac{m+n}{2}+s\right)}{\Gamma(s-2k)\,\Gamma(s+1)}\left(\frac{2}{1-z}\right)^{s-2k-1}=$$

$$=\left(\frac{2}{1-z}\right)^{-2k-1}\frac{\Gamma\left(-k-\frac{m-n}{2}\right)\Gamma\left(-k+\frac{m+n}{2}\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma(-2k)}\,F\left\{-k-\frac{m-n}{2}, -k+\frac{m+n}{2}; -2k; \frac{2}{1-z}\right\},$$

so that, using the relation:

$$\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(-k-\frac{m-n}{2}\right)=\frac{-\pi}{\sin\pi\left(k+\frac{m-n}{2}\right)},$$

we find by (8) and the first part of this section:

$$(9)\quad\begin{cases} Q_k^{m,n}(z)=\dfrac{2^{-k-\frac{m-n}{2}-1}e^{\pi i(m-2k)}\,\pi\cdot\Gamma\left(-k+\frac{m+n}{2}\right)}{\Gamma(-2k)\,\Gamma\left(k-\frac{m-n}{2}+1\right)\sin\pi\left(k+\frac{m-n}{2}\right)}\cdot(z-1)^{k-n/2}(z+1)^{n/2}\cdot\\ \qquad\cdot F\left\{-k-\frac{m-n}{2}, -k+\frac{m+n}{2}; -2k; \frac{2}{1-z}\right\}.\end{cases}$$

On account of

$$(-1)^{2k}\Gamma\left(k-\frac{m-n}{2}+1\right)\sin\pi\left(k+\frac{m-n}{2}\right)=\frac{\pi}{\Gamma\left(-k+\frac{m-n}{2}\right)}$$

we have, for $|z-1|>2$ and the conditions on k, m and n, mentioned above:

$$(10)\quad\begin{cases} Q_k^{m,n}(z)=\dfrac{e^{\pi im}\,2^{-k-\frac{m-n}{2}-1}\,\Gamma\left(-k+\frac{m+n}{2}\right)\Gamma\left(-k+\frac{m-n}{2}\right)}{\Gamma(-2k)}\cdot\\ \qquad\cdot(z-1)^{k-n/2}(z+1)^{n/2}\cdot F\left\{-k-\frac{m-n}{2}, -k+\frac{m+n}{2}; -2k; \frac{2}{1-z}\right\}.\\ \qquad(2k=-2, -3, -4, \ldots)\end{cases}$$

Remarks. 1. The number $-k+\frac{m+n}{2}$ cannot be a negative integer or zero, as can easily be seen.

2. In the case that $k+\frac{m+n}{2}$ is a negative integer, and $-k+\frac{m+n}{2}$ is a positive integer, the expression (10) is meaningful. However for those values of the parameters k, m and n our deduction of (10) is not valid. For: the double limit of this right hand side of (8) (under the conditions of 3), if, $k+\frac{m+n}{2}\to$ negative integer, $2k+2\to$ non-positive integer (with $-k+\frac{m+n}{2}\to$ positive integer) does not exist.

3. If $m=n$, and $k=-1\frac{1}{2}, -2\frac{1}{2}, \ldots$, and$k+m$ is not a negative integer, then we have for $|z-1|>2$:

$$Q_k^m(z)=$$
$$=\frac{e^{\pi i m}\Gamma(-k+m)\Gamma(-k)}{2^{k+1}\Gamma(-2k)}\cdot(z-1)^{k-m/2}(z+1)^{m/2}F\left\{-k,-k+m;-2k;\frac{2}{1-z}\right\}$$

(a special case of the second associated Legendre function).

5. If $k+\frac{m+n}{2}$ is a negative integer then the right side of (1) is infinite so that the factor $\Gamma(k+\frac{m+n}{2}+1)$ must be disregarded if we wish to obtain a finite solution of the differential equation. In the following we assume that $k+\frac{m+n}{2}$ is not integral and negative.

In order to define $Q_k^{m,n}$-functions in those cases in which the preceding definitions do not hold we distinguish the cases:

I. $k-\frac{m-n}{2}$ is an integer, $k+\frac{m-n}{2}$ is not an integer (so $2k$ is not an integer).

(I*) $k-\frac{m-n}{2}$ is a negative integer. Then according to

$$\sin\pi\left(k-\frac{m-n}{2}\right)\Gamma\left(k-\frac{m-n}{2}+1\right)=\frac{-\pi}{\Gamma\left(-k+\frac{m-n}{2}\right)}$$

we define:

$$Q_k^{m,n}(z)=$$
$$=\frac{e^{-2\pi ik}\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(-k+\frac{m-n}{2}\right)}{\pi\cdot 2^{k+\frac{m-n}{2}+3}\sin\pi\left(k+\frac{m-n}{2}\right)}\cdot(z-1)^{m/2}(z+1)^{n/2}\int_C^{(-1+,1+,-1-,1-)}\psi(t)\,dt.$$

(I**) $k-\frac{m-n}{2}$ is a non-negative integer. Now the integral in (1) vanishes as does $\sin\pi\left(k-\frac{m-n}{2}\right)$. Application of de l'Hôpital's rule yields:

$$\mathfrak{Q}_k^{m,n}(z)=\frac{e^{-\pi i\left(k+\frac{m-n}{2}-1\right)}}{\pi\sin\pi\left(k+\frac{m-n}{2}\right)}\cdot\frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}\cdot$$
$$\cdot\frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+3}}\int_C^{(1+,1+,-1-,1-)}\psi(t)\log(t-1)\,dt.$$

For $|z-1|>2$ the relation (8) is valid in both cases.

II. $k+\frac{m-n}{2}$ is an integer, $k-\frac{m-n}{2}$ is not an integer (so $2k$ is not an integer).

(II*) $k+\frac{m-n}{2}$ is a negative integer. Now the right side of (1) is infinite. In order to obtain a finite second solution of the differential equation we have to disregard the factor $\sin\pi\left(k+\frac{m-n}{2}\right)$ in (1) and the factor $\Gamma\left(k+\frac{m-n}{2}+1\right)$ in (8).

(II**) $k+\frac{m-n}{2}$ is a non-negative integer. The integral in (1) vanishes as does the factor $\sin\pi\left(k+\frac{m-n}{2}\right)$, thus we apply de l'Hôpital's rule and define:

$$Q_k^{m,n}(z)=\frac{e^{-\pi i\left(k-\frac{m-n}{2}-1\right)}}{\pi\sin\pi\left(k-\frac{m-n}{2}\right)}\cdot\frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}\cdot\frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+3}}\int_C^{(-1+,1+,-1-,1-)}\psi(t)\log(t+1)\,dt.$$

The expression (8) remains valid.

(III) $k-\frac{m-n}{2}$ and $k+\frac{m-n}{2}$ are integers (so $2k$ is an integer).

(III*) $k-\frac{m-n}{2}$ and $k+\frac{m-n}{2}$ are negative integers.

$$Q_k^{m,n}(z)=\frac{ie^{\pi i\left(k-\frac{m-n}{2}\right)}}{\pi}\cdot\Gamma\left(-k+\frac{m-n}{2}\right)\Gamma\left(k+\frac{m+n}{2}+1\right)\cdot\frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+2}}\int_C^{(-1+,1+)}\psi(t)\,dt.$$

Now the expression (10) is valid for $|z-1|>2$.

(III**) $k-\frac{m-n}{2}$ is a negative integer and $k+\frac{m-n}{2}$ is a non-negative integer.

Now the integral in the preceding definition of $Q_k^{m,n}(z)$ can be reduced to

$$\int_C^{(1+)}\psi(t)\,dt.$$

For $|z-1|>2$ we have either (8) if $2k+2$ is not equal to $0, -1, -2, \ldots$ or (10) if $2k+2=1, 2, \ldots$.

(III***) $k-\frac{m-n}{2}$ is a non-negative integer and $k+\frac{m-n}{2}$ is a negative integer. The right hand side of (3) is infinite. A solution of the differential equation is

$$i\, e^{\pi i\left(k-\frac{m-n}{2}+1\right)} \cdot \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)} \cdot \frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m-n}{2}+2}} \cdot \int_C^{(-1+)} \psi(t)dt.$$

For $|z-1|>2$ we may use as solution (8) if $2k+2$ is not equal to $0, -1, -2, \ldots$ after disregarding the factor $\Gamma\left(k+\frac{m-n}{2}+1\right)$, and (10) if $2k+2$ is equal to $1, 2, \ldots$ after disregarding the factor $\Gamma\left(-k+\frac{m-n}{2}\right)$.

(III****) $k-\frac{m-n}{2}$ and $k+\frac{m-n}{2}$ are non-negative integers (so $2k$ is non-negative). Now we define:

$$Q_k^{m,n}(z) = \frac{1}{\pi i} \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m-n}{2}+1\right)} \cdot \frac{(z-1)^{m/2}(z+1)^{n/2}}{2^{k+\frac{m+n}{2}+2}} \int_C^{(1+)} \psi(t) \log(t-1)dt.$$

In this case (8) remains valid for $|z-1|>2$.

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