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Asymptotic Behavior of Contractions in Banach Spaces

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INTRODUCTION

Let D be a nonempty subset of a Banach space E and let F map D into E. We shall denote by $R(F)$ the range of F, by $\mathrm{cl}(D)$ the closure of D, and by I the identity mapping (on D). A mapping $T: D \to D$ which is nonexpansive, that is $\| Tx - Ty \| \leqslant \| x - y \|$ for all x and y in D, will be called a contraction. Pazy [24, p. 240] has recently established the following interesting result.

THEOREM. *Let C be a nonempty closed convex subset of a Hilbert space. If $T: C \to C$ is a contraction and $A = I - T$, then*

(a) $0 \in R(A)$ *if and only if* $\{T^n x\}$ *is bounded for every* $x \in C$;

(b) $0 \notin \mathrm{cl}(R(A))$ *if and only if* $\lim \| T^n x \|/n > 0$ *for every* $x \in C$;

(c) $0 \in \mathrm{cl}(R(A))$, *but* $0 \notin R(A)$ *if and only if* $\{T^n x\}$ *is unbounded and* $T^n x/n \to 0$ *for every* $x \in C$.

In this note we study more general iteration processes in certain Banach spaces with the purpose of extending this theorem. It turns out that sometimes our aim can indeed be (partially) achieved (see, for example, Theorem 2.10). Several related results, as well as some open problems, are also included. In Section 1 we relate the boundedness of the sequence of iterates with the existence of a fixed point. Section 2 is devoted to a discussion of the crucial "minimum property" of $\mathrm{cl}(R(A))$ (see the definition below). Section 3 contains a result concerning the convergence of a certain sequence of iterates towards a fixed point of T. We shall consider here only real normed linear spaces. This restriction does not cause any loss of generality.

1. The Existence of a Fixed Point

Let E be a normed linear space. We shall denote by B its unit ball $\{x \in E: \| x \| \leqslant 1\}$ and by S its unit sphere $\{x \in E: \| x \| = 1\}$. For $z \in S$ and $0 \leqslant \epsilon \leqslant 2$, we put

$$\delta(\epsilon, z) = \inf\{1 - \tfrac{1}{2} \| x + y \|: x, y \in B, x - y = \lambda z, | \lambda | \geqslant \epsilon\}$$

and

$$\epsilon_0(z) = \sup\{t: 0 \leqslant t \leqslant 2 \text{ and } \delta(t, z) = 0\}.$$

Recall that a nonempty convex subset K of E is said to have normal structure if each bounded convex subset Q of K with a positive diameter d contains a point x which is nondiametral, that is $\sup\{\| x - y \|: y \in Q\} < d$. We shall say that a nonempty subset $D \subset E$ is boundedly (weakly) compact if its intersection with every closed ball is (weakly) compact. In the sequel $B(x, r)$ will stand for the closed ball $\{y \in E: \| x - y \| \leqslant r\}$.

Let N denote the set of all nonnegative integers. Recall that an infinite complex matrix $M = \{t_{m,n}: m, n \in N\}$ is called a Toeplitz matrix (cf. [16, p. 43]) if the following three conditions are fulfilled:

$$\sup \left\{ \sum_{n=0}^{\infty} | t_{m,n} |: m \in N \right\} < \infty; \qquad \lim_{m\to\infty} \left\{ \sum_{n=0}^{\infty} t_{m,n} \right\} = 1;$$

$$\lim_{m\to\infty} t_{m,n} = 0 \qquad \text{for all } n \in N.$$

Let $k \geqslant 1$ be a fixed integer. In this note we shall be concerned with those Toeplitz matrices which enjoy three additional properties, namely

$$t_{m,n} \geqslant 0 \qquad \text{for all } m, n \in N;$$

$$t_{m,n} = 0 \qquad \text{if} \quad n > (m+1)k;$$

$$\sum_{n=0}^{(m+1)k} t_{m,n} = 1 \qquad \text{for all } m \in N.$$

Let x_0 belong to C, a closed convex subset of E, and let $T: C \to C$ be a contraction. Let a sequence $R = \{x_n: n \in N\} \subset C$ be defined inductively by

$$x_{n+1} = t_{n,0}x_0 + \sum_{i=0}^{n} \sum_{j=1}^{k} t_{n,ik+j} T^j x_i, \qquad n \in N. \tag{1.1}$$

It is easy to see that if T has a fixed point, then R is bounded. The following result is an extension of [11, Theorem 1].

Theorem 1.1. *Let C, a convex boundedly weakly compact subset of a Banach space E which satisfies*

$$\sup\{\epsilon_0(z): z \in S\} \leqslant 1, \tag{1.2}$$

possess normal structure. Let $T: C \to C$ be a contraction and let the sequence R be defined by (1.1). *If R is bounded, then T has a fixed point.*

Proof. Let d be the diameter of $\bigcup \{T^j(R): 0 \leqslant j \leqslant k-1\}$, where $T^0 = I$. We may assume that d is positive. Put

$$X_q = \bigcap_{r \geqslant q} \{z \in C: \| z - T^j x_r \| \leqslant d \text{ for all } 0 \leqslant j \leqslant k-1\}$$

where $q, r \in N$. Let X denote the closure of $Y = \bigcup \{X_q: q \geqslant 1\}$. $X \subset C$ is a nonempty weakly compact subset of E. Let $x \in Y$, so that $x \in X_q$ for some $q \geqslant 1$. If $n \geqslant q$, then

$$\begin{aligned}
&\| Tx - x_{n+1} \| \\
&\quad \leqslant t_{n,0} \| Tx - x_0 \| + \sum_{i=0}^{q-1} \sum_{j=1}^{k} t_{n,ik+j} \| x - T^{j-1} x_i \| + d \left(\sum_{i=q}^{n} \sum_{j=1}^{k} t_{n,ik+j} \right) \\
&\quad \leqslant h(n) + d,
\end{aligned}$$

where $h(n) \to 0$ as $n \to \infty$. Also, if $k > 1$, then

$$\| Tx - T^j x_{n+1} \| \leqslant \| x - T^{j-1} x_{n+1} \| \leqslant d \qquad \text{for } 1 \leqslant j \leqslant k-1.$$

Suppose that $Tx \notin X$. Then $B(Tx, e) \cap Y = \varnothing$ for some positive e. Since $Tx \notin X_r$ for all $r \geqslant 1$, we can find a point $u \in R$ such that $t = \| Tx - u \| > d$. Let $z = (Tx - u)/t$. Choose $\frac{1}{2} < c < 1$ so that $\| cTx + (1-c) u - Tx \| \leqslant e$. There exists $n_0 \geqslant q$ such that $t/(h(n) + d) \geqslant \epsilon > 1$ and

$$(1 - 2(1-c)\, \delta(\epsilon, z))\, (h(n) + d) \leqslant d \qquad \text{for all } n \geqslant n_0 .$$

Finally let $x_r \in R$ satisfy $\| cTx + (1-c) u - x_r \| > d$, where $r > n_0$. We have

$$\begin{aligned}
\| u - x_r \| &\leqslant d \leqslant h(r-1) + d, \\
\| Tx - x_r \| &\leqslant h(r-1) + d, \\
Tx - u &= \| Tx - u \| z,
\end{aligned}$$

and

$$\| Tx - u \| \geqslant \epsilon (h(r-1) + d).$$

Therefore,

$$\begin{aligned}
\| cTx + (1-c) u - x_r \| &= \| c(Tx - x_r) + (1-c)(u - x_r) \| \\
&\leqslant (1 - 2(1-c)\, \delta(\epsilon, z))\, (h(r-1) + d) \leqslant d,
\end{aligned}$$

a contradiction. Hence, $Tx \in X$. The continuity of T implies that T maps X into X. Kirk's fixed point theorem [20, p. 1004] can now be applied to yield a fixed point for T.

Although it is not difficult to see that a Banach space for which $\sup\{\epsilon_0(z): z \in S\} < 1$ has normal structure, a recent example of Bynum's [5] implies that a Banach space which satisfies (1.2) may lack normal structure, even if it is reflexive. Another example of his shows that normal structure may be possessed by spaces which do not satisfy (1.2).

If $\sup\{\epsilon_0(z): z \in S\} = 0$, E is said to be uniformly convex in every direction. It is known [9] that many spaces can be renormed so as to become uniformly convex in every direction. For example, all separable spaces and their conjugates have this property. Nevertheless, it might be of interest to determine whether Theorem 1.1 remains true when condition (1.2) is deleted. Here is a (very) partial result in this direction. We omit its proof (cf. [29, p. 10]). Recall that a mapping $F: D \to D$ is a generalized contraction in the sense of Kirk [21] if for each $x \in D$ there is a number $\alpha(x) < 1$ such that

$$\|Fx - Fy\| \leqslant \alpha(x) \|x - y\| \qquad \text{for all } y \in D.$$

PROPOSITION 1.2. *There is no need to assume condition* (1.2) *in Theorem* 1.1 *provided either the matrix M is column-finite, or T is a generalized contraction.*

In fact, if T is a generalized contraction, then the "normal structure" assumption is dispensable, too. Perhaps it is superfluous in general. However, simple examples show that in Theorem 1.1 we cannot merely assume that C is a closed convex subset of a Banach space.

Remark. Theorem 1.1 does remain true without condition (1.2). See [27, p. 253].

2. FIXED POINT FREE CONTRACTIONS

Let $\{c_n: n \in N\}$ be a sequence of real numbers which satisfy

$$0 < c_n \leqslant 1 \qquad \text{for all } n \in N; \tag{2.1}$$

$$\sum_{i=0}^{\infty} c_i \text{ diverges.} \tag{2.2}$$

We define (cf. [28, p. 210]) a Toeplitz matrix by

$$\begin{aligned} t_{m,0} &= \prod_{j=0}^{m} (1 - c_j); \\ t_{m,n} &= c_{n-1} \prod_{j=n}^{m} (1 - c_j), \qquad 1 \leqslant n \leqslant m; \\ t_{m,m+1} &= c_m; \\ t_{m,n} &= 0, \qquad\qquad n > m + 1. \end{aligned}$$

In this special case it seems to be more convenient to write (1.1) in the form

$$x_{n+1} = (1 - c_n)\, x_n + c_n T x_n\,, \qquad n \in N. \tag{2.3}$$

In the sequel we shall denote $\sum_{i=0}^{n} c_i$ by a_n and A will stand for $I - T$ whenever T is a contraction. The convex hull and convex closure of a subset $D \subset E$ will be denoted by co(D) and clco(D), respectively.

PROPOSITION 2.1. *Let x_0 belong to C, a closed convex subset of a Banach space, let $T\colon C \to C$ be a contraction, and let the sequence $\{x_n\}$ be defined by* (2.3). *If $0 \in \mathrm{cl}(R(A))$, then $x_{n+1}/a_n \to 0$.*

Proof. Consider another initial point $y_0 \in C$ together with its associated sequence $\{y_n\}$. If $n \in N$, then

$$\| y_{n+2} - y_{n+1} \| \leqslant (c_{n+1}/c_n)\, \| y_{n+1} - y_n \|$$

and

$$\| Ty_{n+1} - y_{n+1} \| \leqslant \| Ty_n - y_n \|\,.$$

Hence,

$$\| y_{n+1} - y_0 \| \leqslant a_n \| y_0 - Ty_0 \|\,.$$

Also,

$$\| x_{n+1} - y_{n+1} \| \leqslant \| x_0 - y_0 \|\,.$$

Therefore,

$$\| x_{n+1} - x_0 \| \leqslant 2 \| x_0 - y_0 \| + a_n \| y_0 - Ty_0 \|$$

and the result follows.

Recall that the norm of E is said to be Gâteaux differentiable if

$$\lim_{t \to 0} (\| x + ty \| - \| x \|)/t$$

exists for each x and y in S. It is said to be uniformly Gâteaux differentiable if for each y in S this limit is approached uniformly as x varies over S. It is said to be Fréchet differentiable if for each x in S the limit is approached uniformly as y varies over S. A discussion of these concepts can be found in [7]. We shall need the following known result (cf. [12, p. 555]).

LEMMA 2.2. *The dual of a Banach space E has a Fréchet differentiable norm if and only if for any convex set $K \subset E$ every sequence $\{x_n\}$ in K such that $\| x_n \|$ tends to the distance from K to the origin converges.*

Let Q be a nonempty closed subset of a Banach space. We shall say (after Pazy [24, p. 237]) that Q has the minimum property if it contains a point which is an element of least norm for clco(Q).

THEOREM 2.3 [26]. *Let x_0 belong to C, a closed convex subset of a Banach space E whose dual has a Fréchet differentiable norm. Let $T: C \to C$ be a contraction, and let the sequence $\{x_n\}$ be defined by* (2.3). *If* $\text{cl}(R(A))$ *has the minimum property, then $x_{n+1}/a_n \to -v$, where v is the element of least norm in* $\text{cl}(R(A))$.

Proof. Put $d = \| v \|$. The proof of Proposition 2.1 yields

$$\limsup_{n\to\infty} \| x_{n+1} - x_0 \|/a_n \leqslant d.$$

It is not difficult to see that $(x_0 - x_{n+1})/a_n$ belongs to $\text{co}(R(A))$ for all $n \in N$. Therefore, $\| x_0 - x_{n+1} \|/a_n \geqslant d$, so that

$$\lim_{n\to\infty} \| x_0 - x_{n+1} \|/a_n = d.$$

Appealing now to Lemma 2.2 we obtain $(x_0 - x_{n+1})/a_n \to v$, and the result follows.

When does $\text{cl}(R(A))$ enjoy the minimum property? The following assertions provide partial answers to this question.

Recall that the normalized duality mapping J of a normed linear space E into the family of the nonempty subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^*: \langle x, x^* \rangle = \| x \|^2 \text{ and } \| x^* \| = \| x \|\},$$

where $x \in E$. J is single-valued if and only if E has a Gâteaux differentiable norm. In this case J is continuous when E has its strong topology while E^* has its weak star topology.

The next lemma is known (cf. [10, p. 518; 6, p. 206]). It is a consequence of the Hahn–Banach theorem.

LEMMA 2.4. *Let C be a nonempty convex subset of a normed linear space E, and let $x \notin C$. A point $z \in C$ satisfies $\| z - x \| = \inf\{\| y - x \|: y \in C\}$ if and only if there is $j \in J(z - x)$ such that $\langle y - z, j \rangle \geqslant 0$ for all $y \in C$. (In other words, every convex set is a Kolmogorov set.)*

We remark in passing that information and references concerning Kolmogorov sets can be found for example in [1].

Our first partial answer extends [24, Theorem 3].

PROPOSITION 2.5. *Let C be a closed convex subset of a normed linear space whose norm is Gâteaux differentiable, and let $T: C \to C$ be a contraction. If for some $x_0 \in C$, $\{x_n - Tx_n\}$ converges where $\{x_n\}$ is defined by* (2.3), *then* $\text{cl}(R(A))$ *has the minimum property.*

Proof. Let $x_n - Tx_n \to v$. Since

$$x_0 - x_{n+1} = \sum_{i=0}^{n} c_i(x_i - Tx_i), \qquad x_{n+1}/a_n \to -v.$$

Let u belong to C and put $w = Au$.

$$\langle w - x_{n+1} + Tx_{n+1}, J(u - x_{n+1})\rangle \geqslant 0$$

because T is a contraction. After dividing by a_n and letting n tend to infinity, we obtain $\langle w - v, J(v)\rangle \geqslant 0$. Hence,

$$\langle y - v, J(v)\rangle \geqslant 0 \qquad \text{for all } y \in \text{clco}(R(A)).$$

Lemma 2.4 now implies that v is an element of least norm for clco$(R(A))$. Since $v \in \text{cl}(R(A))$, we are done.

We remark that it is clear that here we have used only the accretiveness of A. (Recall that $F: D \to E$ is said to be accretive if for each x and y in D there exists $j \in J(x - y)$ such that $\langle Fx - Fy, j\rangle \geqslant 0$; see [19] for more details.)

THEOREM 2.6. *Let E be a Banach space whose norm is uniformly Gâteaux differentiable. If $T: E \to E$ is a contraction, then* cl$(R(A))$ *is convex.*

Proof. In fact, we shall use only the following consequence of the uniform Gâteaux differentiability of the norm of E (cf. [8, p. 303]):

For each u and z in S and $\epsilon > 0$ there is a positive δ such that if $x, y \in B$, $x - y = \lambda z$ and $|\lambda| < \delta$, then

$$|\langle u, j_1 - j_2\rangle| < \epsilon, \qquad \text{where} \qquad j_1 \in J(x) \quad \text{and} \quad j_2 \in J(y). \tag{2.4}$$

We do not know if (2.4) is essential.

Let $x \in \text{co}(R(A))$. Since T is a contraction there is $y_p \in E$ such that, $x = Ay_p + py_p$ for every positive p. Pick a point $u \in E$. Put $v = Au$ and let $j_p \in J(py_p - pu)$. Then $j_p/p \in J(y_p - u)$, so that

$$\langle Ay_p - Au, j_p\rangle = \langle x - v - pu - p(y_p - u), j_p\rangle \geqslant 0.$$

Consequently,

$$\| py_p - pu \|^2 = \| j_p \|^2 \leqslant \langle x - v - pu, j_p\rangle \leqslant \| j_p \| \, \| x - v - pu \| .$$

This means that $\{j_p\}$ is bounded in E^*. Let a subnet $\{j_r\}$ converge in the weak star topology to z^*. Then

$$\| z^* \|^2 \leqslant \liminf_{r\to 0} \| j_r \|^2 \leqslant \lim_{r\to 0} \langle x - v - ru, j_r\rangle = \langle x - v, z^*\rangle.$$

Consider now an arbitrary point u_1 in E and put $v_1 = Au_1$. This time we start with $\{y_r\}$ and arrive at a subnet $\{y_s\}$ such that $j_{1,s} \in J(sy_s - su_1)$, z_1^* is the weak star limit of $\{j_{1,s}\}$, and $\| z_1^* \|^2 \leqslant \langle x - v_1, z_1^* \rangle$. Condition (2.4) implies that $z^* = z_1^*$. It follows that $\| z^* \|^2 \leqslant \langle x - w, z^* \rangle$ for all w in $R(A)$. In fact, this inequality holds for all $w \in \text{co}(R(A))$. In particular, it holds for x. Hence, $z^* = 0$. Therefore,

$$\limsup_{r \to 0} \| ry_r - ru \|^2 \leqslant \lim_{r \to 0} \langle x - v - ru, j_r \rangle = \langle x - v, z^* \rangle = 0.$$

Thus, $ry_r \to 0$ and $x \in \text{cl}(R(A))$. This completes the proof.

In case E is a Hilbert space, or more generally a Banach space with a uniformly convex dual, this result is due to Pazy [24, p. 238; 25]. A different proof of Theorem 2.6 was outlined in [27]. Note that sometimes $\text{cl}(R(A)) = E$. For example, this happens in any Banach space whenever

$$\inf\{\sup\{\| Tx \| / \| x \| : \| x \| \geqslant r\} : 0 < r < \infty\} < 1.$$

A generalized contraction satisfies this requirement.

Every reflexive space has an equivalent norm which induces a Fréchet differentiable norm in its dual. This follows from [30, p. 177]. (Observe that if E is a Banach space and E^* has a Fréchet differentiable norm, then E is reflexive.) Every Banach space which is generated by a weakly compact set has an equivalent Gâteaux differentiable norm [2, p. 38]. Every separable Banach space has an equivalent uniformly Gâteaux differentiable norm [31, p. 429]. (Of course, there are nonseparable spaces whose norm is uniformly Gâteaux differentiable.) Moreover, Zizler has shown that every reflexive separable Banach space E has an equivalent uniformly Gâteaux differentiable norm which induces a Fréchet differentiable norm in E^*. With Zizler's norm E is uniformly convex in every direction [32, p. 201].

Let K be a nonempty subset of a normed linear space E and let $P: E \to K$ be a retraction. P will be called a sunny retraction if $P(x) = v$ implies $P(v + \lambda(x - v)) = v$ for all $x \in E$ and $\lambda \geqslant 0$. (We prefer this term to the one used by Bruck on p. 385 of [4] because suns already occur in approximation theory. Again we refer to [1] for information and references concerning suns.)

LEMMA 2.7 (cf. [4, Theorem 3]). *Let K be a nonempty subset of a normed linear space E whose norm is Gâteaux differentiable. Let $P: E \to K$ be a retraction. The following are equivalent:*

(a) $\langle Px - x, J(y - Px) \rangle \geqslant 0$ *for all x in E and y in K;*

(b) $\langle z - w, J(Pz - Pw) \rangle \geqslant \| Pz - Pw \|^2$ *for all z and w in E;*

(c) *P is both sunny and nonexpansive.*

Hence, there is at most one sunny nonexpansive retraction on K.

Proof. Suppose (a) holds and let $z, w \in E$. Put $j = J(Pz - Pw)$. Since both $\langle Pw - w, j\rangle$ and $\langle Pz - z, -j\rangle$ are nonnegative,

$$\langle Pw - Pz + z - w, j\rangle \geqslant 0,$$

and (b) follows. Assume now that (b) holds and let $x \in E$ and $y \in K$. Inserting $z = y = Py$ and $w = x$ in (b) we obtain

$$\langle y - x, J(y - Px)\rangle \geqslant \| y - Px \|^2 = \langle y - Px, J(y - Px)\rangle,$$

and (a) follows. Suppose P is both sunny and nonexpansive. Let $x \in E$, $y \in K$ and put $Px = v$. $C = \{v + \lambda(x - v): \lambda \geqslant 0\}$ is convex. If $w \in C$, then

$$\| v - y \| = \| Pw - Py \| \leqslant \| w - y \| .$$

By Lemma 2.4 $\langle x - Px, J(v - y)\rangle \geqslant 0$ and (a) follows. Conversely, suppose (a) holds. P is a contraction by (b). Let $Px = v, \lambda \geqslant 0, w = v + \lambda(x - v)$, and $j = J(Pw - v)$. By (a), both $\langle v - x, j\rangle$ and $\langle w - Pw, j\rangle$ are nonnegative. But $\lambda(v - x) = v - w$. Hence, $\langle v - Pw, j\rangle = - \| Pw - v \|^2 \geqslant 0$, so that $Pw = v$. Finally suppose that both P and Q are sunny nonexpansive retractions. Let $x \in E$ and $j = J(Qx - Px)$. By (a), $\langle Px - x, j\rangle$ and $\langle x - Qx, j\rangle$ are nonnegative. Hence, $\langle Px - Qx, j\rangle \geqslant 0$ and $Px = Qx$.

The proof of the following theorem was inspired by the argument on p. 239 of [24].

THEOREM 2.8. *Let C be a nonempty closed convex subset of a Banach space E and let $T: C \to C$ be a contraction. Suppose that the norm of E is uniformly Gâteaux differentiable while the norm of E^* is Fréchet differentiable. If C is a sunny nonexpansive retract of E, then* cl($R(A)$) *has the minimum property.*

Proof. Denote the sunny nonexpansive retraction by P and define a contraction $U: E \to E$ by $U = TP$. Theorem 2.6 provides us with a point $v \in \mathrm{cl}(R(I - U))$ which is the element of least norm in

$$\mathrm{clco}(R(I - U)) \supset \mathrm{clco}(R(A)).$$

(In fact, $\mathrm{cl}(R(I - U)) = \mathrm{clco}(R(I - U))$ in this case.) Let $v_n = x_n - Ux_n$ converge to v and put $Px_n - TPx_n = z_n \in R(A)$. By Lemma 2.7,

$$\langle z_n - v_n , Jz_n\rangle = \langle Px_n - x_n , J(Px_n - TPx_n)\rangle \leqslant 0.$$

Hence, $\| z_n \| \leqslant \| v_n \|$. An appeal to Lemma 2.2 yields $z_n \to v$. This completes the proof.

Of course, every closed convex set in a Hilbert space is a sunny nonexpansive retract. Now let K be a nonempty closed subset of a Banach space

E. Suppose K is a nonexpansive retract of E. When can we be sure that it is also a sunny nonexpansive retract? A partial answer is provided by the following result which slightly extends the last part of [4, Theorem 3].

A Banach space E is said to satisfy Opial's condition [23, p. 592] if $x_n \rightharpoonup x$ in E implies that

$$\liminf \| x_n - y \| > \liminf \| x_n - x \| \qquad \text{for all } y \neq x.$$

Although a Hilbert space satisfies this condition, not all uniformly convex spaces do [23, pp. 592 and 596]. The duality mapping J is said to be weakly sequentially continuous at zero if $x_n \rightharpoonup 0$ in E implies that $\{Jx_n\}$ converges to zero in the weak star topology of E^*. Note that l^1 satisfies Opial's condition. Its duality mapping is weakly sequentially continuous at zero.

LEMMA 2.9. *Let K be a nonempty nonexpansive retract of a Banach space E whose norm is Gâteaux differentiable. Then each of the following conditions imply that K is a sunny nonexpansive retract.*

(a) *K is boundedly compact;*

(b) *K is boundedly weakly relatively compact, E satisfies Opial's condition, and J is weakly sequentially continuous at zero;*

(c) *E is uniformly convex and J is weakly sequentially continuous at zero.* *When E is reflexive and satisfies either* (b) *or* (c), *it is sufficient to assume that K is the fixed point set of some contraction $Q: E \to E$.*

Proof. Let $Q: E \to E$ be a contraction and let K be its fixed point set. Let $\{b_n\}$ be an arbitrary sequence with $0 < b_n < 1$ whose limit is 1. For each x in E we denote by x_n the fixed point of $b_n Q + (1 - b_n)x$. Let $x \in E$ and $y \in K$. Then

$$(1 - b_n)x_n + b_n(x_n - Qx_n) = (1 - b_n)x$$

and

$$(1 - b_n)y + b_n(y - Qy) = (1 - b_n)y.$$

Since Q is a contraction it follows that

$$\| x_n - y \|^2 \leqslant \langle x - y, J(x_n - y)\rangle \qquad \text{and} \qquad \langle x_n - x, J(y - x_n)\rangle \geqslant 0.$$

The sequence $\{Qx_n\}$ is bounded because

$$\| Qx_n - Qy \| \leqslant \| x_n - y \| \leqslant \| x - y \|.$$

If Q is a retraction and (a) holds, we denote the strong limit of a subsequence of $\{Qx_n\} \subset K$ by Px. The corresponding subsequence of $\{x_n\}$ also converges to

Px. Clearly P is a retraction onto K and Lemma 2.7 implies that it is sunny as well as nonexpansive. Suppose now that (b) or (c) holds. This time we denote, for each $x \in E$, by Px the weak limit of a subsequence $\{Qx_k\}$ of $\{Qx_n\}$. The sequence $\{x_n - Qx_n\}$ converges strongly to zero because

$$x_n - Qx_n = (1 - b_n)\,x - (1 - b_n)\,Qx_n\,.$$

Both Opial's condition and uniform convexity imply that $Px \in K$. Therefore,

$$\| x_k - Px \|^2 \leqslant \langle x - Px, J(x_k - Px)\rangle \to 0.$$

In other words, the corresponding subsequence $\{x_k\}$ of $\{x_n\}$ converges strongly to Px. Therefore, $\langle Px - x, J(y - Px)\rangle \geqslant 0$, and again Lemma 2.7 enables us to conclude the proof.

The l^p spaces, $1 < p < \infty$, satisfy the conditions imposed on E in (b) and (c). [22] and [14] contain more information concerning these conditions. Nonexpansive retracts are discussed in [13]. Karlovitz [18] has recently constructed sunny nonexpansive retractions in two-dimensional spaces.

Combining Theorems 1.1, 2.3, and 2.8, as well as Lemma 2.9, we obtain the following extension of Pazy's theorem.

THEOREM 2.10. *Let C be a nonempty closed convex subset of a Banach space E which satisfies Opial's condition and is uniformly convex in every direction. Suppose that the norm of E is uniformly Gâteaux differentiable while the norm of E^* is Fréchet differentiable. Assume further that C is the fixed point set of a nonexpansive self-mapping of E. If T: $C \to C$ is a contraction and $\{x_n\}$ is defined by* (2.3), *then*

(a) $0 \in R(A)$ *if and only if* $\{x_n\}$ *is bounded for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (2.1) *and* (2.2);

(b) $0 \notin \mathrm{cl}(R(A))$ *if and only if* $\lim \| x_{n+1} \|/a_n > 0$ *for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (2.1) *and* (2.2);

(c) $0 \in \mathrm{cl}(R(A))$, *but* $0 \notin R(A)$ *if and only if* $\{x_n\}$ *is unbounded and* $x_{n+1}/a_n \to 0$ *for every* x_0 *in* C *and every sequence* $\{c_n\}$ *which satisfies* (2.1) *and* (2.2).

Proof. J is weakly sequentially continuous at zero because E satisfies Opial's condition and has a uniformly Gâteaux differentiable norm [14].

Theorem 2.10 is applicable to the l^p spaces, $1 < p < \infty$, as well as to smooth strictly convex finite-dimensional spaces.

Remark. In Theorem 2.10, "satisfies Opial's condition and is uniformly convex in every direction" can be replaced by "has normal structure." See [27, pp. 251 and 253].

3. Convergence

We have already observed that when a contraction T has a fixed point the sequence $\{x_n\}$ defined by (2.3) is bounded. However, if the initial point x_0 is not a fixed point of T, then it may fail to converge. In fact, there is a nonexpansive self-mapping of l^2 which has the origin as its unique fixed point such that $\{x_n\}$ does not converge for all $x_0 \neq 0$ even though $c_n = \frac{1}{2}$ for all $n \in N$ [17, p. 535]. Therefore, it is of interest to consider other iteration processes.

Let $\{k_n: n \in N\}$, $0 < k_n \leqslant 1$, be a sequence whose limit is 1. Define a Toeplitz matrix by

$$\begin{aligned} t_{m,0} &= 1 - k_m\,, && m \in N; \\ t_{m,m+1} &= k_m\,, && m \in N; \\ t_{m,n} &= 0 && \text{otherwise.} \end{aligned}$$

In other words, given a point $x_0 \in C$, a closed convex subset of a Banach space, and a contraction $T: C \to C$, a sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - k_n)\, x_0 + k_n T x_n\,, \qquad n \in N. \tag{3.1}$$

Theorem 3.1. *Let $k_n = 1 - (n + 2)^{-t}$, where $0 < t < 1$ and $n \in N$. Let $x_0 \in C$, a weakly compact convex subset of a Banach space E whose norm is Gâteaux differentiable, let $T: C \to C$ be a contraction, and let $\{x_n\}$ be defined by* (3.1). *Suppose J is weakly sequentially continuous at zero. If E satisfies Opial's condition or is uniformly convex, then $\{x_n\}$ converges to a fixed point of T.*

Proof. Let K be the nonempty fixed point set of T. Let $\{b_n\}$ be an arbitrary sequence with $0 < b_n < 1$ whose limit is 1. Finally let $\{y_n\} \subset C$ satisfy $y_n = (1 - b_n)\, x_0 + b_n T y_n$. By the proof of Lemma 2.9 every weakly convergent subsequence of $\{y_n\}$ converges strongly to Px_0, where $P: C \to K$ is the unique sunny nonexpansive retraction onto K. It follows that $\{y_n\}$ itself converges to $Px_0 \in K$. Now the proof of [15, Theorem 3] implies that $x_n \to Px_0$.

In the course of this proof we have established the following assertion.

Theorem 3.2. *Suppose the hypotheses of Theorem* 3.1 *hold. Define for each $0 < k < 1$ a point y_k in C by $y_k = (1 - k)\, x_0 + kTy_k$. Then the net $\{y_k\}$ converges strongly to a fixed point of T as $k \to 1$.*

This result improves upon [3, Theorem 3].

Additional information concerning the sequence defined by (3.1) can be found in [29]. Kaniel [17] discovered an ingenious non-Toeplitz process.

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