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ON SOME WEAKLY COMPACT SPACES AND THEIR PRODUCTS*

Stephen H. HECHLER
Queens College, City University of New York, Flushing, N.Y. 11367, U.S.A.

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We consider independence results concerning two topological problems. First, a space is defined to be *weakly Lindelöf* iff every open cover admits a subcover of cardinality less than c. We introduce a topological hypothesis H and show that it implies that every weakly Lindelöf regular separable T_1 space is countably compact iff it is compact. We then show that H follows from Martin's axiom and is, therefore, consistent with the negation of the continuum hypothesis. We also note that it is consistent with the negation of the continuum hypothesis that there exist a separable normal countably-compact T_1 space of cardinality $\aleph_1$ (and thus weakly Lindelöf) which is not compact.

In another direction, we define uncountable cardinals $K_t \leq K_c \leq c$, and we prove that every product of fewer than K_t (sequentially compact) strongly $\aleph_0$-compact spaces is itself (sequentially compact) strongly $\aleph_0$-compact and that any product of no more than K_t such spaces is countably compact. On the other hand, we show that *no* product of K_c or more non-$\aleph_0$-bounded spaces can be strongly $\aleph_0$-compact. We then show that it is consistent with the negation of the continuum hypothesis both that $K_t = K_c = \aleph_1$ and that $K_t = K_c = c$.

We conclude with some open questions.

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Lindelöf	countably compact
compact	sequentially compact
$\aleph_0$-bounded	strongly $\aleph_0$-compact
Martin's axiom	continuum hypothesis

1. Introduction

In this paper we shall consider independence results concerning some well known generalizations of compactness. One set of results will generalize the obvious fact that every countably-compact Lindelöf space is compact. We suppose that the continuum hypothesis fails and note that we may reasonably define a space to be *weakly Lindelöf* iff every

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open cover admits a subcover of cardinality strictly less than c. It is *not* true (given the negation of the continuum hypothesis) that every weakly-Lindelöf countably-compact space is compact; the ordinal space $[0, \Omega)$ is a convenient counterexample. However, since every countable subset of this space is relatively compact, we are led to consider separable spaces, and, for technical reasons, we shall also require regularity. Even these will not be sufficient in all models of set theory, but what we shall prove is that given a certain topological hypothesis H which we shall show is a consequence of Martin's axiom [7] and which is, therefore, consistent with the negation of the continuum hypothesis [12], then every countably-compact weakly-Lindelöf regular separable T_1 space is compact. On the other hand, we shall prove that $\aleph_1 < c < 2^{\aleph_1}$ implies that there exists a non-compact such space.

We shall also consider a generalization of sequential compactness. We remember that a space is *sequentially compact* iff every sequence in it admits a convergent subsequence and that Scarborough and Stone [11] have proven that while sequential compactness is countably productive, no product of c or more T_1 spaces each containing at least two elements is ever sequentially compact. Following Saks and Stephenson [10], we define a space to be *strongly* $\aleph_0$*-compact* iff for every infinite subset S there is an infinite subset $T \subseteq S$ which is relatively compact.[1] Clearly, sequential compactness implies strong $\aleph_0$-compactness in Hausdorff spaces (but not necessarily in T_1 spaces where a sequence may have infinitely many limit points). Frolík [5], when he first introduced the notion, proved it was countably productive and found a family of c strongly $\aleph_0$-compact spaces whose product was not strongly $\aleph_0$-compact. A direct analogue of the second Scarborough–Stone theorem is, of course, impossible since compactness implies strong $\aleph_0$-compactness and is productive. In fact, if, again following Saks and Stephenson, we define a space to be $\aleph_0$*-bounded* iff each of its countable subsets is relatively compact, then we see that $\aleph_0$-boundedness is also productive and implies strong $\aleph_0$-compactness. Thus any analogue of this theorem must necessarily exclude such spaces. We show that excluding such spaces is sufficient in that we prove that there exists an uncountable cardinal $K_c \leqslant c$ such that no product of K_c or more non-$\aleph_0$-bounded Hausdorff spaces can be strongly $\aleph_0$-compact. We also extend the first Scarborough–Stone theorem by proving that there ex-

[1] This definition differs slightly from that given in [10], but it is equivalent in Hausdorff spaces.

ists an uncountable cardinal $K_t \leq c$ such that every product of fewer than K_t sequentially compact or strongly $\aleph_0$-compact spaces is itself sequentially compact or strongly $\aleph_0$-compact, respectively.

In what follows, when we use the terms "regular" and "normal", we shall mean them to *include* the property of being T_1. Finally, we shall assume the Axiom of Choice throughout and without further mention, and whenever we speak of "consistency" we shall be referring to consistency with respect to the axioms of Zermelo–Fraenkel set theory. This latter is simply a convenience based on the fact that almost all independence proofs in the literature are with respect to this system.

We wish to thank both the referee and Franklin Tall for suggestions which resulted in a strengthening of 3.1 and 3.5, and we would like to thank the referee for suggesting that we consider the implication H → S in 3.8 and for many other useful suggestions.

2. Products of some weakly compact spaces

We begin with some combinatorial notions concerning the family of subsets of the set N of natural numbers which we shall use both in this and the following section. For any two sets $A, B \subseteq \mathrm{N}$ we define

$$A \subseteq^* B \quad \text{iff} \quad A - B \text{ is finite,}$$

$$A =^* B \quad \text{iff} \quad A \subseteq^* B \text{ and } B \subseteq^* A, \text{ and}$$

$$A \subset^* B \quad \text{iff} \quad A \subseteq^* B \text{ but } A \neq^* B.$$

We then define a family $\{F_\alpha \subset \mathrm{N}: \alpha < \kappa\}$ to be a *κ-tower* iff

$$\alpha < \beta < \kappa \to \emptyset \neq^* F_\beta \subset^* F_\alpha$$

and to be an *inverse κ-tower* iff

$$\alpha < \beta < \kappa \to F_\alpha \subset^* F_\beta \neq^* \mathrm{N}.$$

Such an (inverse) tower is defined to be *maximal* iff there is no set $F_\kappa \subset \mathrm{N}$ such that the family $\{F_\alpha: \alpha \leq \kappa\}$ is an (inverse) $(\kappa + 1)$-tower. We note that the family $\{F_\alpha: \alpha < \kappa\}$ is a maximal κ-tower iff the family $\{\mathrm{N} - F_\alpha: \alpha < \kappa\}$ is a maximal inverse κ-tower, and we define K_t to be the smallest cardinal κ such that there exists a maximal κ-tower.

Booth [1,2] has noted that Martin's axiom implies that $K_t = c$ while Rothberger [9] has proven that

$$\aleph_0 \leqslant \kappa < K_t \rightarrow 2^\kappa = c \,.$$

The converse of Rothberger's theorem is not true, however. If we look at Cohen's [3] original models of the negation of the continuum hypothesis, we see that if we take any family of $\aleph_1$ generic sets and use it to construct an $\aleph_1$-tower by the standard diagonalization techniques, using only constructible reorderings of countable ordinals, the resulting tower will be maximal. On the other hand, it is well known that in these models we have $\aleph_0 \leqslant \kappa < c \rightarrow 2^\kappa = c$.

We shall need another notion which, to the author's knowledge, has not been dealt with before. Define a family $\mathcal{F}$ of functions from N into N to be *complete* iff for every infinite set $S \subseteq \mathrm{N}$ there exists a function $f \in \mathcal{F}$ such that $f[S] = \mathrm{N}$, and define K_c to be the cardinality of the smallest complete family. Using an argument similar to that above, we can easily see that in the Cohen models mentioned above $K_c = \aleph_1$, and since we shall show later (2.8) that $K_c \geqslant K_t$, it follows that Martin's axiom also implies that $K_c = c$.

We are now ready to consider products of strongly $\aleph_0$-compact spaces and some generalizations of the following three theorems due to Scarborough and Stone.

2.1. Theorem. *Every product of at most $\aleph_0$ sequentially compact spaces is sequentially compact* [11, proof of 5.2].

2.2. Theorem. *Every product of at most $\aleph_1$ sequentially compact spaces is countably compact* [11 5.5].

2.3. Theorem. *No product of c or more T_1 spaces each containing at least two points is sequentially compact* [11, 5.3, 5.4].

In his original paper Frolik [5] proved 2.1 for strongly $\aleph_0$-compact spaces, and Saks and Stephenson [10] proved that 2.2 also holds for these spaces. We observe that these latter proofs as well as the original proofs of 2.1 and 2.2 require only that one be able to extend any countable α-tower. Thus we see immediately:

2.4. Theorem. [2] *Every product of fewer than K_t (sequentially compact) strongly $\aleph_0$-compact spaces is itself (sequentially compact) strongly $\aleph_0$-compact, and every product of no more than K_t such spaces is countably compact.* □

2.5. Corollary. *It is consistent with the negation of the continuum hypothesis that every product of fewer than c (sequentially compact) strongly $\aleph_0$-compact spaces is (sequentially compact) strongly $\aleph_0$-compact and that every product of no more than c such spaces is countably compact.*

Proof. As we mentioned, it is consistent with the negation of the continuum hypotheses that $K_t = c$. □

With respect to 2.3, Booth [1,2] has found an uncountable cardinal less than or equal to c which is equal to $\aleph_1$ in Cohen models and can be used to replace c, and Frolík [5] has exhibited a particular product of c strongly $\aleph_0$-compact spaces which is not itself strongly $\aleph_0$-compact. As we have already pointed out, every product of $\aleph_0$-bounded spaces is $\aleph_0$-bounded and, therefore, strongly $\aleph_0$-compact. Thus the following is, in a sense, best possible.

2.6. Theorem. *If $\{T_\alpha : \alpha < K_c\}$ is any collection of non-$\aleph_0$-bounded Hausdorff spaces, then $T^* = \Pi T_\alpha$ is not strongly $\aleph_0$-compact.*

Proof. Since each T_α is non-$\aleph_0$-bounded, it must contain a countably-infinite non-relatively-compact subset which we may, without loss of generality, assume to be N (although not necessarily with the discrete topology). We now define a set $\Phi = \{\phi_n : n \in \mathbf{N}\} \subseteq T^*$ as follows. Let $\mathcal{F} = \{f_\alpha : \alpha < K_c\}$ be any complete family of functions from N into N, and for each $n \in \mathbf{N}$ and each $\alpha < K_c$ define

$$\phi_n(\alpha) = f_\alpha(n) .$$

Now let Ψ be any infinite subset of Φ, and let

$$S = \{n : \phi_n \in \Psi\} .$$

[2] This theorem with respect to sequentially compact spaces was probably known to David Booth.

Then the projection of Ψ onto T_α is simply $f[S]$. But because $\mathcal{F}$ is complete, there must be at least one $\beta < K_c$ such that $f_\beta[S] = \mathbf{N}$. Thus the projection of Ψ onto T_β is not relatively compact, and since T_β is Hausdorff, this implies that Ψ itself cannot be relatively compact in T^*. We have, therefore, shown that no infinite subset of the countable set $\Phi \subseteq T^*$ is relatively compact, so T^* is not strongly $\aleph_0$-compact. □

Next, noting that if a product of spaces is strongly $\aleph_0$-compact, then so must be the factors, we see:

2.7. Corollary. *If a strongly $\aleph_0$-compact Hausdorff space can be expressed as a product of spaces, then all but fewer than K_c of the factors must be $\aleph_0$-bounded.* □

Now, using the consistency of $K_c = \aleph_1$ with the negation of the continuum hypothesis, we have:

2.8. Corollary. *It is consistent with the negation of the continuum hypothesis that no product of $\aleph_1$ or more non-$\aleph_0$-bounded Hausdorff spaces be strongly $\aleph_0$-compact, and that any strongly $\aleph_0$-compact Hausdorff space expressible as a product admits fewer than K_c non-$\aleph_0$-bounded factors.* □

Finally, combining 2.4 with 2.6, we obtain:

2.9. Corollary. $K_t \leqslant K_c$. □

3. Weakly Lindelöf spaces

Earlier, we defined a space to be *weakly Lindelöf* iff every open cover admits a subcover of cardinality less than c, and, similarly, we define a space to be *weakly separable* iff it contains a dense set of cardinality less than c, *weakly first countable* iff each point has a neighborhood base of cardinality less than c, and $\aleph_0$-*weakly Lindelöf* iff the closure of each countable subset is weakly Lindelöf.

We begin with a topological hypothesis H which will imply most of the results in this section and which we shall show is equivalent to a known consequence of Martin's axiom [7] and is therefore consistent with the negation of the continuum hypothesis [12]. We shall also show that it implies that $K_t = c$ and that every non-compact countably-compact weakly-first-countable separable regular space has cardinality exactly equal to c.

> H. *If D is any countable dense subset of a* T_1 *space and there exists an open cover* $\mathcal{U}$ *of the space such that the set* $\{U \cap D\colon U \in \mathcal{U}\}$ *has cardinality less than c and admits no finite subcover of D, then there exists an infinite closed discrete subset of D.*

Although H is rather technical in form, it has rather surprising consequences. Perhaps the most striking is the following.

3.1. Theorem. H *implies that a weakly-Lindelöf separable regular space is countably compact iff it is compact.*

Proof. Assume H holds, let T be any weakly-Lindelöf countably-compact separable regular space, and let D be any countable dense subset of T. Now choose any open cover of T. Because T is weakly Lindelöf, we may assume that this cover has cardinality less than c, and thus, by H, there exists either an infinite closed discrete subset of D or a finite subcover of D. But T is countably compact, so it cannot contain an infinite closed discrete subset. We have, therefore, shown that every open cover of T admits a finite subcover of the dense subset D, and it is well known that in regular spaces this implies compactness. □

If we drop separability and look instead at the closures of the countable subsets of a space, we have:

3.2. Corollary. H *implies that every* $\aleph_0$*-weakly-Lindelöf regular space is countably compact iff it is* $\aleph_0$*-bounded.* □

Although, as we shall show later, regularity is actually needed in the above, we have developed elsewhere [6] analogues to the compactness notions which we have referred to earlier. These notions, which we have called e-compactness, e-relative compactness, and e-$\aleph_0$-boundedness, reduce to the standard notions in regular spaces and may be used to replace these standard notions in the above theorems and corollaries when non-regular T_1 spaces are considered. In Hausdorff spaces e-compactness implies (but is strictly stronger than) absolute closure, so conclusions concerning absolute closure may easily be obtained from the above.

Also, it is shown that e-$\aleph_0$-boundedness is productive and implies countable compactness, so although it is well known [8] that countable compactness is not even finitely productive, we have:

3.3. Corollary. H *implies that every product of* $\aleph_0$*-weakly-Lindelöf countably-compact* T_1 *spaces is countably compact.*

Proof. If the spaces are regular, then by 3.2 they are $\aleph_0$-bounded, and $\aleph_0$-boundedness is easily seen to be productive and to imply countable compactness. For non-regular T_1 spaces the same proof applies using e-$\aleph_0$-boundedness in place of $\aleph_0$-boundedness.

To see that both regularity and *some* set theoretical or topological hypotheses are needed to obtain conclusions such as those of 3.1, we look at some very useful spaces recently found by Franklin and Rajagopalan. In their paper [4] Franklin and Rajagopalan show how to use any inverse κ-tower (where κ is any cardinal of cofinality greater than ω) to construct a non-regular non-compact sequentially-compact (and, therefore, countably-compact) separable regular space of cardinality κ. Furthermore they note that if the inverse tower is maximal, then it is possible to delete one point from the space in such a way that the space remains non-compact and sequentially compact but becomes normal. (Explicit constructions of these spaces can be found in [6].) Thus we have:

3.4. Theorem. *If the continuum hypothesis fails, then there exists a countably-compact weakly-Lindelöf separable Hausdorff space which is not compact. Furthermore, it is consistent (with the negation of the continuum hypothesis) that there exist such a space which is normal as well.* □

Combining this with 3.1, we obtain:

3.5. Corollary. *It is independent of the negation of the continuum hypothesis as to whether or not every weakly-Lindelöf countably-compact separable regular (or normal) space is compact.* □

In particular, since $K_t > \aleph_1 \to 2^{\aleph_1} = 2^{\aleph_0}$, we have:

3.6. Corollary. *If* $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$, *then there exists a weakly-Lindelöf countably-compact normal separable space (of cardinality* $\aleph_1$*) which is not compact.* □

Furthermore, if we compare 3.1 with the second part of 3.4 and note that the spaces referred to in the latter exist if $K_t < c$, we see:

3.7. Corollary. H *implies that*:
(a) $K_t = c$, *and*
(b) $\aleph_0 \leqslant \kappa < c \to 2^\kappa = c$. □

Using this, we see:

3.8. Theorem. H *implies that every weakly separable weakly-first-countable Hausdorff space has cardinality at most c and that every non-compact countable-compact weakly-first-countable regular separable space has cardinality exactly equal to c.*

Proof. The standard proof that separable first-countable Hausdorff spaces have cardinality at most c makes use of the fact that $c^{\aleph_0} = c$. The same proof when generalized to weak separability and weak first countability requires only that for $\kappa < c$ we have $c^\kappa = c$. But this follows immediately from H by 3.7b. The second part then follows directly from the first part and 3.1 because a space of cardinality less than c is necessarily weakly Lindelöf. □

Finally, to prove the consistency of H, we consider the following hypothesis, which is a slight weakening of the hypothesis $S_\aleph$ introduced in [7] where it was shown to be a consequence of Martin's axiom and, therefore, consistent with the negation of the continuum hypothesis.

> S. *If $\mathcal{F}$ is any family of fewer than c subsets of* N *and no finite subfamily of $\mathcal{F}$ covers all but finitely many members of* N, *then there exists an infinite set $M \subseteq$ N such that no member of $\mathcal{F}$ contains infinitely many members of M.*

This hypothesis is important to us because:

3.9. Theorem. *The hypotheses* S *and* H *are equivalent.*

Proof. S → H. Let D and $\mathcal{U}$ be as in the hypotheses of H. Then the set $\{U \cap D: U \in \mathcal{U}\}$ satisfies the hypotheses of S with respect to D, so there must exist an infinite set $M \subseteq D$ such that for each $U \in \mathcal{U}$ the set $U \cap M$ is finite. But because $\mathcal{U}$ is an open cover and the space in question is T_1, M is clearly closed and discrete.

H → S. Let $\mathcal{F}$ satisfy the hypotheses of S, and assume that $\bigcup\mathcal{F} =$ N and that each member of $\mathcal{F}$ is infinite. Then consider the topological space $T = \mathcal{F} \cup \mathrm{N}$ with the topology generated by the base $\mathcal{B}$ defined as follows:

(a) Every subset of N belongs to $\mathcal{B}$.

(b) If $F \in \mathcal{F}$, $M \subseteq \mathrm{N}$, and $M =^* F$, then $M \cup \{F\} \in \mathcal{B}$.

It is easily seen that T is a T_1 space and that N is a dense subset of T. Now let $\mathcal{U} = \{F \cup \{F\}: F \in \mathcal{F}\}$. Clearly, $\mathcal{U}$ is an open cover of T of cardinality $|\mathcal{F}|$ which we have assumed to be less than c. Also, by our conditions on $\mathcal{F}$, no finite subset of $\mathcal{U}$ can cover N, so we may apply H to obtain an infinite closed discrete subset M of N. However, the fact that M is closed ensures that $F \cap M$ is finite for all $F \in \mathcal{F}$. □

4. Open problems

(1) Booth [1] has shown that there exists a combinatorially defined cardinal K such that a product of κ copies of the two-element discrete space is sequentially compact iff $\kappa < K$, and he has proven the consistency of $\aleph_1 < c = K < 2^{\aleph_1}$. Can results of this kind be obtained for products of arbitrary sequentially compact spaces or products of strongly $\aleph_0$-compact but non-$\aleph_0$-bounded spaces? In particular, can the hypothesis $K_t = c$ be replaced by weaker hypotheses which do not imply $2^{\aleph_0} = 2^{\aleph_1}$?

(2) Is it consistent that K_c be strictly greater than K_t?

(3) Is it consistent that there exist a family of fewer than K_c strongly-$\aleph_0$-compact spaces whose product is not strongly $\aleph_0$-compact? This would, of course, imply K_t less than K_c.

(4) Does H imply Martin's axiom?

(5) Do any of the consequences of H mentioned imply H? In particular, does $K_t = c$ imply H?

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