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Heteroclinic solutions of boundary value problems on the real line involving singular Φ-Laplacian operators

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ABSTRACT

We discuss the solvability of the following strongly nonlinear BVP:

$$\begin{cases} \big(a\big(x(t)\big)\Phi\big(x'(t)\big)\big)' = f\big(t, x(t), x'(t)\big), & t \in \mathbb{R}, \\ x(-\infty) = \alpha, \qquad x(+\infty) = \beta \end{cases}$$

where $\alpha < \beta$, $\Phi : (-r, r) \to \mathbb{R}$ is a general increasing homeomorphism with bounded domain (*singular Φ-Laplacian*), a is a positive, continuous function and f is a Carathéodory nonlinear function. We give conditions for the existence and non-existence of heteroclinic solutions in terms of the behavior of $y \mapsto f(t, x, y)$ and $y \mapsto \Phi(y)$ as $y \to 0$, and of $t \mapsto f(t, x, y)$ as $|t| \to +\infty$. Our approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

1. Introduction

In this paper we discuss the solvability of certain boundary value problems on the whole real line, associated to a differential equation involving the mixed differential operator $(a(x)\Phi(x'))'$, where a is a positive, continuous function and Φ is the so-called *singular Φ-Laplacian*.

More precisely, we investigate the existence and the non-existence of solutions to the following boundary value problem

$$\begin{cases} \big(a\big(x(t)\big)\Phi\big(x'(t)\big)\big)' = f\big(t, x(t), x'(t)\big) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = \alpha, \qquad x(+\infty) = \beta \end{cases} \tag{P}$$

where $\alpha < \beta$ are given constants, $\Phi : (-r, r) \to \mathbb{R}$, $r > 0$, is a general increasing homeomorphism with $\Phi(0) = 0$, a is positive and continuous, and f is a Carathéodory nonlinear function.

Differential equations governed by nonlinear differential operators have been widely studied. In this setting, the most investigated operator is the classical p-Laplacian, that is $\Phi_p(y) := y|y|^{p-2}$ with $p > 1$, which, in recent years, has been generalized to other types of differential operators that preserve the monotonicity of the p-Laplacian, but are not homogeneous. These more general operators, which are usually referred to as Φ-Laplacian, are involved in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$\big(\Phi\big(x'\big)\big)' = f\big(t, x, x'\big),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\Phi(0) = 0$. For a comprehensive bibliography on this subject, see e.g. [7].

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More recently, equations involving other types of differential operators have been studied for instance by Bereanu and Mawhin. They considered the case in which the increasing homeomorphism Φ is defined on the whole real line but is not surjective (see e.g. [1,2,4]), and the case in which Φ is defined only on a bounded domain (see [1,3,5]). In this case such an operator is also called *singular Φ-Laplacian*, and this is also the case we are investigating here.

A different point of view arising from other types of models, e.g. reaction–diffusion equations with non-constant diffusivity and porous media equations, leads to consider nonlinear differential operators of the type $(a(x)x')'$, where a is a positive and continuous function. For references see again [7].

In this paper we are interested in the case of mixed differential operators. In particular, we consider the following strongly nonlinear equation:

$$\big(a(x)\Phi\big(x'\big)\big)' = f\big(t, x, x'\big),$$

and we discuss the existence of heteroclinic solutions on the whole real line to the above equation. The same problem was addressed recently by Cupini, Marcelli and Papalini in [7], in the case in which $\Phi:\mathbb{R}\to\mathbb{R}$ is a generic increasing homeomorphism, and by Marcelli and Papalini in [9] when $\Phi(y)\equiv y$.

In the papers [6,7,9] the authors linked the solvability of (P) to the relative behaviors of $f(t,x,\cdot)$ and $\Phi(\cdot)$ as $y\to 0$, and of $f(\cdot,x,y)$ as $|t|\to+\infty$. Moreover, it was shown that (apart the special case when $f(t,x,y)\sim\frac{1}{t}$ as $|t|\to+\infty$) the presence of the function a in the differential operator and the dependence on x in the right-hand side do not play any role for the solvability of (P).

Here we are concerned with the same problem, but in the case of the singular Φ-Laplacian. Our approach is based on fixed point techniques suitably combined to the method of upper and lower solutions. First, we give an existence result for an auxiliary problem on a sequence of compact intervals, then by a diagonal process we achieve the existence of heteroclinic solutions on the whole real line. In this way we obtain our existence result for problem (P).

The main difference between our results and those in [7] is the following: since we are dealing with a singular Φ-Laplacian operator, in our case suitable *a priori* bounds on the derivative of the solutions are already guaranteed by the structure of Φ. Then, the Nagumo-type condition on the growth of Φ as $y\to+\infty$, which is needed in [7], here is not replaced by any assumption. Hence, here we are able to reach similar conclusions as in [7] but under milder hypotheses. We stress that a similar feature has been pointed out in [3].

As in [7], in our case the sufficient conditions guaranteeing the solvability of problem (P) are rather sharp and cannot be improved, in the following sense: in many concrete situations they are both necessary and sufficient for the existence of solutions.

For instance, when the operator Φ is asymptotic to a power $|y|^{\mu}$ as $y\to 0$, and the right-hand side has the product structure

$$f(t,x,y) = h(t)g(x)c(y)$$

where $h\in L^q_{loc}(\mathbb{R})$, for some $1\leqslant q\leqslant\infty$, satisfies $t\cdot h(t)\leqslant 0$ for every t, the map g is positive in $[\alpha,\beta]$, and $c(y)>0$ for $y\neq 0$, we obtain the following result (see Corollary 4.3 and Remark 4.4): if there are $\delta>-1$, $\nu>0$ such that

$$\big|h(t)\big|\sim \text{const.}|t|^{\delta} \quad \text{as } |t|\to+\infty \quad \text{and} \quad \big|c(y)\big|\sim\text{const.}|y|^{\nu} \quad \text{as } y\to 0$$

then

$$(P) \text{ admits solutions} \quad \Leftrightarrow \quad \nu<\delta+\mu+1.$$

This necessary and sufficient condition for the existence of solutions emphasizes the crucial relation between the infinitesimal order ν of $c(y)$ as $y\to 0$ and the rate δ of $h(t)$ as $|t|\to+\infty$.

Observe that the behavior of both the right-hand side f and the differential operator $(a(x)\Phi(x'))'$ with respect to x does not affect the solvability of (P), which results to be completely independent of $a(x)$ and $g(x)$. Similar conclusions have been reached in [6,7,9].

The paper is organized as follows. In Section 2 we give two preliminary results: the first one is related to an auxiliary boundary value problem on a compact interval, while the second one is a convergence result. Section 3 is devoted to our main result, Theorem 3.1, together with a non-existence result, Theorem 3.2. In Section 4 we present some operative criteria for the solvability of (P) when the right-hand side has the product structure

$$f(t,x,y) = b(t,x)c(x,y),$$

and we conclude the paper with some examples.

2. Preliminary results

In this section we give two preliminary results. The first one is an existence result for an auxiliary boundary value problem on a compact interval, while the second one is a convergence result.

Let $I=[a,b]\subset\mathbb{R}$ be a compact interval, $A:C^1(I)\to C(I)$, $x\mapsto A_x$, and $F:C^1(I)\to L^1(I)$, $x\mapsto F_x$, two continuous maps and, given $r>0$, let $\Phi:(-r,r)\to\mathbb{R}$ be an increasing homeomorphism such that $\Phi(0)=0$.

Given $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, consider the following auxiliary problem on I:

$$\begin{cases} \big(A_u(t)\Phi\big(u'(t)\big)\big)' = F_u(t), & \text{a.e. on } I, \\ u(a) = \alpha, \qquad u(b) = \beta. \end{cases} \tag{Q}$$

By a *solution* of problem (Q) we mean a function $u \in C^1(I)$ such that $|u'(t)| < r$ for all $t \in I$ and $A_u \cdot (\Phi \circ u') \in W^{1,1}(I)$, which satisfies $u(a) = \alpha$, $u(b) = \beta$ and $(A_u(t)\Phi(u'(t)))' = F_u(t)$ a.e. on I.

Our first result, Theorem 2.1, provides a necessary and sufficient condition for the existence of a solution of problem (Q) in terms of the constants r, α, β and the length of the interval I. This existence result has been proved in Theorem 1 of [8]. Here we prove a further estimate of the derivative of the solution in terms of the same constants.

Theorem 2.1. *Assume that the following conditions hold*:

(F1) *there exist $m, M > 0$ such that $m \leqslant A_x(t) \leqslant M$ for every $x \in C^1(I)$, $t \in I$;*
(F2) *the map A sends bounded sets of $C^1(I)$ into uniformly continuous sets in $C(I)$, i.e., for every bounded set $D \subset C^1(I)$ and every $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon) > 0$ such that*

$$\big|A_x(t_1) - A_x(t_2)\big| < \varepsilon \quad \textit{for any } x \in D \textit{ and any } t_1, t_2 \in I \textit{ with } |t_1 - t_2| < \rho;$$

(F3) *there exists $\eta \in L^1(I)$ such that $|F_x(t)| \leqslant \eta(t)$, a.e. on I, for any $x \in C^1(I)$.*

Then, problem (Q) *admits a solution u if and only if $b - a > \frac{\beta-\alpha}{r}$.*
Moreover, there exists a decreasing function $k : (\frac{\beta-\alpha}{r}, +\infty) \to (0, r)$, which depends only on m, M, η, such that

$$\big|u'(t)\big| \leqslant k(b-a) \quad \textit{for every } t \in I. \tag{1}$$

Proof. The fact that there exists a solution u of (Q) if and only if $b - a > \frac{\beta-\alpha}{r}$ has been proved in [8, Theorem 1].

Moreover, the argument in [8, Theorem 1] shows that problem (Q) has a solution u, which is obtained as a fixed point of a suitable multivalued operator (which in our case is single-valued), as follows.

Let

$$D = \big\{y \in C(I): \ \|y\|_C \leqslant \|\eta\|_1\big\},$$

where $\|\cdot\|_C$ and $\|\cdot\|_1$ denote the norms in $C(I)$ and $L^1(I)$ respectively. In Claim I of the proof of [8, Theorem 1], the authors establish the following fact: for every $x \in C^1(I)$ and every $y \in D$ there is a constant $I_{xy} \in \mathbb{R}$ such that

$$\int_a^b \Phi^{-1}\left(\frac{I_{xy} + y(t)}{A_x(t)}\right) \mathrm{d}t = \beta - \alpha.$$

Then, they define an operator $g : C^1(I) \times D \to C^1(I)$ by

$$g_{(x,y)}(t) = \alpha + \int_a^t \Phi^{-1}\left(\frac{I_{xy} + y(s)}{A_x(s)}\right) \mathrm{d}s, \quad \text{for all } x \in C^1(I),\ y \in D \text{ and all } t \in I$$

and define the following multivalued operator:

$$\Gamma : S \to 2^S, \quad \Gamma_x = g_{(x,\hat{F}_x)}$$

where the domain S is a suitable convex and compact subset of $C^1(I)$ and, for $x \in C^1(I)$,

$$\hat{F}_x = \left\{y \in C(I): \ \exists f \in F_x \text{ with } y(t) = \int_0^t f(s)\,\mathrm{d}s,\ \forall t \in I\right\}.$$

Then, in [8], the solution u of problem (Q) is obtained as a fixed point of Γ.

Notice that, in our case, since the operator F is single-valued, we have simply

$$\hat{F}_x(t) = \int_0^t F_x(s)\,\mathrm{d}s, \quad t \in I.$$

In order to estimate $|u'(t)|$, $t \in I$, we will give an estimate of the derivative with respect to t of the operator $g_{(x,y)}(t)$ for any given $x \in C^1(I)$ and $y \in D$.

For this purpose, recall first that, in Claim I of the proof of [8, Theorem 1], by the mean value theorem the authors deduce the existence of $\bar{t} \in I$ such that

$$(b-a)\Phi^{-1}\left(\frac{I_{xy}+y(\bar{t})}{A_x(\bar{t})}\right) = \beta - \alpha$$

for every $x \in C^1(I)$ and $y \in D$. Thus,

$$\frac{I_{xy}+y(\bar{t})}{A_x(\bar{t})} = \Phi\left(\frac{\beta-\alpha}{b-a}\right).$$

Put, for simplicity, $\sigma := b-a$. It follows that, for every $x \in C^1(I)$ and every $y \in D$, we have

$$|I_{xy}| \leqslant \|\eta\|_1 + M\Phi\left(\frac{\beta-\alpha}{\sigma}\right). \tag{2}$$

Consequently, for every $x \in C^1(I)$, $y \in D$ and every $t \in I$, we have

$$\left|\frac{I_{xy}+y(t)}{A_x(t)}\right| \leqslant \frac{1}{m}\left(M\Phi\left(\frac{\beta-\alpha}{\sigma}\right) + 2\|\eta\|_1\right).$$

Now, by definition of the operator $g_{(x,y)}(t)$, for every $x \in C^1(I)$, $y \in D$ and every $t \in I$, we have

$$g'_{(x,y)}(t) = \Phi^{-1}\left(\frac{I_{xy}+y(t)}{A_x(t)}\right).$$

Therefore,

$$\left|g'_{(x,y)}(t)\right| \leqslant \left|\Phi^{-1}\left(\frac{I_{xy}+y(t)}{A_x(t)}\right)\right|. \tag{3}$$

Now observe that, since Φ is an increasing homeomorphism, given $\theta_0 > 0$, for all θ such that $|\theta| \leqslant \theta_0$ one has $|\Phi^{-1}(\theta)| \leqslant \max\{\Phi^{-1}(\theta_0), -\Phi^{-1}(-\theta_0)\}$. Consequently, from inequalities (2) and (3) it follows that there is a function

$$k = k(\sigma) := \max\left\{\Phi^{-1}\left(\frac{M}{m}\Phi\left(\frac{\beta-\alpha}{\sigma}\right) + \frac{2\|\eta\|_1}{m}\right), -\Phi^{-1}\left(-\frac{M}{m}\Phi\left(\frac{\beta-\alpha}{\sigma}\right) - \frac{2\|\eta\|_1}{m}\right)\right\}$$

such that

$$\left|g'_{(x,y)}(t)\right| \leqslant k < r \tag{4}$$

for every $x \in C^1(I)$, $y \in D$ and every $t \in I$. Clearly, k is defined on $(\frac{\beta-\alpha}{r}, +\infty)$ with values in $(0, r)$, depends only on m, M, η, and is increasing as a function of σ.

Finally, since $u = g_{(u,\hat{F}_u)}$, from the estimate (4) we get that $|u'(t)| \leqslant k < r$ for every $t \in I$, and the assertion follows. □

Let now $a : \mathbb{R} \to \mathbb{R}$ be a positive continuous function and $f : \mathbb{R}^3 \to \mathbb{R}$ a Carathéodory function. Let us consider the problem

$$\begin{cases} \left(a\left(x(t)\right)\Phi\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right) & \text{a.e. } t \in \mathbb{R}, \\ x(-\infty) = \alpha, \qquad u(+\infty) = \beta. \end{cases} \tag{P}$$

By a *solution* of problem (P) we mean a function $x \in C^1(\mathbb{R})$ such that $|x'(t)| < r$ for all $t \in \mathbb{R}$ and $(a \circ x) \cdot (\Phi \circ x') \in W^{1,1}(\mathbb{R})$, which satisfies $x(-\infty) = \alpha$, $x(+\infty) = \beta$ and $(a(x(t))\Phi(x'(t)))' = f(t, x(t), x'(t))$ for a.e. $t \in \mathbb{R}$.

As pointed out in the introduction, in our main results we will investigate the existence and non-existence of solutions to problem (P).

To prove the existence we will apply a sequential approach. Roughly speaking we will restrict equation

$$\left(a\left(x(t)\right)\Phi\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right) \quad \text{for a.e. } t \in \mathbb{R},$$

on a sequence of compact intervals exhausting $\mathbb{R}$, then we will use Theorem 2.1 in order to prove the existence of a solution in any compact interval and finally we will show that these solutions converge on the whole $\mathbb{R}$ to a solution of (P).

We state the convergence result that we will need in Lemma 2.2 below. This lemma is analogous to Lemma 2.2 in [7]. Therefore we omit the proof since it can be carried out as in [7]. Let us stress that the estimates on the derivative of the solution x can be proved using a convergence argument together with the boundedness assumption on the derivatives of u_n.

Lemma 2.2. *For all $n \in \mathbb{N}$ let $I_n := [-n, n]$ and let $(u_n)_n$ be a sequence in $C^1(I_n)$ which verifies the following conditions: $|u_n'(t)| < r$ for all $t \in I_n$, $(a \circ u_n)(\Phi \circ u_n') \in W^{1,1}(I_n)$, $u_n(-n) = \alpha$, $u_n(n) = \beta$, the sequence $(u_n(0))_n$ is bounded and*

$$\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)' = f\big(t, u_n(t), u_n'(t)\big) \quad \text{for a.e. } t \in I_n.$$

Assume that there exist two functions $H, \gamma \in L^1(\mathbb{R})$, with H continuous, such that:

$$\big|u_n'(t)\big| \leqslant H(t) < r \quad \text{for all } t \in I_n, \text{ for all } n \in \mathbb{N}, \tag{5}$$

and

$$\big|a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big| \leqslant \gamma(t) \quad \text{a.e. on } I_n, \text{ for all } n \in \mathbb{N}. \tag{6}$$

Then, the sequence $(x_n)_n$ of piecewise C^1 functions on $\mathbb{R}$ defined by

$$x_n(t) := \begin{cases} u_n(t) & \text{for } t \in I_n, \\ \beta & \text{for } t > n, \\ \alpha & \text{for } t < -n \end{cases}$$

admits a subsequence uniformly convergent in $\mathbb{R}$ to a function $x \in C^1(\mathbb{R})$, which is a solution of problem (P) *with the additional property $|x'(t)| \leqslant H(t) < r$ for all $t \in \mathbb{R}$.*

3. Existence and non-existence theorems

In this section we investigate the existence of solutions to problem (P). We will give both an existence result, Theorem 3.1, and a non-existence result, Theorem 3.2. Our approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

We will make the following assumption on the function f:

there exist two constants $\alpha < \beta$ such that

$$f(t, \alpha, 0) \leqslant 0 \quad \text{and} \quad f(t, \beta, 0) \geqslant 0 \quad \text{for a.e. } t \in \mathbb{R}. \tag{H}$$

Clearly, assumption (H) implies that the constant functions α and β respectively are constant lower and upper solutions of equation

$$\big(a\big(x(t)\big)\Phi\big(x'(t)\big)\big)' = f\big(t, x(t), x'(t)\big).$$

Throughout this section we will adopt the following notation:

$$m := \min_{x \in [\alpha, \beta]} a(x) > 0, \qquad M := \max_{x \in [\alpha, \beta]} a(x).$$

Moreover, in what follows $[x]^+$ and $[x]^-$ respectively will denote the positive and negative part of the real number x, and we will set $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$.

The following is our main existence result. Its proof follows the same outline as in Theorem 3.2 of [7], in which the authors consider a similar problem with a nonsingular Φ-Laplacian operator, but with some differences. The main difference is that, in our case, due to the presence of a singular Φ-Laplacian operator, here we do not need a Nagumo-type assumption as in [7]. Roughly speaking, that assumption is needed in order to guarantee that the derivative of the solution is not too large. In our case this condition is ensured by the additional properties of the solutions which we have established in Theorem 2.1 above.

Theorem 3.1. *Assume that there is $\mu > 0$ such that*

$$\liminf_{y \to 0} \frac{|\Phi(y)|}{|y|^\mu} > 0. \tag{7}$$

Assume moreover that there are $L > 0$ and $\gamma > 1$, a function $\eta \in L^1(\mathbb{R})$ and a function $K \in W^{1,1}_{loc}([0, +\infty))$, null in $[0, L]$ and strictly increasing in $[L, +\infty)$, with the following properties:

$$\int^{+\infty} K(t)^{-\frac{1}{\mu(\gamma-1)}} \, \mathrm{d}t < +\infty, \tag{8}$$

$$\begin{cases} f(t, x, y) \leqslant -K'(t)\Phi\big(|y|\big)^\gamma, \\ f(-t, x, y) \geqslant K'(t)\Phi\big(|y|\big)^\gamma \end{cases} \quad \text{for a.e. } t \geqslant L, \text{ every } x \in [\alpha, \beta], \ |y| \leqslant \Theta, \tag{9}$$

with $\Theta := \Phi^{-1}(\frac{M}{m}\Phi(k_1))$, *and* $k_1 := k(2L) < r$, *where the function* k *is the same as in the assertion of Theorem* 2.1 *above. Further,*

$$\left|f(t,x,y)\right| \leqslant \eta(t) \quad \text{if } x \in [\alpha,\beta],\ |y| \leqslant N(t),\ \text{for a.e. } t \in \mathbb{R} \tag{10}$$

where

$$N(t) := \Phi^{-1}\left(\frac{M}{m}\left\{\left(\Phi(k_1)\right)^{-(\gamma-1)} + \frac{\gamma-1}{M}K\left(|t|\right)\right\}^{-\frac{1}{\gamma-1}}\right).$$

Then, there exists a function $x \in C^1(\mathbb{R})$, *with* $\alpha \leqslant x(t) \leqslant \beta$ *for every* $t \in \mathbb{R}$, *which is a solution of problem* (P).

Proof. Without loss of generality, assume $L > \frac{\beta-\alpha}{2r}$.

Given $n \in \mathbb{N}$ with $n \geqslant L$, set $I_n := [-n,n]$. Define a truncation operator $T : W^{1,1}(I_n) \to W^{1,1}(I_n)$, $x \mapsto T_x$, by

$$T_x(t) := \left[\beta \wedge x(t)\right] \vee \alpha, \quad t \in I_n. \tag{11}$$

The operator T is well defined and we have $T_x'(t) = x'(t)$ for a.e. $t \in I_n$ such that $\alpha < x(t) < \beta$, whereas $T_x'(t) = 0$ for a.e. t such that $x(t) \leqslant \alpha$, and $T_x'(t) = 0$ for a.e. t such that $x(t) \geqslant \beta$.

Given $x \in W^{1,1}_{loc}(\mathbb{R})$, define

$$Q_x(t) := -N(t) \vee \left[T_x'(t) \wedge N(t)\right]. \tag{12}$$

Moreover, define a penalty function $w : \mathbb{R} \to \mathbb{R}$ by

$$w(x) := [x-\beta]^+ - [x-\alpha]^-.$$

Clearly, $w(x) = 0$ if $\alpha \leqslant x \leqslant \beta$.

Consider the following auxiliary boundary value problem on the compact interval I_n:

$$\begin{cases} \left(a\left(T_x(t)\right)\Phi\left(x'(t)\right)\right)' = f\left(t, T_x(t), Q_x(t)\right) + \arctan\left(w\left(x(t)\right)\right), & \text{a.e. in } I_n, \\ x(-n) = \alpha, \qquad x(n) = \beta. \end{cases} \tag{P_n^*}$$

Let us prove that problem (P_n^*) admits solutions for every $n \geqslant L$. For this purpose, let $A : C^1(I_n) \to C(I_n)$, $x \mapsto A_x$, and $F : C^1(I_n) \to L^1(I_n)$, $x \mapsto F_x$, be defined by

$$A_x(t) := a\left(T_x(t)\right) \quad \text{and} \quad F_x(t) := f\left(t, T_x(t), Q_x(t)\right) + \arctan\left(w\left(x(t)\right)\right), \quad t \in I_n.$$

It is not difficult to show that the maps A and F are well defined and continuous and satisfy assumptions $(F1)$–$(F3)$ of Theorem 2.1. Moreover, for $n \geqslant L$ the condition $2n > (\beta-\alpha)/r$ is satisfied too. Therefore, for $n \geqslant L$ Theorem 2.1 applies to problem (P_n^*) yielding the existence of a solution $u_n \in C^1(I_n)$ which satisfies the estimate $|u_n'(t)| \leqslant k(2n) < r$ for all $t \in I_n$.

In order to apply Lemma 2.2 we need to show that for any $n \geqslant L$ the function u_n is actually a solution of equation

$$\left(a\left(x(t)\right)\Phi\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right) \quad \text{for a.e. } t \in I_n.$$

For this purpose, from now on the proof will be split into steps.

Step 1. We have $\alpha \leqslant u_n(t) \leqslant \beta$ for all $t \in I_n$, hence $T_{u_n}(t) \equiv u_n(t)$ and $w(u_n(t)) \equiv 0$.

First we show that $\alpha \leqslant u_n(t)$ for every $t \in I_n$. If t_0 is such that $u_n(t_0) - \alpha := \min(u_n(t) - \alpha) < 0$, by the boundary conditions in (P_n^*), t_0 belongs to a compact interval $[t_1,t_2] \subset I_n$ satisfying $u_n(t_1) = u_n(t_2) = \alpha$ and $u_n(t) < \alpha$ for every $t \in (t_1,t_2)$. Hence, $T_{u_n}(t) \equiv \alpha$ and $Q_{u_n}(t) \equiv 0$ in $[t_1,t_2]$, and by assumption (H), for a.e. $t \in (t_1,t_2)$ we have

$$\left(a(\alpha)\Phi\left(u_n'(t)\right)\right)' = a(\alpha)\left(\Phi\left(u_n'(t)\right)\right)' = f(t,\alpha,0) + \arctan\left(u_n(t) - \alpha\right) < 0.$$

Thus, the function $t \mapsto \Phi(u_n'(t))$ is strictly decreasing in (t_1,t_2), so since $u_n'(t_0) = 0$ we have

$$\Phi\left(u_n'(t)\right) < \Phi\left(u_n'(t_0)\right) = 0$$

for every $t \in (t_0,t_2)$. Since Φ is strictly increasing, we deduce that $u_n'(t) < 0$ in (t_0,t_2), which implies $u_n(t_0) > u_n(t_2) = \alpha$, in contradiction with the definition of t_0.

Similarly one can show that $u_n(t) \leqslant \beta$ for every $t \in I_n$.

Step 2. The function $t \mapsto a(u_n(t))\Phi(u_n'(t))$ is decreasing in $[-n,-L]$ and in $[L,n]$.

In fact, since u_n is a solution to (P_n^*) and $N(t) \leqslant \Phi^{-1}(\frac{M}{m}\Phi(k_1))$ for every $t \in \mathbb{R}$, by (12) we get $|Q_{u_n}(t)| \leqslant \Theta$ (see (9)). Using Step 1 and assumption (9) we have that for a.e. $t \geqslant L$

$$\left(a\left(u_n(t)\right)\Phi\left(u_n'(t)\right)\right)' = f\left(t, u_n(t), Q_{u_n}(t)\right) \leqslant -K'(t)\Phi\left(\left|Q_{u_n}(t)\right|\right)^{\gamma} \leqslant 0 \tag{13}$$

and we get the claim in $[L,n]$. Analogously we can prove the monotonicity in $[-n,-L]$.

Step 3. We have $u_n'(t) \geqslant 0$ whenever $L \leqslant |t| \leqslant n$.

Suppose $u_n'(\bar{t}) < 0$ for some $\bar{t} \in [L, n)$. Then, since $\Phi(0) = 0$, by the previous step we have

$$a\big(u_n(t)\big)\Phi\big(u_n'(t)\big) \leqslant a\big(u_n(\bar{t})\big)\Phi\big(u_n'(\bar{t})\big) < 0 \quad \text{for every } t \in [\bar{t}, n]$$

and then, by the sign of a and Φ, we infer $u_n'(t) < 0$ for every $t \in [\bar{t}, n]$. Thus, since u_n solves the boundary conditions in (P_n^*), using Step 1 we get

$$\beta = u_n(n) < u_n(\bar{t}) \leqslant \beta,$$

a contradiction. Similarly, we can show that $u_n'(t) \geqslant 0$ in $[-n, -L]$.

Step 4. If $u_n'(t_0) = 0$ for some $t_0 \in [L, n)$ then $u_n'(t) \equiv 0$ in $[t_0, n)$.

Indeed, if $u_n'(t_0) = 0$ for some $t_0 \in [L, n)$, since $\Phi(0) = 0$, by Step 2 we have $a(u_n(t))\Phi(u_n'(t)) \leqslant 0$ in $[t_0, n]$, hence $u_n'(t) \leqslant 0$ in $[t_0, n]$ and the claim follows from Step 3.

Step 5. We have $|u_n'(t)| \leqslant N(t)$ for a.e. $t \in I_n$.

First notice that $|u_n'(t)| < k_1$ for every $t \in [-L, L]$ and $n \geqslant L$.

In fact, by Theorem 2.1, since the function k is decreasing we have

$$\big|u_n'(t)\big| < k(2n) \leqslant k(2L) = k_1,$$

for every $t \in [-L, L]$ and $n \geqslant L$. Now, in the interval $[-L, L]$ the function N is constantly equal to $\Theta \geqslant k_1$. Thus, $|u_n'(t)| < k_1 \leqslant N(t)$ for $t \in [-L, L]$.

Moreover, in force of Step 3, we have $u_n'(t) \geqslant 0$ for every $t \in I_n \setminus [-L, L]$. Hence, in order to prove the claim, it remains to show that $u_n'(t) \leqslant N(t)$ for every $t \in I_n \setminus [-L, L]$.

For this purpose, let $\hat{t} := \sup\{t > L\colon\ u_n'(\tau) < N(\tau) \text{ in } [L, t]\}$. By Step 5, $\hat{t}$ is well defined. Assume, by contradiction, $\hat{t} < n$. In view of Step 4, we have $u_n'(t) > 0$ in $[L, \hat{t}]$. Moreover, by Step 1 and the definition of Q_{u_n}, we have

$$\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)' = f\big(t, T_{u_n}(t), Q_{u_n}(t)\big) = f\big(t, u_n(t), u_n'(t)\big) \quad \text{a.e. in } [L, \hat{t}].$$

Since u_n' is nonnegative in $[L, n)$, by (13) we have

$$\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)' \leqslant -K'(t)\Phi\big(u_n'(t)\big)^\gamma \leqslant -\frac{K'(t)}{M^\gamma}\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)^\gamma$$

for a.e. $t \in [L, \hat{t}]$. Then, recalling that $K(L) = 0$ and $u_n'(t) > 0$ in $[L, \hat{t}]$, we get

$$\frac{1}{1-\gamma}\Big[\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)^{1-\gamma} - \big(a\big(u_n(L)\big)\Phi\big(u_n'(L)\big)\big)^{1-\gamma}\Big] = \int_L^t \frac{(a(u_n(s))\Phi(u_n'(s)))'}{(a(u_n(s))\Phi(u_n'(s)))^\gamma}\,\mathrm{d}s \leqslant -\frac{K(t)}{M^\gamma}$$

for every $t \in [L, \bar{t}]$.

Now, $u_n'(L) < k_1$ and consequently $\Phi(u_n'(L)) < \Phi(k_1)$. Thus, recalling that a is positive, we obtain

$$\begin{aligned}\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)^{1-\gamma} &\geqslant \big(a\big(u_n(L)\big)\Phi\big(u_n'(L)\big)\big)^{1-\gamma} + \frac{\gamma-1}{M^\gamma}K(t) \\ &> \big(M\Phi(k_1)\big)^{1-\gamma} + \frac{\gamma-1}{M^\gamma}K(t)\end{aligned}$$

which implies

$$u_n'(t) < \Phi^{-1}\left(\frac{M}{m}\left\{\Phi(k_1)^{1-\gamma} + \frac{\gamma-1}{M}K(t)\right\}^{1/(1-\gamma)}\right) = N(t)$$

for every $t \in [L, \hat{t}]$, in contradiction with the definition of $\hat{t}$. So, $\hat{t} = n$ and the claim is proved. The same argument works in the interval $[-n, -L]$ too.

Summarizing, taking into account the properties proved in Steps 1–5, we get

$$\big(a\big(u_n(t)\big)\Phi\big(u_n'(t)\big)\big)' = f\big(t, u_n(t), u_n'(t)\big) \quad \text{a.e. } t \in I_n$$

for every $n \geqslant L$.

Therefore, the sequence $(u_n)_n$ satisfies all the assumptions of Lemma 2.2, with $H(t) = N(t)$ and $\gamma(t) = \eta(t)$, $t \in \mathbb{R}$. Indeed, $\eta \in L^1(\mathbb{R})$ by assumption. Moreover by definition the function N is continuous and such that $N(t) \leqslant \Theta < r$ for all $t \in \mathbb{R}$. Further, assumption (7) implies that $\limsup_{\xi \to 0} \frac{|\Phi^{-1}(\xi)|}{|\xi|^{1/\mu}} < +\infty$. Hence, from assumption (8) we get $N \in L^1(\mathbb{R})$.

Finally, Lemma 2.2 implies the existence of a solution x of problem (P). □

The assumptions of the previous existence result are not improvable, in the following sense: if conditions (7) and (9) are satisfied with the reversed inequalities and the summability condition (8) does not hold, then problem (P) does not admit solutions. This fact is stated in the next theorem.

Theorem 3.2. *Assume that there is $\mu > 0$ such that*

$$\limsup_{y \to 0} \frac{|\Phi(y)|}{|y|^{\mu}} < +\infty. \tag{14}$$

Suppose that there are $L \geqslant 0$, $0 < \rho < r$, $\gamma > 1$ and a positive strictly increasing function $K \in W^{1,1}_{loc}([L, +\infty))$ which satisfies

$$\int^{+\infty} K(t)^{-\frac{1}{\mu(\gamma-1)}}\, \mathrm{d}t = +\infty \tag{15}$$

along with one of the following conditions:

$$f(t, x, y) \geqslant -K'(t)\Phi\big(|y|\big)^{\gamma} \quad \text{for a.e. } t \geqslant L, \text{ every } x \in [\alpha, \beta],\ |y| < \rho \tag{16}$$

or

$$f(t, x, y) \leqslant K'(-t)\Phi\big(|y|\big)^{\gamma} \quad \text{for a.e. } t \leqslant -L, \text{ every } x \in [\alpha, \beta],\ |y| < \rho. \tag{17}$$

Assume moreover that

$$tf(t, x, y) \leqslant 0 \quad \text{for a.e. } |t| \geqslant L, \text{ every } x \in \mathbb{R},\ |y| < \rho. \tag{18}$$

Then, problem (P) *can only admit solutions which are constant in $[L, +\infty)$ (when* (16) *holds) or constant in $(-\infty, -L]$ (when* (17) *holds). Therefore, if both* (16) *and* (17) *hold and $L = 0$, then problem* (P) *does not admit solutions. More precisely, no function $x \in C^1(\mathbb{R})$, with $(a \circ x)(\Phi \circ x')$ almost everywhere differentiable, exists satisfying the boundary conditions and the differential equation in* (P).

Proof. The proof is quite similar to that of [9, Theorem 4]. However, we give it for completeness.

Suppose that (16) holds (the proof being similar if (17) holds). Let $x \in C^1(\mathbb{R})$, with $(a \circ x)(\Phi \circ x')$ almost everywhere differentiable (not necessarily belonging to $W^{1,1}(\mathbb{R})$), be a solution of problem (P).

First of all, let us prove that $\lim_{t \to +\infty} x'(t) = 0$. Indeed, since $x(+\infty) = \beta \in \mathbb{R}$, we have $\limsup_{t \to +\infty} x'(t) \geqslant 0$ and $\liminf_{t \to +\infty} x'(t) \leqslant 0$. If $\liminf_{t \to +\infty} x'(t) < 0$, then there exists an interval $[t_1, t_2] \subset [L, +\infty)$ such that $-\rho < x'(t) < 0$ in $[t_1, t_2]$ and $0 > \Phi(x'(t_2)) > \frac{m}{M}\Phi(x'(t_1))$. However, by virtue of assumption (18) we deduce that $a(x(t))\Phi(x'(t))$ is decreasing in $[t_1, t_2]$, and thus

$$\Phi\big(x'(t_2)\big) \leqslant \frac{1}{M} a\big(x(t_2)\big)\Phi\big(x'(t_2)\big) \leqslant \frac{1}{M} a\big(x(t_1)\big)\Phi\big(x'(t_1)\big) \leqslant \frac{m}{M}\Phi\big(x'(t_1)\big),$$

a contradiction. So, necessarily $\liminf_{t \to +\infty} x'(t) = 0$.

Similarly we get $\limsup_{t \to +\infty} x'(t) = 0$, and consequently $\lim_{t \to +\infty} x'(t) = 0$. Hence, we can define $t^* := \inf\{t \geqslant L\colon |x'(t)| < \rho \text{ in } [t, +\infty)\}$.

Let us now prove that $x'(t) \geqslant 0$ for every $t \geqslant t^*$. Indeed, if $x'(\hat{t}) < 0$ for some $\hat{t} \geqslant t^*$, being $a(x(t))\Phi(x'(t))$ decreasing in $[t^*, +\infty)$, we get

$$\Phi\big(x'(t)\big) \leqslant \frac{1}{M} a\big(x(t)\big)\Phi\big(x'(t)\big) \leqslant \frac{1}{M} a\big(x(\hat{t})\big)\Phi\big(x'(\hat{t})\big) \leqslant \frac{m}{M}\Phi\big(x'(\hat{t})\big) < 0,$$

for every $t \geqslant \hat{t}$, in contradiction with the boundedness of x.

Let us define $\tilde{t} := \inf\{t \geqslant t^*\colon\ x(\tau) \geqslant \alpha \text{ in } [t, +\infty)\} \geqslant t^*$.

Assume by contradiction that $x'(\bar{t}) > 0$ for some $\bar{t} \geqslant \tilde{t}$. Put $T := \sup\{t \geqslant \bar{t}\colon\ x'(\tau) > 0 \text{ in } [\bar{t}, t]\}$, and observe that $T = +\infty$. Indeed, if $T < +\infty$, since $0 < x'(t) < \rho$ in $[\bar{t}, T]$, by (16) we have

$$\big(a\big(x(t)\big)\Phi\big(x'(t)\big)\big)' = f\big(t, x(t), x'(t)\big) \geqslant -K'(t)\Phi\big(\big|x'(t)\big|\big)^{\gamma} \quad \text{for a.e. } t \in [\bar{t}, T]. \tag{19}$$

Without loss of generality, assume ρ so small that $\Phi(\rho) \leqslant 1$. Being $\gamma > 1$, we get

$$\big(a\big(x(t)\big)\Phi\big(x'(t)\big)\big)' \geqslant -K'(t)\Phi\big(x'(t)\big) \geqslant -\frac{K'(t)}{\tilde{m}} a\big(x(t)\big)\Phi\big(x'(t)\big),$$

where $\tilde{m} := \min\{a(\xi) \mid \xi \in [x(\bar{t}), x(T)]\}$. Then, integrating in $[t, T]$, with $t < T$, and taking into account that $x'(T) = 0$, we obtain

$$a\bigl(x(t)\bigr)\Phi\bigl(x'(t)\bigr) \leqslant \int_t^T \frac{K'(\tau)}{\tilde{m}} a\bigl(x(\tau)\bigr)\Phi\bigl(x'(\tau)\bigr)\,\mathrm{d}\tau \quad \text{for every } t \in (\bar{t}, T].$$

By the Gronwall's inequality we get $a(x(t))\Phi(x'(t)) \leqslant 0$, i.e. $x'(t) \leqslant 0$ in the same interval, in contradiction with the definition of T. Hence, $T = +\infty$.

Observe now that by (19) we get

$$\frac{1}{1-\gamma}\bigl[a\bigl(x(t)\bigr)\Phi\bigl(x'(t)\bigr)^{1-\gamma} - a\bigl(x(t)\bigr)\Phi\bigl(x'(t)\bigr)^{1-\gamma}\bigr] = \int_{\bar{t}}^{t} \frac{(a(x(s))\Phi(x'(s)))'}{(a(x(s))\Phi(x'(s)))^{\gamma}}\,\mathrm{d}s \geqslant \frac{1}{\tilde{m}}\bigl(K(\bar{t}) - K(t)\bigr)$$

therefore, putting $\tilde{M} := \max\{a(\xi) \mid \xi \in [x(\bar{t}), x(T)]\}$, for a.e. $t \geqslant \bar{t}$ we have

$$\tilde{M}^{1-\gamma} x'(t)^{1-\gamma} \leqslant a\bigl(x(t)\bigr)\Phi\bigl(x'(t)\bigr)^{1-\gamma} \leqslant a\bigl(x(\bar{t})\bigr)\Phi\bigl(x'(\bar{t})\bigr)^{1-\gamma} + \frac{(\gamma-1)}{\tilde{m}^{\gamma}}\bigl(K(t) - K(\bar{t})\bigr)$$

then

$$x'(t) \geqslant \frac{1}{\tilde{M}}\left(a\bigl(x(\bar{t})\bigr)\Phi\bigl(x'(\bar{t})\bigr)^{1-\gamma} + \frac{(\gamma-1)}{\tilde{m}^{\gamma}}\bigl(K(t) - K(\bar{t})\bigr)\right)^{\frac{1}{1-\gamma}}.$$

By virtue of (15) we deduce that $x(+\infty) - x(\bar{t}) = \int_{\bar{t}}^{+\infty} x'(\tau)\,\mathrm{d}\tau = +\infty$, in contradiction with the boundedness of x.

Therefore, $x'(t) \equiv 0$ in $[\tilde{t}, +\infty)$ and by the definition of $\tilde{t}$ this implies $\tilde{t} = t^*$. So, $x'(t) \equiv 0$ in $[t^*, +\infty)$ and by the definition of t^* this implies $t^* = L$. □

Remark 3.3. If the sign condition in (18) is satisfied with the reversed inequality, i.e., if

$$tf(t, x, y) \geqslant 0 \quad \text{for a.e. } |t| \geqslant L,\ \text{every } x \in \mathbb{R},\ |y| < \rho \tag{20}$$

then it is possible to prove that $\lim_{x\to\pm\infty} x'(t) = 0$ and $x'(t) \leqslant 0$ for $|t| \geqslant L$. So, since $\alpha < \beta$, when $L = 0$ problem (P) does not admit solutions.

4. Criteria for right-hand side of the type $f(t, x, y) = b(t, x)c(x, y)$

In this section, following [7], we will give some operative criteria which can be applied when the right-hand side has the following product structure

$$f(t, x, y) = b(t, x)c(x, y).$$

We will show the link between the local behaviors of $c(x, \cdot)$ at $y = 0$ and of $b(\cdot, x)$ at infinity which plays a key role for the existence or non-existence of solutions.

We will assume that b is a Carathéodory function and c is a continuous function satisfying

$$c(x, y) > 0 \quad \text{for every } y \neq 0 \text{ and } x \in [\alpha, \beta]; \qquad c(\alpha, 0) = c(\beta, 0) = 0,$$

for suitable constants $\alpha < \beta$. Clearly this implies that assumption (H) on p. 671 is satisfied.

As in the previous section, we denote

$$m := \min_{x\in[\alpha,\beta]} a(x) > 0, \qquad M := \max_{x\in[\alpha,\beta]} a(x).$$

Our first result, which is a consequence of Theorem 3.1, provides sufficient conditions for the existence of solutions to problem (P) when f has the above product structure.

Theorem 4.1. *Suppose that*

$$t \cdot b(t, x) \leqslant 0 \quad \text{for a.e. } t \text{ such that } |t| \geqslant \bar{t},\ \text{every } x \in [\alpha, \beta] \tag{21}$$

for some $\bar{t} \geqslant 0$, and there exists a function $\lambda \in L^q_{loc}(\mathbb{R})$, $1 \leqslant q \leqslant +\infty$, such that

$$\bigl|b(t, x)\bigr| \leqslant \lambda(t) \quad \text{for a.e. } t \in \mathbb{R},\ \text{every } x \in [\alpha, \beta]. \tag{22}$$

Moreover, assume that there exist real constants $-1 < \delta_1 \leqslant \delta_2$, $0 < \gamma_2 \leqslant \gamma_1$, satisfying

$$\gamma_1 > 1, \qquad \gamma_2(\delta_1 + 1) > (\gamma_1 - 1)(\delta_2 + 1), \tag{23}$$

such that for every $x \in [\alpha,\beta]$ we have

$$h_1|t|^{\delta_1} \leqslant \big|b(t,x)\big| \leqslant h_2|t|^{\delta_2}, \quad \text{a.e. } |t|>L, \tag{24}$$

$$q_1\Phi\big(|y|\big)^{\gamma_1} \leqslant c(x,y) \leqslant q_2\Phi\big(|y|\big)^{\gamma_2}, \quad \text{whenever } |y|<\rho, \tag{25}$$

for certain positive constants h_1, h_2, q_1, q_2 and for $L \geqslant \bar{t}$ and $\rho < r$.

Finally, assume that (7) *holds for some positive constant $\mu < \frac{\delta_1+1}{\gamma_1-1}$.*

Then, problem (P) *admits solutions.*

Proof. Let $\Theta := \Phi^{-1}(\frac{M}{m}\Phi(k_1))$ and $k_1 := k(2L) < r$ as in the statement of Theorem 3.1. Moreover, let

$$\sigma := \max\{\rho, \Theta\}$$

and

$$m_0 := \min\big\{c(x,y)\colon\ x\in[\alpha,\beta],\ \rho \leqslant |y| \leqslant \sigma\big\}.$$

Observe that $m_0 > 0$ since $c(x,y) > 0$ for $y \neq 0$. Finally, let

$$\psi_0 := \min\left\{\frac{m_0}{\Phi(\sigma)^{\gamma_1}}, q_1\right\}$$

and

$$K(t) := \begin{cases} \psi_0 \int_L^t \min\{\min_{x\in[\alpha,\beta]} b(-\tau,x), -\max_{x\in[\alpha,\beta]} b(\tau,x)\}\,d\tau & \text{for } t \geqslant L,\\ 0 & \text{for } 0 \leqslant t \leqslant L.\end{cases}$$

Assumption (22) implies that $K \in W^{1,1}_{loc}([0,+\infty))$ and by (24) we get $K(t) \to +\infty$ as $t \to +\infty$. Finally from (21) and (24) it follows that the function K is strictly increasing on $[L,+\infty)$.

Observe that from the definition of ψ_0 and assumption (25), it follows that

$$c(x,y) \geqslant \psi_0\Phi\big(|y|\big)^{\gamma_1} \quad \text{for every } x\in[\alpha,\beta],\ y\in[-\sigma,\sigma].$$

Therefore, by (21) we get

$$f(t,x,y) = b(t,x)c(x,y) \leqslant \psi_0 b(t,x)\Phi\big(|y|\big)^{\gamma_1} \leqslant -K'(t)\Phi\big(|y|\big)^{\gamma_1}$$

and

$$f(-t,x,y) = b(-t,x)c(x,y) \geqslant \psi_0 b(-t,x)\Phi\big(|y|\big)^{\gamma_1} \geqslant K'(t)\Phi\big(|y|\big)^{\gamma_1}$$

for a.e. $t \geqslant L$, every $x\in[\alpha,\beta]$ and every $y\in[-\sigma,\sigma]$. Thus, assumption (9) of Theorem 3.1 holds with $\gamma=\gamma_1$.

Now, assumption (24) implies that $h_1\psi_0 t^{\delta_1} \leqslant K'(t)$ for a.e. $t \geqslant L$. Consequently,

$$K(t) \geqslant \frac{h_1\psi_0}{\delta_1+1}\big(t^{\delta_1+1} - L^{\delta_1+1}\big). \tag{26}$$

Hence, by the upper bound on the constant μ we obtain that condition (8) is satisfied with $\gamma=\gamma_1$.

Moreover, since $\lim_{|t|\to+\infty} N(t) = 0$, there is a constant $L^* > L$ such that $N(t) \leqslant \rho$ for every $|t| \geqslant L^*$. Let

$$\eta(t) := \begin{cases} \max_{x\in[\alpha,\beta]} |b(t,x)| \cdot \max_{(x,y)\in[\alpha,\beta]\times[-\sigma,\sigma]} c(x,y) & \text{if } |t| \leqslant L^*,\\ h_2 q_2 |t|^{\delta_2}\Phi(N(t))^{\gamma_2} & \text{if } |t| > L^*.\end{cases}$$

By assumptions (24) and (25), for every y such that $|y| \leqslant N(t)$ for a.e. $t\in\mathbb{R}$, and every $x\in[\alpha,\beta]$, we have

$$\big|f(t,x,y)\big| = \big|b(t,x)\big|c(x,y) \leqslant \eta(t).$$

Let us show that $\eta \in L^1(\mathbb{R})$. Indeed, by (22) and the continuity of c we get $\eta \in L^1([-L^*,L^*])$. Moreover, when $|t| > L^*$, by (26) we have

$$0 < \eta(t) \leqslant h_2q_2|t|^{\delta_2}\left\{\frac{M}{m}\left(\Phi(k_1)^{-(\gamma_1-1)} + \frac{\gamma_1-1}{M}K\big(|t|\big)\right)^{-\frac{1}{\gamma_1-1}}\right\}^{\gamma_2}$$

$$\leqslant h_2q_2|t|^{\delta_2}\left\{\frac{M}{m}\left(\Phi(k_1)^{-(\gamma_1-1)} + \frac{\gamma_1-1}{M}\frac{h_1\psi_0}{\delta_1+1}\big(|t|^{\delta_1+1} - L^{\delta_1+1}\big)\right)^{-\frac{1}{\gamma_1-1}}\right\}^{\gamma_2}.$$

The last term belongs to $L^1(\mathbb{R}\setminus[-L^*,L^*])$. In fact, by the bound on μ, we have $\frac{\gamma_2(\delta_1+1)}{\gamma_1-1} - \delta_2 > 1$. This shows that $\eta \in L^1(\mathbb{R})$.

Hence, Theorem 3.1 applies yielding the existence of a solution to problem (P). □

We state now a non-existence result, which follows from Theorem 3.2.

Theorem 4.2. *Suppose that*

$$t \cdot b(t,x) \leqslant 0 \quad \text{for a.e. } t \in \mathbb{R} \text{ and every } x \in [\alpha,\beta]. \tag{27}$$

Assume that there are real constants $\delta > -1$, $\gamma > 1$, $\Lambda \geqslant 0$, and a positive function $\ell \in L^1([0,\Lambda])$ such that

$$\big|b(t,x)\big| \leqslant \lambda_1 |t|^\delta, \quad \text{for every } x \in [\alpha,\beta],\ \text{a.e. } |t| > \Lambda, \tag{28}$$

$$\big|b(t,x)\big| \leqslant \ell\big(|t|\big) \quad \text{for a.e. } |t| \leqslant \Lambda, x \in [\alpha,\beta], \tag{29}$$

$$c(x,y) \leqslant \lambda_2 \Phi\big(|y|\big)^\gamma, \quad \text{for every } x \in [\alpha,\beta],\ |y| < \rho \tag{30}$$

for some positive constants λ_1, λ_2 and for some positive $\rho < r$. Finally, assume that condition (14) *holds for some positive constant μ, with $\mu \geqslant \frac{\delta+1}{\gamma-1}$.*

Then, problem (P) *does not admit solutions.*

Proof. Let

$$g(t) := \begin{cases} \ell(t) & \text{for } 0 \leqslant t \leqslant \Lambda, \\ \lambda_1 t^\delta & \text{for } t < \Lambda. \end{cases}$$

Moreover, let $K(t) := \lambda_2 \int_0^t g(\tau)\,d\tau$ for $t \geqslant 0$. Observe that $K(t)$ is a strictly increasing function and belongs to $W^{1,1}_{loc}([0,+\infty))$. Further, for every $t \geqslant \Lambda$ we have

$$K(t) = \lambda_2 \int_0^\Lambda \ell(\tau)\,d\tau + \lambda_1\lambda_2 \int_\Lambda^t \tau^\delta\,d\tau = \lambda_2 \int_0^\Lambda \ell(\tau)\,d\tau + \frac{\lambda_1\lambda_2}{\delta+1}\big(t^{\delta+1} - \Lambda^{\delta+1}\big).$$

Hence, by the lower bound on the constant μ we get (15). Moreover, by (28), (29) and (30) we obtain that both (16) and (17) of Theorem 3.2 hold with $L = 0$. Thus, Theorem 3.2 applies, and this completes the proof. □

A more effective result can be stated if the right-hand side has the product structure

$$f(t,x,y) = h(t)g(x)c(y).$$

In fact, in this case the link between the local behaviors of $c(\cdot)$ at $y = 0$ and of $h(\cdot)$ at infinity can be expressed in a quite simple way. The next result expresses the existence and non-existence of solutions in this case.

Corollary 4.3. *Let $f(t,x,y) = h(t)g(x)c(y)$, where $h \in L^q_{loc}(\mathbb{R})$, for some $1 \leqslant q \leqslant +\infty$, c is continuous in $\mathbb{R}$ and g is continuous and positive in $[\alpha,\beta]$.*

Assume that $c(y) > 0$ for $y \neq 0$; $t \cdot h(t) \leqslant 0$ for every t and suppose that

$$\lim_{|t|\to+\infty} \big|h(t)\big||t|^{-\delta} \in (0,+\infty) \quad \text{for some } \delta > -1, \tag{31}$$

$$\lim_{y\to 0} c(y)|y|^{-\nu} \in (0,+\infty) \quad \text{for some } \nu > 0. \tag{32}$$

Then, the following assertions hold:

- *if condition* (14) *holds with a positive constant $\mu \leqslant \nu - \delta - 1$, then* (P) *has no solution*;
- *if we further assume that condition* (7) *holds for some positive constant μ, with $\nu - \delta - 1 < \mu < \nu$, then* (P) *admits solutions.*

Proof. Put $b(t,x) := h(t)g(x)$. Let us prove the first assertion using Theorem 4.2. Suppose that condition (14) holds true and that $\mu \leqslant \nu - \delta - 1$. Then, $\gamma := \frac{\nu}{\mu} > 1$ and it is not difficult to show that assumptions (31) and (32) imply that conditions (27)–(30) hold for suitable positive constants λ_1, λ_2, Λ, ρ. Thus, Theorem 4.2 applies and consequently problem (P) does not admit solutions.

Assume now that (7) holds with $\nu - \delta - 1 < \mu < \nu$, and let us prove that in this case all the assumptions of Theorem 4.1 are satisfied. Observe first that conditions (21) and (22) are satisfied. Moreover, assumption (31) implies that (24) holds with $\delta_1 = \delta_2 = \delta$ and a suitable choice of the positive constants h_1, h_2, L. Finally, let $\gamma_2 := \frac{\nu}{\mu}$. It is possible to choose $\gamma_1 \in (\gamma_2, \gamma_2 + 1)$ in such a way that conditions (23) and (25) hold. Hence Theorem 4.1 applies yielding the existence of solutions to (P). □

Remark 4.4. When the function Φ behaves as a power as $y \to 0$, that is there exist $\mu, c_1, c_2, y_0 > 0$ such that

$$c_1|y|^\mu \leqslant \left|\Phi(y)\right| \leqslant c_2|y|^\mu \quad \text{for every } y \text{ with } |y| \in (0, y_0) \tag{33}$$

then the two assertions of Corollary 4.3 can be joined into a necessary and sufficient condition whenever $\nu > \mu$. In fact, in this case problem (P) admits solutions *if and only if* $\nu - \delta - 1 < \mu$.

Remark 4.5. Notice that in the previous criteria the functions a and g do not play any role for the existence or non-existence of solutions to (P).

We close the paper with the following illustrating examples.

Example 4.6. Consider the increasing homeomorphism $\Phi : (-r, r) \to \mathbb{R}$ given by

$$\Phi(y) := \frac{\operatorname{sign}(y)|y|^\mu}{(r^2 - y^2)^\kappa}$$

with $\mu, \kappa > 0$, and let

$$f(t, x, y) := -\frac{|t|^\lambda}{t} g(x)|y|^\nu$$

with $\lambda, \nu > 0$ and g a generic continuous positive function. Let us discuss the existence of solutions to (P) using Corollary 4.3 with $h(t) = -\frac{|t|^\lambda}{t}$ and $c(y) = |y|^\nu$. First notice that assumption (32) holds, and (31) holds with $\delta = \lambda - 1$. Therefore, if $\nu \geqslant \mu + \lambda$, then (P) has no solutions.

As for the existence, if $\mu < \nu < \mu + \lambda$, as a consequence of Remark 4.4 we have that problem (P) admits solutions.

Example 4.7. Consider the map $\Phi : (-1, 1) \to \mathbb{R}$ given by

$$\Phi(y) := \frac{y}{\sqrt{1 - y^2}}$$

and let

$$f(t, x, y) := -tg(x)|y|^\nu$$

with $\nu > 1$ and g continuous and positive. In this particular case our equation becomes

$$\left(\frac{a(x(t))x'(t)}{\sqrt{1 - x'(t)^2}}\right)' = -tg\big(x(t)\big)\left|x'(t)\right|^\nu$$

with a continuous and positive.

By Remark 4.4, problem (P) admits solutions if and only if $1 < \nu < 2$.

Example 4.8. Let $\Phi : (-1, 1) \to \mathbb{R}$ be given by

$$\Phi(y) := \frac{y^3 - y^2 + y}{\sqrt{1 - y^2}},$$

so that Φ is a non-symmetric increasing homeomorphism, and let $f(t, x, y)$ be the same as in the previous example. Again by Remark 4.4, problem (P) admits solutions if $1 < \nu < 2$ while it has no solution for $\nu \geqslant 2$.

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