JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **226,** 82–100 (1998)
ARTICLE NO. AY986077

Robust Stability of C_0-Semigroups and an Application to Stability of Delay Equations

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Submitted by G. F. Webb

Received October 14, 1997

Let A be a closed linear operator on a complex Banach space X and let $\lambda \in \varrho(A)$ be a fixed element of the resolvent set of A. Let U and Y be Banach spaces, and let $D \in \mathscr{L}(U, X)$ and $E \in \mathscr{L}(X, Y)$ be bounded linear operators. We define $r_\lambda(A; D, E)$ by

$$\sup\{r \geq 0: \lambda \in \varrho(A + D\Delta E) \quad \text{for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \leq r\}$$

and prove that

$$r_\lambda(A; D, E) = \frac{1}{\|ER(\lambda, A)D\|}.$$

We give two applications of this result. The first is an exact formula for the so-called stability radius of the generator of a C_0-semigroup of linear operators on a Hilbert space; it is derived from a precise result about robustness under perturbations of uniform boundedness in the right half-plane of the resolvent of an arbitrary semigroup generator. The second application gives sufficient conditions on the norm of the operators $B_j \in \mathscr{L}(X)$ such that the classical solutions of the delay equation

$$\dot{u}(t) = Au(t) + \sum_{j=1}^{n} B_j u(t - h_j), \qquad t \geq 0,$$

are exponentially stable in $L^p([-h, 0]; X)$.

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† This research has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

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0022-247X/98 $25.00

Key Words: perturbation; stability radius, robust stability; infinite-dimensional linear systems; stability of delay equations; functional differential equations

0. INTRODUCTION

In this paper we investigate the robustness of certain properties of a closed linear operator A on a Banach space X under small additive perturbations. Some "structure" in the perturbation will be allowed, in the following sense: we fix Banach spaces U and Y and two operators $D \in \mathscr{L}(U, X)$ and $E \in \mathscr{L}(X, Y)$ (or even $E \in \mathscr{L}(\mathscr{D}(A), Y)$), and consider perturbations of the form $D\Delta E$, with $\Delta \in \mathscr{L}(Y, U)$. The question we address is the following.

> If A has a certain property (P), what is the supremum of all $r \geq 0$ with the following property: for all bounded linear operators $\Delta \in \mathscr{L}(Y, U)$ with norm $\|\Delta\| \leq r$, the perturbed operator $A + D\Delta E$ has property (P) as well.

Among the properties we consider are the following: containment of a given complex number $\lambda \in \mathbb{C}$ in the resolvent set of the operator, containment of a given set $\Omega \subset \mathbb{C}$ in the resolvent set, and uniform boundedness of the resolvent on Ω. For these properties we give a precise answer to the above question in terms of the so-called *transfer function* $\lambda \mapsto ER(\lambda, A)D$, where $R(\lambda, A) := (\lambda - A)^{-1}$ is the resolvent of A.

In two subsequent sections, we give two applications of the abstract results of Section 1. In Section 2 we prove some new results on robust stability. Among others, we obtain an exact formula for the stability radius for generators of Hilbert space semigroups. In Section 3 we study the delay equation

$$\dot{u}(t) = Au(t) + \sum_{j=1}^{n} B_j u(t - h_j), \qquad t \geq 0,$$

where A is the generator of a C_0-semigroup on a Banach space X. Regarding the bounded operators B_j as a perturbation of an appropriate Cauchy problem corresponding to the absence of delays, we obtain sufficient conditions on A and B_j for exponential stability of classical solutions.

1. THE ABSTRACT PERTURBATION RESULTS

Throughout this section, X, U, and Y are fixed complex Banach spaces, A is a closed linear operator on X with domain $\mathscr{D}(A)$, and $D \in \mathscr{L}(U, X)$ and $E \in \mathscr{L}(\mathscr{D}(A), Y)$ are bounded linear operators; we regard $\mathscr{D}(A)$ as a Banach space with respect to the graph norm $\|\cdot\|_{\mathscr{D}(A)}$.

PROPOSITION 1. *Let A be a closed linear operator on X and suppose $\lambda \in \varrho(A)$. If $\Delta \in \mathscr{L}(Y, U)$ satisfies*

$$\|\Delta\| \leq (1-\delta)\frac{1}{\|ER(\lambda, A)D\|} \tag{1.1}$$

for some $\delta \in (0,1)$, then $\lambda \in \varrho(A + D\Delta E)$, and

$$\|R(\lambda, A + D\Delta E)\| \leq \|R(\lambda, A)\|\left(1 + \frac{1}{\delta}\|D\|\,\|\Delta E R(\lambda, A)\|\right).$$

Proof. Fix $\lambda \in \varrho(A)$. From $\|\Delta E R(\lambda, A)D\| \leq 1 - \delta$ we see that $I - \Delta E R(\lambda, A)D$ is invertible. Using the Neumann series we estimate

$$\left\|(I - \Delta E R(\lambda, A)D)^{-1}\right\| \leq \sum_{n=0}^{\infty}(1-\delta)^n = \frac{1}{\delta}.$$

It follows that $I - D\Delta E R(\lambda, A)$ is invertible as well, and its inverse is given by

$$(I - D\Delta E R(\lambda, A))^{-1} = I + D(I - \Delta E R(\lambda, A)D)^{-1}\Delta E R(\lambda, A).$$

By the above estimate,

$$\left\|(I - D\Delta E R(\lambda, A))^{-1}\right\| \leq 1 + \frac{1}{\delta}\|D\|\,\|\Delta E R(\lambda, A)\|.$$

From the identity $\lambda - A - D\Delta E = (I - D\Delta E R(\lambda, A))(\lambda - A)$ we see that $\lambda - A - D\Delta E$ is closed, being the composition of a closed operator and a bounded invertible operator. It also shows that $\lambda - A - D\Delta E$ maps $\mathscr{D}(A)$ injectively onto X. Hence, the inverse mapping $(\lambda - A - D\Delta E)^{-1}$ is well defined on X, and being the inverse of a closed operator, it is closed. Hence by the closed graph theorem, $(\lambda - A - D\Delta E)^{-1}$ is bounded, which means that $\lambda \in \varrho(A + D\Delta E)$. By the previous estimate, we obtain

$$\begin{aligned}\|R(\lambda, A + D\Delta E)\| &= \left\|R(\lambda, A)(I - D\Delta E R(\lambda, A))^{-1}\right\| \\ &\leq \|R(\lambda, A)\|\left(1 + \frac{1}{\delta}\|D\|\,\|\Delta E R(\lambda, A)\|\right).\end{aligned}$$

∎

This result shows that the property "$\lambda \in \varrho(A)$" is stable under small perturbations. Next we show that the bound (1.1) is actually the best possible. To this end, for $\lambda \in \varrho(A)$ we introduce the quantity

$$r_\lambda(A; D, E) := \sup\{r \geq 0: \quad \lambda \in \varrho(A + D\Delta E) \text{ for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \leq r\}.$$

THEOREM 1.2. *Let A be a closed linear operator on X. Then for all $\lambda \in \varrho(A)$ we have*

$$r_\lambda(A; D, E) = \frac{1}{\|ER(\lambda, A)D\|}.$$

Proof. If $0 \leq r < \|ER(\lambda, A)D\|^{-1}$ and $\|\Delta\| \leq r$, then $\lambda \in \varrho(A + D\Delta E)$ by Proposition 1.1. Hence, $r_\lambda(A; D, E) \geq \|ER(\lambda, A)D\|^{-1}$. To prove the converse inequality, let us fix $\varepsilon > 0$. Choose $u \in U$, $\|u\| = 1$, such that

$$\frac{1}{\|ER(\lambda, A)Du\|} \leq \frac{1}{\|ER(\lambda, A)D\|} + \varepsilon.$$

By the Hahn–Banach theorem we may choose $y^* \in Y^*$, $\|y^*\| = 1$, such that

$$\left\langle \frac{ER(\lambda, A)Du}{\|ER(\lambda, A)Du\|}, y^* \right\rangle = 1.$$

Define $\Delta \in \mathscr{L}(Y, U)$ by

$$\Delta y := \frac{\langle y, y^* \rangle u}{\|ER(\lambda, A)Du\|}, \qquad y \in Y.$$

Then $\Delta E R(\lambda, A)Du = u$ and

$$\|\Delta\| \leq \frac{1}{\|ER(\lambda, A)D\|} + \varepsilon.$$

Set $v := R(\lambda, A)Du$. Then $\Delta Ev = u \neq 0$, so $v \neq 0$, and

$$(\lambda - A - D\Delta E)v = Du - D\Delta E R(\lambda, A)Du = Du - Du = 0.$$

This shows that $\lambda - A - D\Delta E$ is not injective, which implies $\lambda \in \sigma(A + D\Delta E)$. ∎

We remark that the proofs of Proposition 1.1 and Theorem 1.2 are based entirely on techniques in a paper of Latushkin, Montgomery-Smith, and Randolph [13], where they are used to obtain the two-sided bounds (2.4) below for robust stability.

For a subset $\Omega \subset \varrho(A)$ we define

$$r_{\Omega}(A; D, E) := \sup\{r \geq 0: \quad \Omega \subset \varrho(A + D\Delta E) \text{ for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \leq r\}.$$

We then have the following straightforward generalization of Theorem 1.2:

COROLLARY 1.3. *Let A be a closed linear operator on X. If $\Omega \subset \varrho(A)$, then*

$$r_{\Omega}(A; D, E) = \inf_{\lambda \in \Omega} \frac{1}{\|ER(\lambda, A)D\|}.$$

We may also impose uniform boundedness of the resolvent on the set Ω by defining, for a subset $\Omega \subset \varrho(A)$ such that $\sup_{\lambda \in \Omega}\|R(\lambda, A)\| < \infty$,

$$r_{\Omega}^{\infty}(A; D, E) := \sup\Big\{r \geq 0: \quad \Omega \subset \varrho(A + D\Delta E) \text{ and } \sup_{\lambda \in \Omega} \|R(\lambda, A + D\Delta E)\| < \infty \text{ for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \leq r\Big\}.$$

COROLLARY 1.4. *Let A be a closed linear operator on X and assume that E extends to a bounded operator from X into Y. If $\Omega \subset \varrho(A)$ with $\sup_{\lambda \in \Omega}\|R(\lambda, A)\| < \infty$, then*

$$r_{\Omega}^{\infty}(A; D, E) = \frac{1}{\sup_{\lambda \in \Omega}\|ER(\lambda, A)D\|}.$$

Proof. It is clear from the definition that $r_{\Omega}^{\infty}(A; D, E) \leq r_{\Omega}(A; D, E)$. Hence by Corollary 1.3 we only need to prove the inequality $r_{\Omega}^{\infty}(A; D, E) \geq \inf_{\lambda \in \Omega}\|ER(\lambda, A)D\|^{-1}$. But this inequality follows immediately from Proposition 1.1, since $\|\Delta E\, R(\lambda, A)\| \leq \|\Delta E\|\, \|R(\lambda, A)\|$ and $\sup_{\lambda \in \Omega}\|R(\lambda, A)\| < \infty$. ∎

2. APPLICATION TO ROBUST STABILITY OF C_0-SEMIGROUPS

Throughout this section we fix complex Banach spaces X, U, and Y, and bounded linear operators $D \in \mathscr{L}(U, X)$ and $E \in \mathscr{L}(X, Y)$. We further consider a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ of bounded linear operators on X, and denote by A its generator. Our terminology concerning semigroups is standard; for more information we refer to [16] and [18].

In this section and the next we will be concerned with the behavior under small perturbations of the following four quantities (see [17, Chap. 1]):

- The *spectral bound* $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$
- The *abscissa of uniform boundedness* $s_0(A)$ of the resolvent of A,

$$s_0(A) := \inf\Big\{\omega \in \mathbb{R}: \{\operatorname{Re}\lambda > \omega\} \subset \varrho(A) \text{ and } \sup_{\operatorname{Re}\lambda > \omega} \|R(\lambda, A)\| < \infty\Big\}$$

- The *growth bound* $\omega_1(A)$,

$$\omega_1(A) := \inf\Big\{\omega \in \mathbb{R}: \text{there exists } M > 0 \text{ such that } \|T(t)x\| \le Me^{\omega t}\|x\|_{\mathscr{D}(A)} \text{ for all } x \in \mathscr{D}(A) \text{ and } t \ge 0\Big\}$$

- The *uniform growth bound* $\omega_0(A)$,

$$\omega_0(A) := \inf\Big\{\omega \in \mathbb{R}: \text{there exists } M > 0 \text{ such that } \|T(t)\| \le Me^{\omega t} \text{ for all } t \ge 0\Big\}.$$

It is well known [17, Sect. 1.2, 4.1] that

$$-\infty \le s(A) \le \omega_1(A) \le s_0(A) \le \omega_0(A) < \infty. \tag{2.1}$$

If $\omega_0(A) < 0$ (resp. $\omega_1(A) < 0$), then $\mathbf{T}$ is said to be *uniformly exponentially stable* (resp. *exponentially stable*). Below we will use the following simple fact concerning $s_0(A)$: if $\{\operatorname{Re}\lambda > 0\} \subset \varrho(A)$ and $\sup_{\operatorname{Re}\lambda > 0}\|R(\lambda, A)\| < \infty$, then $s_0(A) < 0$; see [17, Lemma 2.3.4].

We start by studying the behavior of the abscissa of uniform boundedness under small additive perturbations. To this end, for a semigroup with $s_0(A) < 0$ we define

$$r_{s_0}(A; D, E) := \sup\Big\{r \ge 0: \quad s_0(A + D\Delta E) < 0 \text{ for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \le r\Big\}.$$

Recalling that the suprema along vertical lines $\operatorname{Re}\lambda = c$ of a bounded holomorphic X-valued function on $\{\operatorname{Re}\lambda > 0\}$ decrease as c increases, an application of Corollary 1.4 to $\Omega = \{\operatorname{Re}\lambda > 0\}$ shows the following:

THEOREM 2.1. *Suppose A is the generator of a C_0-semigroup on X. If $s_0(A) < 0$, then*

$$r_{s_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}}\|ER(i\omega, A)D\|}.$$

For a uniformly exponentially stable C_0-semigroup we now define

$$r_{\omega_0}(A; D, E) := \sup\Big\{r \ge 0: \quad \omega_0(A + D\Delta E) < 0 \text{ for all } \Delta \in \mathscr{L}(Y, U) \text{ with } \|\Delta\| \le r\Big\}.$$

It is a well-known theorem of Gearhart [4] (cf. [17], Corollary 2.2.5]) that for C_0-semigroups on a Hilbert space, the abscissa of uniform boundedness of the resolvent and the uniform growth bound always coincide. Hence if X is isomorphic to a Hilbert space, Theorem 2.1 assumes the following form:

COROLLARY 2.2. *Suppose A is the generator of a C_0-semigroup on X. If X is isomorphic to a Hilbert space, and if $\omega_0(A) < 0$, then*

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \| ER(i\omega, A)D\|}.$$

Remark. It is not assumed that U and Y are isomorphic to Hilbert spaces.

The quantity $r_{\omega_0}(A; D, E)$ is called the *stability radius* of A with respect to the "perturbation structure" (D, E) and was introduced, in the finite-dimensional setting, by Hinrichsen and Pritchard [5]; see also their survey paper [6]. To state some known results about the stability radius, for $p \in [1, \infty)$ we define the *input–output operator* $\mathbb{L}_p(A; D, E) \in \mathscr{L}(L^p(\mathbb{R}_+; U), L^p(\mathbb{R}_+; Y))$ by

$$\mathbb{L}_p(A; D, E) f(s) := E \int_0^s T(s-t) D f(t)\, dt \qquad s \geq 0, \quad f \in L^p(\mathbb{R}_+; U).$$

This operator is easily seen to be bounded if $\omega_0(A) < 0$; conversely, if $U = Y = X$, then boundedness of $\mathbb{L}_p(A; I, I)$ implies $\omega_0(A) < 0$ [17, Theorem 3.3.1]. The following results are well known:

- If X, U, and Y are finite-dimensional, then

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \| ER(i\omega, A)D\|}. \tag{2.2}$$

- If X is a Banach space, and U and Y are Hilbert spaces, then

$$\frac{1}{\|\mathbb{L}_2(A; D, E)\|} = r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega \in \mathbb{R}} \| ER(i\omega, A)D\|}. \tag{2.3}$$

- If X, U, and Y are arbitrary Banach spaces, then for all $p \in [1, \infty)$,

$$\frac{1}{\|\mathbb{L}_p(A; D, E)\|} \leq r_{\omega_0}(A; D, E) \leq \frac{1}{\sup_{\omega \in \mathbb{R}} \| ER(i\omega, A)D\|}. \tag{2.4}$$

The identities (2.2) and (2.3) are due to Hinrichsen and Pritchard [6] and Pritchard and Townley [19] (where a more general setup is considered), respectively. Notice that in some sense our Corollary 2.2 complements the second identity in (2.3).

The inequalities (2.4) were obtained by Latushkin, Montgomery-Smith, and Randolph [13] by using the theory of evolutionary semigroups; this further enabled them to extend certain results on time-varying systems due to Hinrichsen and Pritchard [7]. They also showed that the inequality between the first and third terms in (2.4) may be strict. More results on the time-varying case may be found in [2].

In the case of positive semigroups, Theorem 2.1 and Corollary 2.2 simplify somewhat:

COROLLARY 2.3. *If X, U, and Y are Banach lattices, $D \in \mathscr{L}(U, X)$ and $E \in \mathscr{L}(X, Y)$ are positive, and A is the generator of a positive C_0-semigroup on X with $s_0(A) < 0$, then*

$$r_{s_0}(A; D, E) = \frac{1}{\|EA^{-1}D\|}.$$

If, in addition, X is isomorphic to a Hilbert space, then the same result holds for the uniform growth bound.

Proof. From

$$|ER(i\omega, A)Du| \le E|R(i\omega, A)|D|u| \le ER(0, A)D|u|$$

[16, Corollary C-III-1.3] it follows that $\|ER(i\omega, A)D\| \le \|ER(0, A)D\| = \|EA^{-1}D\|$ for all $\omega \in \mathbb{R}$. Accordingly, the supremum in the expressions in Theorem 2.1 and Corollary 2.2 is taken for $\omega = 0$. ∎

For a detailed treatment of the theory of positive semigroups we refer to [16].

The next application is concerned with semigroups that are uniformly continuous for $t > 0$. First we recall that if A is the generator of a C_0-semigroup that is uniformly continuous for $t > t_0$ for some $t_0 \ge 0$, then the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = \exp(t\sigma(A)) \setminus \{0\}$$

holds for all $t \ge 0$ [16, Theorem A-III-6.6], [17, Theorem 2.3.2]. In particular, this implies that $s(A) = s_0(A) = \omega_0(A)$. We will combine Theorem 2.1 with the following simple observation [16, Theorem A-II-1.30], the proof of which is included for the reader's convenience.

LEMMA 2.4. *If A is the generator of a C_0-semigroup $\mathbf{T}$ on X that is uniformly continuous for $t > 0$, and if B is a bounded linear operator on X, then the semigroup generated by $A + B$ is uniformly continuous for $t > 0$.*

Proof. Let $\mathbf{S} = \{S(t)\}_{t \geq 0}$ denote the semigroup generated by $A + B$. Put $V_0(t) := T(t)$, $t \geq 0$, and define the operators $V_n(t)$ inductively by

$$V_{n+1}(t)x := \int_0^t T(t-s)BV_n(s)x\,ds, \qquad n \in \mathbb{N}, \quad x \in X, \quad t \geq 0.$$

As is well known [18, Section 3.1], if $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then

$$\|V_n(t)\| \leq Me^{\omega t}\frac{M^n\|B\|^n t^n}{n!}, \qquad n \in \mathbb{N}, \quad t \geq 0,$$

and

$$S(t) = \sum_{n=0}^{\infty} V_n(t), \qquad t \geq 0,$$

the convergence being uniform on compact subsets of $[0, \infty)$. Fix $n \geq 0$ and positive real numbers $0 < \varepsilon < \delta_0 < \delta_1 < \infty$. For $\delta_0 \leq t \leq t' \leq \delta_1$, from

$$V_{n+1}(t')x - V_{n+1}(t)x = \int_t^{t'} T(t'-s)BV_n(s)x\,ds$$
$$+ \int_0^t (T(t'-s) - T(t-s))BV_n(s)x\,ds$$

we obtain, by splitting the second integral as $\int_0^t = \int_{t-\varepsilon}^t + \int_0^{t-\varepsilon}$,

$$\|V_{n+1}(t') - V_{n+1}(t)\|$$
$$\leq C_n\Big((t'-t) + \varepsilon + \sup_{s\in[0,t-\varepsilon]}\|T(t'-s) - T(t-s)\|\Big),$$

where C_n is a finite constant depending on M, ω, $\|B\|$, δ_0, δ_1, and n only. It follows that

$$\limsup_{t' \downarrow t}\|V_{n+1}(t') - V_{n+1}(t)\| \leq C_n\varepsilon,$$

and since ε can be taken arbitrarily small, we see that $V_{n+1}(\cdot)$ is uniformly continuous for $t > 0$. Therefore the same is true for $S(\cdot)$. ∎

COROLLARY 2.5. *Suppose A is the generator of a uniformly exponentially stable C_0-semigroup on X that is uniformly continuous for $t > 0$. Then*

$$r_{\omega_0}(A; D, E) = \frac{1}{\sup_{\omega\in\mathbb{R}}\|ER(i\omega, A)D\|}.$$

This result applies to compact semigroups, differentiable semigroups, and analytic semigroups, since each of these is uniformly continuous for $t > 0$.

3. DELAY EQUATIONS IN $L^p([-h,0];X)$

Throughout this section, we fix a C_0-semigroup $\mathbf{T}$ with generator A on a complex Banach space X. We also fix $p \in [1,\infty)$ and nonnegative real numbers $0 \le h_1 < \cdots < h_n =: h$.

Given bounded linear operators $B_1, \dots, B_n$ on X, we will study the delay equation

$$(DE_{B_1,\dots,B_n}) \qquad \begin{aligned} \dot{u}(t) &= Au(t) + \sum_{j=1}^{n} B_j u(t-h_j), \qquad t \ge 0, \\ u(0) &= x, \\ u(t) &= f(t), \qquad t \in [-h,0). \end{aligned}$$

Here, $x \in X$ is the initial value and $f \in L^p([-h,0];X)$ is the "history" function. This equation has been investigated by Nakagiri [14, 15]; see also [1, 3, 8, 9, 12, 21] for related studies.

A *mild solution* of $(\mathrm{DE}_{B_1,\dots,B_n})$ is a function $u(\cdot;x,f) \in L^p_{\mathrm{loc}}([-h,\infty);X)$ satisfying

$$u(t;x,f) = \begin{cases} T(t)x + \int_0^t T(t-s)\sum_{j=1}^n B_j u(s-h_j;x,f)\,ds, & t \ge 0, \\ f(t), & t \in [-h,0). \end{cases}$$

It follows form [14, Theorem 2.1] that for all $x \in X$ and $f \in L^p([-h,0];X)$ a unique mild solution $u(\cdot;x,f)$ exists; this solution is continuous on $[0,\infty)$ and exponentially bounded. To study the asymptotic behavior of these solutions by semigroup methods, we introduce the product space $\mathscr{X} := X \times L^p([-h,0];X)$ and define bounded linear operators $\mathscr{T}_{B_1,\dots,B_n}(t)$ on $\mathscr{X}$ as follows. Given a function $u \in L^p_{\mathrm{loc}}([-h,\infty);X)$, for each $t \ge 0$ we define $u_t \in L^p([-h,0];X)$ by $u_t(s) := u(t+s)$, $s \in [-h,0]$. Denoting the unique mild solution of $(\mathrm{DE}_{B_1,\dots,B_n})$ by $u(\cdot;x,f)$, we now define

$$\mathscr{T}_{B_1,\dots,B_n}(t)(x,f) := \big(u(t;x,f), u_t(\cdot;x,f)\big), \qquad t \ge 0.$$

By [15, Proposition 3.1] we have

PROPOSITION 3.1. *The family* $\mathcal{T}_{B_1,\ldots,B_n} = \{\mathcal{T}_{B_1,\ldots,B_n}(t)\}_{t\geq 0}$ *defines a* C_0-*semigroup of linear operators on* $\mathcal{X}$. *Its generator* $\mathcal{A}_{B_1,\ldots,B_n}$ *is given by*

$$\mathcal{D}\left(\mathcal{A}_{B_1,\ldots,B_n}\right) = \left\{(x,f)\in\mathcal{X}:\quad f\in W^{1,p}([-h,0];X),\, f(0)=x\in\mathcal{D}(A)\right\},$$

$$\mathcal{A}_{B_1,\ldots,B_n}(x,f) = \left(Ax + \sum_{j=1}^{n} B_j f(-h_j), f'\right), \qquad (x,f)\in\mathcal{D}\left(\mathcal{A}_{B_1,\ldots,B_n}\right).$$

Here $W^{1,p}([-h,0];X)$ is the space of absolutely continuous X-valued functions f on $[-h,0]$ that are strongly differentiable a.e. with derivative $f'\in L^p([-h,0];X)$.

Whenever the operators $B_1,\ldots,B_n$ are understood, we will drop them from the notation and simply write $\mathcal{T}$ and $\mathcal{A}$.

The spectrum and resolvent of $\mathcal{A}$ are described by [15, Theorem 6.1] as follows.

PROPOSITION 3.2. *We have* $\lambda\in\varrho(A)$ *if and only if* $\lambda\in\varrho(A+\sum_{j=1}^{n}e^{-\lambda h_j}B_j)$. *In this case the resolvent of* $\mathcal{A}$ *is given by*

$$R(\lambda,\mathcal{A}) = E_\lambda R\left(\lambda, A + \sum_{j=1}^{n} e^{-\lambda h_j}B_j\right)H_\lambda F + T_\lambda,$$

where $E_\lambda\in\mathcal{L}(X,\mathcal{X})$, $H_\lambda\in\mathcal{L}(\mathcal{X},X)$, $F\in\mathcal{L}(\mathcal{X},\mathcal{X})$, *and* $T_\lambda\in\mathcal{L}(\mathcal{X},\mathcal{X})$ *are defined by*

$$E_\lambda x := (x, e^{\lambda\cdot}x);$$

$$H_\lambda(x,f) := x + \int_{-h}^{0} e^{\lambda s} f(s)\,ds;$$

$$F(x,f) := \left(x, \sum_{j=1}^{n}\chi_{[-h_j,0]}(\cdot)B_j f(-h_j-\cdot)\right);$$

$$T_\lambda(x,f) := \left(0, \int_{\cdot}^{0} e^{\lambda(\cdot-\xi)} f(\xi)\,d\xi\right).$$

Our first result relates the abscissa $s_0(\mathcal{A})$ to $s_0(A)$:

THEOREM 3.3. *Assume that* $s(A)<0$. *If*

$$\sup_{\omega\in\mathbb{R}}\left\|\sum_{j=1}^{n} e^{i\omega h_j}B_j\right\| < \frac{1}{\sup_{\omega\in\mathbb{R}}\|R(i\omega,A)\|},$$

then $s_0(\mathcal{A})<0$.

Proof. Choose $\delta \in (0,1)$ such that

$$\sup_{\omega\in\mathbb{R}}\left\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \right\| \leq (1-\delta)\frac{1}{\sup_{\omega\in\mathbb{R}}\| R(i\omega, A)\|}.$$

Recalling that the suprema along vertical lines $\operatorname{Re}\lambda = c$ of bounded analytic functions decrease as c increases, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ we have

$$\left\| \sum_{j=1}^{n} e^{-\lambda h_j} B_j \right\| \leq \sup_{\omega\in\mathbb{R}}\left\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \right\| \leq (1-\delta)\frac{1}{\sup_{\omega\in\mathbb{R}}\| R(i\omega, A)\|}$$
$$\leq (1-\delta)\frac{1}{\| R(\lambda, A)\|}.$$

Therefore by Proposition 1.1, $\{\operatorname{Re}\lambda > 0\} \subset \varrho(A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j)$, and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ we have

$$\left\| R\left(\lambda, A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j\right)\right\| \leq \| R(\lambda, A)\|\left(1 + \frac{1}{\delta}\left\| \sum_{j=1}^{n} e^{-\lambda h_j} B_j R(\lambda, A)\right\|\right)$$
$$\leq \| R(\lambda, A)\|\left(1 + \frac{1-\delta}{\delta}\right) = \frac{1}{\delta}\| R(\lambda, A)\|.$$

Hence by Proposition 3.2, $\{\operatorname{Re}\lambda > 0\} \subset \varrho(\mathscr{A})$ and

$$\| R(\lambda, \mathscr{A})\| = \left\| E_\lambda R\left(\lambda, A + \sum_{j=1}^{n} e^{-\lambda h_j} B_j\right) H_\lambda F + T_\lambda \right\|$$
$$\leq \frac{1}{\delta}\| R(\lambda, A)\|(1 + h^{1/p})(1 + h^{1/q})\left(1 + \sum_{j=1}^{n}\|B_j\|\right) + \|T_0\|$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$; $1/p + 1/q = 1$. Therefore, $\sup_{\operatorname{Re}\lambda > 0}\|R(\lambda, \mathscr{A})\| < \infty$, which implies $s_0(\mathscr{A}) < 0$. ∎

Note that by (2.1) in particular we have $\omega_1(\mathscr{A}) < 0$, which means that the semigroup $\mathscr{T}$ is exponentially stable. This, in turn, implies that there exist $M > 0$ and $\omega > 0$ such that for all $(x, f) \in \mathscr{D}(\mathscr{A}) = \{(x, f) \in X \times W^{1,p}([-h, 0]; X)\colon f(0) = x \in \mathscr{D}(A)\}$ we have

$$\| u(t; x, f)\| \leq Me^{-\omega t}\|(x, f)\|_{\mathscr{D}(\mathscr{A})}.$$

COROLLARY 3.4. *Suppose $p = 2$ and X is isomorphic to Hilbert space. If $\omega_0(A) < 0$ and*

$$\sup_{\omega \in \mathbb{R}} \left\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \right\| < \frac{1}{\sup_{\omega \in \mathbb{R}} \| R(i\omega, A) \|},$$

then $\omega_0(\mathscr{A}) < 0$.

Another situation in which information about $\omega_0(\mathscr{A})$ may be obtained from $s_0(\mathscr{A})$ is described in the following proposition.

PROPOSITION 3.5. *If the semigroup generated by A is uniformly continuous for $t > 0$, then the semigroup generated by $\mathscr{A}$ is uniformly continuous for $t > h$.*

Proof. We proceed as in the proof of [15, Proposition 3.1]. For $(x, f) \in \mathscr{X}$ we define

$$k(s; x, f) := \sum_{j=1}^{n} B_j u(s - h_j; x, f), \qquad s \geq 0,$$

where $u(\cdot; x, f)$ is the unique mild solution of $(\mathrm{DE}_{B_1, \ldots, B_n})$ with initial value (x, f). For $t > 0$ set

$$Q_t(x, f) := \int_0^t T(t - s) k(s; x, f)\, ds, \qquad (x, f) \in \mathscr{X},$$

and for $\varepsilon \in (0, t]$ set

$$Q_{t,\varepsilon}(x, f) := \int_0^{t-\varepsilon} T(t - s) k(s; x, f)\, ds, \qquad (x, f) \in \mathscr{X}.$$

The argument in [15] shows that there exist constants $M_t > 0$ and $N_t > 0$, both increasing with t, such that

$$\| k(\cdot; x, f) \|_{L^p([0,t]; X)} \leq M_t \| (x, f) \|_{\mathscr{X}}$$

and

$$\| Q_{t,\varepsilon}(x, f) - Q_t(x, f) \| \leq \varepsilon^{1/q} N_t \| (x, f) \|_{\mathscr{X}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Define $K_h := \sup_{\sigma\in[0,h]}\|T(\sigma)\|$. Fix $h+\varepsilon \le t \le t' \le 2h+\varepsilon$. Then, for all $s\in[-h,0]$,

$$\begin{aligned}
&\|u(t'+s;x,f)-u(t+s;x,f)\|\\
&\quad\le \|T(t'+s)x-T(t+s)x\|\\
&\qquad+\left\|\int_{t+s}^{t'+s}T(t'+s-\sigma)k(\sigma;x,f)\,d\sigma\right\|\\
&\qquad+\left\|\int_{t+s-\varepsilon}^{t+s}(T(t'+s-\sigma)-T(t+s-\sigma))k(\sigma;x,f)\,d\sigma\right\|\\
&\qquad+\left\|\int_{0}^{t+s-\varepsilon}(T(t'+s-\sigma)-T(t+s-\sigma))k(\sigma;x,f)\,d\sigma\right\|\\
&\quad\le \|T(t'+s)-T(t+s)\|\,\|(x,f)\|_{\mathscr{X}}\\
&\qquad+(t'-t)^{1/q}N_{2h+\varepsilon}\|(x,f)\|_{\mathscr{X}}\\
&\qquad+(K_h+1)\varepsilon^{1/q}N_{2h+\varepsilon}\|(x,f)\|_{\mathscr{X}}+(2h)^{1/q}M_{2h}\|(x,f)\|_{\mathscr{X}}\\
&\qquad\times\sup_{\sigma\in[0,t+s-\varepsilon]}\|T(t'+s-\sigma)-T(t+s-\sigma)\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{t'\downarrow t}\left(\sup_{\|(x,f)\|_{\mathscr{X}}\le 1}\ \sup_{s\in[-h,0]}\|u(t'+s;x,f)-u(t+s;x,f)\|\right)\\
&\qquad\le (K_h+1)\varepsilon^{1/q}N_{2h+\varepsilon},
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\limsup_{t'\downarrow t}\left(\sup_{\|(x,f)\|_{\mathscr{X}}\le 1}\|u_{t'}(x,f)-u_t(x,f)\|_{L^p([-h,0];X)}\right)\\
&\qquad\le (K_h+1)\varepsilon^{1/q}h^{1/p}N_{2h+\varepsilon}.
\end{aligned}$$

It follows that

$$\limsup_{t'\downarrow t}\|\mathscr{T}(t')-\mathscr{T}(t)\|\le (K_h+1)\varepsilon^{1/q}(1+h^{1/p})N_{2h+\varepsilon}.$$

Since the choice of $\varepsilon>0$ was arbitrary and $N_{2h+\varepsilon}$ decreases with ε, this proves that $\lim_{t'\downarrow t}\|\mathscr{T}(t')-\mathscr{T}(t)\| = 0$. ∎

COROLLARY 3.6. *Assume that A generates a uniformly exponentially stable C_0-semigroup that is uniformly continuous for $t > 0$. If*

$$\sup_{\omega\in\mathbb{R}} \left\| \sum_{j=1}^{n} e^{i\omega h_j} B_j \right\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \| R(i\omega, A) \|},$$

then $\omega_0(\mathscr{A}) < 0$, i.e., $\mathscr{T}$ is uniformly exponentially stable.

We now consider the case $n = 1$ in more detail and return to the notation $\mathscr{T}_B$ and $\mathscr{A}_B$ to denote the semigroup on $\mathscr{X}$ and its generator governing the solutions of the problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + Bu(t-h), \qquad t \geq 0, \\ u(0) &= x, \\ u(t) &= f(t), \qquad t \in [-h, 0). \end{aligned}$$

Assuming that $s_0(\mathscr{A}_0) < 0$ ($\mathscr{A}_0$ being the generator $\mathscr{A}_B$ corresponding to the zero operator $B = 0$), we define

$$r_{s_0}(\mathscr{A}_0) := \sup\{r \geq 0: \quad s_0(\mathscr{A}_B) < 0 \text{ for all } B \in \mathscr{L}(X) \text{ with } \|B\| \leq r\}.$$

With this notation, the case $n = 1$ of Theorem 3.3 says that

$$r_{s_0}(\mathscr{A}_0) \geq \frac{1}{\sup_{\omega\in\mathbb{R}} \| R(i\omega, A) \|}.$$

In fact, we have the following more precise result.

THEOREM 3.7. *If $s_0(A) < 0$, then*

$$r_{s_0}(\mathscr{A}_0) = r_{s_0}(A; I, I) = \frac{1}{\sup_{\omega\in\mathbb{R}} \| R(i\omega, A) \|}.$$

Proof. It only remains to prove the inequality

$$r_{s_0}(\mathscr{A}_0) \leq \frac{1}{\sup_{\omega\in\mathbb{R}} \| R(i\omega, A) \|}.$$

Fix $\varepsilon > 0$ and choose $\omega_0 \in \mathbb{R}$ such that

$$\frac{1}{\| R(i\omega_0, A) \|} \leq \frac{1}{\sup_{\omega\in\mathbb{R}} \| R(i\omega, A) \|} + \varepsilon.$$

Define operators $E \in \mathscr{L}(\mathscr{D}(\mathscr{A}_0), X)$ and $D \in \mathscr{L}(X, \mathscr{X})$ by

$$E(x, f) := f(-h), \qquad Dx := (x, 0).$$

Using Proposition 3.2 it is easily verified that

$$ER(i\omega_0, \mathscr{A}_0)Dx = e^{-i\omega_0 h}R(i\omega_0, A)x, \qquad x \in X,$$

and therefore $\|ER(i\omega_0, \mathscr{A}_0)D\| = \|R(i\omega_0, A)\|$. By Theorem 1.2 there exists $B_0 \in \mathscr{L}(X)$ such that

$$\|B_0\| \le \frac{1}{\|ER(i\omega_0, \mathscr{A}_0)D\|} + \varepsilon$$

and

$$i\omega_0 \in \sigma(\mathscr{A}_0 + DB_0E).$$

Noting that $\mathscr{A}_{B_0} = \mathscr{A}_0 + DB_0E$, this means that $i\omega_0 \in \sigma(\mathscr{A}_{B_0})$. Therefore $\mathscr{A}_{B_0}$ cannot have a uniformly bounded resolvent on $\{\operatorname{Re}\lambda > 0\}$. The estimate

$$\|B_0\| \le \frac{1}{\|ER(i\omega_0, \mathscr{A}_0)D\|} + \varepsilon = \frac{1}{\|R(i\omega_0, A)\|} + \varepsilon$$
$$\le \frac{1}{\sup_{\omega\in\mathbb{R}}\|R(i\omega, A)\|} + 2\varepsilon$$

then shows that

$$r_{s_0}(\mathscr{A}_0) \le \frac{1}{\sup_{\omega\in\mathbb{R}}\|R(i\omega, A)\|} + 2\varepsilon.$$

∎

In particular, if $p = 2$ and X is isomorphic to a Hilbert space, or if **T** is uniformly continuous for $t > 0$, it follows that

$$r_{\omega_0}(\mathscr{A}_0) = r_{\omega_0}(A; I, I) = \frac{1}{\sup_{\omega\in\mathbb{R}}\|R(i\omega, A)\|},$$

where $r_{\omega_0}(\mathscr{A}_0)$ is defined in the obvious way.

For the generator A of a positive semigroup on a Banach lattice X we have $s(A) = \omega_1(A) = s_0(A)$; moreover, $s(A) \in \sigma(A)$ whenever $s(A) >$

$-\infty$ [16, Theorem C-III-1.1]. This will be used to prove the following versions of Theorems 3.3 and 3.7:

THEOREM 3.3′. *Let A generate a positive C_0-semigroup on a Banach lattice X, and assume that the operators B_j are positive, $j = 1, \dots, n$. Then the semigroup $\mathcal{T}_{B_1,\dots,B_n}$ is positive. If $s_0(A) < 0$ and*

$$\left\| \sum_{j=1}^{n} B_j \right\| < \frac{1}{\|A^{-1}\|},$$

then $s_0(\mathcal{A}_{B_1,\dots,B_n}) < 0$.

Proof. It is an easy consequence of Proposition 3.2 that $R(\lambda, \mathcal{A}_{B_1,\dots,B_n}) \geq 0$ for sufficiently large real λ. Then $\mathcal{T}_{B_1,\dots,B_n}(t) \geq 0$ for all $t \geq 0$ by the exponential formula [18, Theorem 1.8.3].

Since $\|R(\lambda, A)\| \leq \|R(\operatorname{Re}\lambda, A)\|$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > s(A)$, for some $\delta \in (0,1)$ and all $\lambda \geq 0$ we have

$$\left\| \sum_{j=1}^{n} e^{-\lambda h_j} B_j \right\| \leq \left\| \sum_{j=1}^{n} B_j \right\| \leq (1-\delta) \frac{1}{\|R(0,A)\|} \leq (1-\delta) \frac{1}{\|R(\lambda,A)\|}.$$

Hence, $[0,\infty) \subset \varrho(\mathcal{A}_{B_1,\dots,B_n})$ by Propositions 1.1 and 3.2. Since $\mathcal{A}_{B_1,\dots,B_n}$ generates a positive semigroup, this implies that

$$s_0\big(\mathcal{A}_{B_1,\dots,B_n}\big) = s\big(\mathcal{A}_{B_1,\dots,B_n}\big) < 0.$$

∎

If $n = 1$ we have

THEOREM 3.7′. *Let A generate a positive C_0-semigroup on a Banach lattice X with $s_0(A) < 0$. Then*

$$r_{s_0}(\mathcal{A}_0) = r_{s_0}(A; I, I) = \frac{1}{\|A^{-1}\|}.$$

The identities $r_{s_0}(\mathcal{A}_0) = r_{s_0}(A; I, I)$ in Theorems 3.7 and 3.7′ can be interpreted as saying that the stability radius for boundedness of the resolvent for the delay problem is independent of the delay h (and equals to stability radius for boundedness of the resolvent for the undelay problem).

In the situation of Theorem 3.7′, if in addition B is assumed to be positive, then we further have $s_0(A+B) = \omega_1(A+B)$ and $s_0(\mathcal{A}_B) =$

$\omega_1(\mathscr{A}_B)$, so that we can reformulate this observation in terms of exponential stability of the semigroups involved. In the state space $C([-h, 0]; X)$ this is a well known phenomenon (cf. [16, Corollary B-IV-3.10], where different methods are used). For further results on the stability of delay equations in $C([-h, 0]; X)$, the reader might consult [10, 11, 20].

ACKNOWLEDGMENT

We thank Yuri Latushkin for kindly sending us the preprints [2] and [13], and Sjoerd Verduyn Lunel for pointing out to us the work by Nagakiri [14, 15].

REFERENCES

1. J. A. Burns, T. L. Herdman, and H. W. Stech, Linear functional-differential equations as semigroups on product spaces, *SIAM J. Math. Anal.* **14** (1983), 98–116.
2. S. Clark, Yu. Latushkin, and T. Randolph, Evolution semigroups and stability of time-varying systems on Banach spaces, preprint.
3. R. Datko, Representation of solutions and stability of linear differential-difference equation in a Banach space, *J. Differential Equations* **29** (1978), 105–166.
4. L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, *Trans. Amer. Math. Soc.* **236** (1978), 385–394.
5. D. Hinrichsen and A. J. Pritchard, Stability radius for structured perturbations and the algebraic Riccati equation, *Systems Control Lett.* **8** (1986), 105–113.
6. D. Hinrichsen and A. J. Pritchard, Real and complex stability radii: a survey, *in* "Control of Uncertain Systems" (D. Hinrichsen and B. Martensson, Eds.), pp. 119–162, Birkhäuser, Basel/Boston/Berlin, 1990.
7. D. Hinrichsen and A. J. Pritchard, Robust stability of linear evolution operators on Banach spaces, *SIAM J. Control Optim.* **32** (1994), 1503–1541.
8. F. Kappel, Semigroups and delay equations, *in* "Semigroups, Theory and Applications" (H. Brezis, M. Crandall and F. Kappel, Eds.), Vol. II (Trieste 1984), pp. 136–176, Pitman Research Notes 152, Pitman, London, 1986.
9. F. Kappel and K. P. Zhang, Equivalence of functional-differential equations of neutral type and abstract Cauchy problems, *Monatsh. Math.* **101** (1986), 115–133.
10. W. Kerscher and R. Nagel, Asymptotic behavior of one-parameter semigroups of positive operators, *Acta Appl. Math.* **2** (1984), 297–309.
11. W. Kerscher and R. Nagel, Positivity and stability for Cauchy problems with delay, *in* "Partial Differential Equations" (Rio de Janeiro 1986), pp. 216–235, Lecture Notes in Math, 1324, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1988.
12. K. Kunisch and W. Schappacher, Necessary conditions for partial differential equations with delay to generate C_0-semigroups, *J. Differential Equations* **50** (1983), 49–79.
13. Yu. Latushkin, S. Montgomery-Smith, and T. Randolph, Evolution semigroups and robust stability of evolution operators on Banach spaces, preprint.
14. S. Nagakiri, Optimal control of linear retarded systems in Banach spaces, *J. Math. Anal. Appl.* **120** (1986), 169–210.
15. S. Nagakiri, Structural properties of functional differential equations in Banach spaces, *Osaka J. Math.* **25** (1988), 353–398.

16. R. Nagel (Ed.), ‘‘One-Parameter Semigroups of Positive Operators,’’ Springer Lecture Notes in Mathematics 1184, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1986.
17. J. M. A. M. van Neerven, The asymptotic behavior of semigroups of linear operators, *in* ‘‘Operator Theory: Advances and Applications,’’ Vol. 88, Birkhäuser Verlag, Basel/Boston/Berlin, 1996.
18. A. Pazy, ‘‘Semigroups of Linear Operators and Applications to Partial Differential Equations,’’ Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.
19. A. J. Pritchard and S. Townley, Robustness of linear systems, *J. Differential Equations* **77** (1989), 254–286.
20. C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974), 395–418.
21. G. F. Webb, Functional differential equations and nonlinear semigroups in L^p-space, *J. Differential Equations* **20** (1976), 71–89.