The Inhomogeneous Dirichlet Problem for $\Delta^2$ in Lipschitz Domains

Vilhelm Adolfsson

Department of Mathematics, University of Göteborg and Chalmers University of Technology, S-412 96 Göteborg, Sweden

and

Jill Pipher

Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912

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We study the inhomogeneous Dirichlet problem for the bi-Laplacian with data given in Sobolev and Besov spaces on non-smooth domains. © 1998 Academic Press

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INTRODUCTION

The purpose of this paper is to initiate a study of the inhomogeneous Dirichlet problem for higher order elliptic equations in non-smooth domains, with data in Sobolev and Besov spaces. We will specifically study the Dirichlet problem for the bi-Laplacian,

\[
\begin{align*}
\Delta^2 u &= f & \text{in } \Omega, \\
(u, Vu) &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where $f$ is given in a Sobolev or Besov space on $\Omega$, a $C^1$ domain in $\mathbb{R}^n$, a Lipschitz domain in $\mathbb{R}^3$, or a Lipschitz domain in $\mathbb{R}^n$ in case the estimates are measured in $L^p$ with $p$ close to 2. The study of the above problem is naturally reduced to the homogeneous problem via the fundamental solution. As is well-known, the $L^p$-theory for the homogeneous Dirichlet problem of the Laplacian has to be restricted to the range $p > 2 - \varepsilon$ for a Lipschitz domain in $\mathbb{R}^n$. In contrast to this situation, the $L^p$-theory for the bi-Laplacian will also depend on the dimension, [PV1]. This feature is connected with an Agmon-Miranda maximum principle, [PV3], and explains the restriction to $n = 3$ in the case of a Lipschitz domain and “general” $p$.

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The inhomogeneous Dirichlet problem for the Laplacian with data in Sobolev and Besov spaces has been studied in [JK]. Many of the results of our paper are analogous to the situation for the Laplacian. In [JK] they also derived the sharpness of the results obtained. The counterexamples produced there relied either on considering domains in the exterior of a cone, or using conformal mappings and properties of the Green function. We will obtain similar results for the class of counterexamples produced using the exterior of a cone. Even though it is possible to prove sharpness results for more ranges of indices, $p \leq 2$, the main techniques employed in [JK] have no immediate counterpart in the case of the bi-Laplacian. The obstructions in this direction are connected to the change of sign of the Green function of the bi-Laplacian so these questions remain a point left open.

As mentioned above, the inhomogeneous problem is naturally reduced to a study of the homogeneous problem, and is the path taken in [JK] for the Laplacian. In the case $p = 2$ their theorem states that for $1 < \alpha < 3/2$, the inhomogeneous problem

$$\begin{cases} Au = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

has a unique solution $u \in L^2_{\alpha-2}(\Omega)$ and $\|u\|_{L^2_{\alpha}(\Omega)} \leq C \|f\|_{L^2_{\alpha-2}(\Omega)}$ for every $f$. This problem is reduced to a homogeneous problem by extending $f \in L^2_{\alpha-2}(\Omega)$ to an element $f \in L^2_{\alpha-2}(\mathbb{R}^n)$ with compact support and putting $w = f \ast N$ where $N(x) = c_n |x|^{2-\alpha}$, so that $Aw = f$ and $w \in L^2_{\alpha}(\Omega)$. Let $g = \text{Tr} w$, the trace of $w$ on $\partial \Omega$. Then $g \in L^2_{\alpha-1/2}(\partial \Omega)$ if $1/2 < \alpha < 3/2$. Let now $v$ be the solution to

$$\begin{cases} Av = 0 & \text{in } \Omega \\ v = g & \text{on } \partial \Omega. \end{cases} \quad (*)$$

Then $v \in L^2_{\alpha}(\Omega)$ if $1/2 \leq \alpha \leq 3/2$. So $u = w - v$ is the solution to the inhomogeneous problem. However, in the range $\alpha \leq 1/2$, (or more generally $\alpha \leq 1/p$ in case $p \neq 2$), the trace mapping is no longer continuous, and uniqueness in the homogeneous problem fails. Thus, we will consider only the range $\alpha > 1/2$, (or $\alpha > 1/p$). Furthermore, unlike the case of a smooth domain, the trace operator $\text{Tr}$ on $L^2_{\alpha}(\Omega)$ is larger than the space $L^2_{\alpha}(\Omega)$. This explains the restriction $1/2 < \alpha < 3/2$. That $(*)$ is valid in the full range $1/2 < \alpha < 3/2$ follows from the solvability of the Dirichlet problem and the Regularity problem, $\alpha = 1/2$ and $\alpha = 3/2$ respectively, and interpolating on one hand the spaces on the boundary and on the other the solid spaces, i.e., the spaces defined in the interior. In the case of the bi-Laplacian we follow the same approach but now the data on the boundary involves the function

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and its gradient. For a $C^1$ or a Lipschitz domain, the boundary is insufficiently smooth to let us discuss in a straightforward manner higher (tangential) derivatives on the boundary, i.e., to straightforwardly specify data for the regularity problem. A way out of this problem is to use the notion of (Whitney) arrays introduced in Verchota’s work [V2] to solve boundary value problems for the polyharmonic equation. That means looking at arrays with entries being functions satisfying a compatibility condition. The entries of course correspond to the function and its partial derivatives. Thus we require more smoothness on the derivatives instead of on the function itself.

In case of the bi-Laplacian, to carry out the reduction sketched above for the Laplacian, we have to have solvability of the endpoint cases, i.e. the Dirichlet and regularity problems for the bi-Laplacian in various $L^p$, and related, spaces. This is provided by the results in [DKV; PV1,2,3; V1,2] and results in this paper. Further, we need extension and restriction theorems for solid and boundary Sobolev and Besov spaces. Results along these lines are proved, for more general domains, in [JW]. However, even though their restriction/extension theorems work also for integer values of smoothness for the spaces on the boundary, their boundary spaces do not coincide with our array spaces. It is for these last spaces, the natural generalization of $L^1(\partial \Omega)$, which we have solvability of the boundary value problem. Nevertheless, we establish here that the array spaces on the boundary are an interpolation scale and that for non-integer smoothness indices, these array spaces coincide with the boundary spaces of [JW]. These results are proved in Section 1. Thus, we are able to carry through the reduction to the homogeneous problem. In connection with the interpolation procedure, it is well-known that real interpolation of Sobolev spaces gives Besov spaces, but for the subspace of solutions, i.e. bi-harmonic functions, these spaces coincide and give us therefore results also in the Sobolev scale. This is the content of Proposition 5 in Section 2. Our main results are Theorems 2.1 and 2.2 in Section 2. In Section 3 we have put some results for the boundary value problem with data in certain atomic smoothness spaces being in this context the natural replacement of $L^1$. Section 4 contains applications of our results to Bergman spaces, (see [CC]), and the last Section, Section 5, concludes by showing sharpness of our results for certain ranges of indices. These results apply to show the sharpness of the results on Bergman spaces of the preceding Section.

1. SOBOLEV AND BESOV SPACES

We let $R_\Omega f$ denote the restriction to $\Omega$ of the function $f$ defined in $\mathbb{R}^n$. Recall that an open set $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain if for each $Q \in \partial \Omega$
there exists a rectangular coordinate system, \((x, s), x \in \mathbb{R}^{n-1}, s \in \mathbb{R}\), a neighborhood, \(U(Q) \equiv U \subset \mathbb{R}^n\) containing \(Q\), and a function \(\varphi_Q \equiv \varphi: \mathbb{R}^{n-1} \to \mathbb{R}\) such that

(i) \(|\varphi(x) - \varphi(y)| \leq C_Q |x - y|\) for all \(x, y \in \mathbb{R}^{n-1}\), \(C_Q < \infty\);

(ii) \(U \cap \Omega = \{(x, s): s > \varphi(x)\} \cap U\).

The coordinate systems \((x, s)\), may always be taken to be a rotation and translation of the standard rectangular coordinates for \(\mathbb{R}^n\).

**Definition 1.1.** For \(1 \leq p \leq \infty\) and \(\alpha \geq 0\) we define \(L^p_\alpha(\Omega) = R_\alpha L^p_\alpha\), with the usual quotient norm

\[
\|f\|_{L^p_\alpha(\Omega)} = \inf \{ \|g\|_{L^p_\alpha}: R_\alpha g = f \}.
\]

For each non-negative integer \(k\) we put

\[
W^{k,p}_\alpha(\Omega) = \left\{ f: \frac{\partial^\beta f}{\partial x^\beta} \in L^p(\Omega), |\beta| \leq k \right\},
\]

with norm

\[
\|f\|_{k,p} \equiv \left( \sum_{|\beta| \leq k} \left( \int_{\Omega} \left| \frac{\partial^\beta f}{\partial x^\beta} \right|^p dx \right)^{1/p} \right)^{1/p}.
\]

Standard Calderón-Zygmund estimates for singular integrals show that for \(1 < p < \infty\) and non-negative integers \(k\), \(L^p_\alpha(\mathbb{R}^n) = W^{k,p}_\alpha(\mathbb{R}^n)\). The extension results of Calderón and Stein give for \(\Omega\) a bounded Lipschitz domain that \(L^p_\alpha(\Omega) = W^{k,p}_\alpha(\Omega), 1 < p < \infty\), and that \(L^p_\alpha\) is a complex interpolation scale for \(\alpha \geq 0\) and \(1 < p < \infty\); i.e.,

\[
\left[ L^{p_0}_\alpha(\Omega), L^{p_1}_\alpha(\Omega) \right]_\theta = L^{p_\theta}_\alpha(\Omega),
\]

where \(1 < p_0, p_1 < \infty\), \(\alpha = (1 - \theta) \alpha_0 + \theta \alpha_1\) and \(1/p = (1 - \theta)/p_0 + \theta/p_1\). In particular we note that complex interpolation of the Sobolev spaces \(W^{k,p}_\alpha(\Omega)\) gives the potential spaces \(L^{p_\theta}_\alpha(\Omega)\). Real interpolation of the Sobolev spaces gives the Besov spaces.

**Definition 1.2.** Let \(\phi \in \mathcal{S}'\), the space of rapidly decreasing functions, be such that \(\text{supp } \phi = \{ \xi: 2^{-1} < |\xi| < 2\}, \phi > 0\) in \(\{ \xi: 2^{-1} < |\xi| < 2\}\), and

\[
\sum_{r = -\infty}^{\infty} \phi(\xi/2^r) = 1
\]
for \( \xi \neq 0 \). Put \( \hat{\phi}_n(\xi) = \hat{\phi}(\xi/2^n) \) and \( \hat{\xi}/\hat{\phi}(\xi) = \sum_{n=-\infty}^{\infty} \hat{\phi}_n(\xi) \). We then define

\[
B^p_q(\mathbb{R}^n) = \left\{ f \in \mathcal{S} : \| f \|_{B^p_q(\mathbb{R}^n)} = \| \hat{\psi} \ast f \|_{L^p} + \left( \sum_{r=0}^{\infty} (2^r \| \hat{\phi}_r \ast f \|_{L^p})^q \right)^{1/q} < \infty \right\},
\]

where \( 0 \leq p, q \leq \infty, \alpha \in \mathbb{R} \) and \( \| f \|_{B^p_q(\mathbb{R}^n)} \) denotes the norm.

We will use the notational convention that \( B^p_q = B^p_{\infty,q} \). There are many different characterizations of the Besov spaces, see [BL], [JK] and [JW].

We define the Besov space \( B^p_q(\Omega) \) as the restriction to \( \Omega \) of \( B^p_q(\mathbb{R}^n) \) with the quotient norm. Then it is well-known that real interpolation gives

\[
\left[ L^p(\Omega), W^k_p(\Omega) \right]_{\alpha,q} = B^p_q(\Omega)
\]

for \( 1 \leq p, q \leq \infty, 0 < \alpha < 1 \) and \( k \) a positive integer. Furthermore, Stein’s extension operator is a bounded operator from \( B^p_q(\Omega) \) to \( B^p_q(\mathbb{R}^n) \) for all \( \alpha \geq 0 \), and real interpolation of the spaces \( B^p_q(\Omega) \) stays within the class, see [JK, Prop. 2.17]. Other useful facts, consequences of the structure of Stein’s extension operator, are:

1-1) Suppose \( \alpha \geq 0 \) and \( 1 < p < \infty \). Then \( f \in L^p_{\alpha+1}(\Omega) \) if and only if \( f \in L^p(\Omega) \) and \( \forall \phi \in L^p_{\alpha+1}(\Omega) \).

1-2) Suppose \( \alpha > 0 \) and \( 1 \leq p, q \leq \infty \). Then \( f \in B^p_q(\alpha,1)(\Omega) \) if and only if \( f \in L^p(\Omega) \) and \( \forall \phi \in B^p_q(\alpha,1)(\Omega) \).

We will next consider spaces on the boundary. Suppose \( \tilde{f} \) is an extension with compact support in \( \mathbb{R}^n \), of \( f \) defined in \( \Omega \). Our interest in Besov and Sobolev spaces on the boundary stems from the fact that, for the biharmonic operator, the fundamental solution potential of \( f \) will have traces on \( \partial \Omega \) in these spaces. We will solve the homogeneous problem with data being these traces. To solve this last problem we rely on the solvability of the Dirichlet problem, [PV1], where the data can be seen to be taken in the array-spaces \( WA^p_\alpha(\partial \Omega) \) and \( WA^p_\gamma(\partial \Omega) \), to be defined below, and then we obtain the solutions for more general data via interpolation. To achieve this we will have to establish two facts; the first is that our array spaces can be identified with the approximation spaces of Jonsson and Wallin, [JW], for non-integer smoothness index. This enables us to use restriction and extension theorems for our spaces. The next needed fact is that our array-spaces are stable under interpolation, i.e. \( (WA^p_\alpha(\partial \Omega), WA^p_\gamma(\partial \Omega))_{\alpha,p} = WA^p_\alpha(\partial \Omega) \), for certain ranges of \( \alpha \) and \( p \). In this section we prove the last two stated facts.
We define for $1 \leq p \leq \infty$ spaces on the boundary of a Lipschitz domain $\Omega$ in $\mathbb{R}^n$, by
\[
L^p(\partial \Omega) = W^p_\gamma(\partial \Omega) = \{ f \in L^p(\partial \Omega); \nabla \gamma f \in L^p(\partial \Omega) \},
\]
where $\nabla \gamma$ denotes tangential derivatives, and
\[
B^p(\partial \Omega) = \left\{ f \in L^p(\partial \Omega); \int_{\partial \Omega} \int_{\partial \Omega} \frac{|f(P) - f(Q)|^p}{|P - Q|^{n-1+\alpha}} \, d\sigma(P) \, d\sigma(Q) < \infty \right\}.
\]
The norms are the standard ones. We will often use the notation $\| f \| \cdots \|$ to mean the norm of $f$ in the space indicated by $\cdots$.

We would like to be able to speak of higher smoothness of the functions on the boundary in spite of the fact that the boundary is only Lipschitz. For our purposes mainly two viewpoints will be considered. The first definition describes a version of the spaces of boundary data for which we in the endpoint cases, i.e. $\alpha = 0$ and $\alpha = 1$, have solvability of the homogeneous Dirichlet problem.

**Definition 1.3.** A sequence $\hat{f} = (f_0, f_1, \ldots, f_n)$ belongs to the Whitney array space $WA^p_{\alpha}(\partial \Omega)$, $1 \leq p \leq \infty$, $0 < \alpha < 1$ if

(i) $f_0 \in L^p(\partial \Omega),$

(ii) $f_j \in B^p(\partial \Omega), j \geq 1,$

and for a.e. $Q \in \partial \Omega$:

(iii) $(N_i(\Omega)(\partial \partial Q_j) - N_j(\partial \partial Q_i)) \, f_0(Q) = N_i(\Omega) f_i(Q) - N_j(\Omega) f_j(Q)$

where $N_i(\Omega) = (N_i(Q), \ldots, N_n(Q))$ is the unit normal to the boundary of $\Omega$. In the endpoint cases $\alpha = 0$ and $\alpha = 1$ we define $B^p(\partial \Omega)$ as $L^p(\partial \Omega)$ and $L^p_\gamma(\partial \Omega)$, respectively. Condition (iii) is understood in the following sense. Take any extension $\tilde{f}_0(x)$ of $f_0(Q)$ to a neighborhood of $\partial \Omega$. Then the expression $(N_i(\Omega)(\partial \partial Q_j) - N_j(\partial \partial Q_i))$ is a tangential derivative and is thus independent of the extension. The norm on $WA^p_{\alpha}(\partial \Omega)$ is taken to be the sum of the norms of each component in the array.

Condition (iii) above is equivalent to saying that if $(Z, \varphi)$ is any coordinate chart on the boundary, then the $n-1$ compatibility conditions,
\[
\frac{\partial}{\partial x_j} (f_0(x, \varphi(x))) = f_j(x, \varphi(x)) + \frac{\partial \varphi}{\partial x_j} f_0(x, \varphi(x)),
\]
is satisfied. We denote this last condition by $\hat{f} \in CC$, for short. Furthermore, given $f = (f_0, f_1, \ldots, f_n)$ we put $\hat{f} = (f_1, \ldots, f_n)$. An alternative definition is the following.
**Definition 1.4.** The array \( f \in WA^p_{\alpha + \frac{1}{2}}(\partial \Omega) \), \( 1 \leq p \leq \infty \), \( 0 \leq \alpha \leq 1 \), if \( f = (f_0, f_1, \ldots, f_n) \) satisfies (i) and (ii) of Definition 1.3 and for each coordinate chart \((Z, \varphi)\), there exist \( C^\infty_0(\mathbb{R}^n) \) functions \( \{F_m\} \) such that

\[
\|f_0(\cdot, \varphi(\cdot)) - F_m(\cdot, \varphi(\cdot))\|_{L^p(\mathbb{R}^{n-1})} \to 0,
\]

and

\[
\|f_j(\cdot, \varphi(\cdot)) - (D_j F_m)(\cdot, \varphi(\cdot))\|_{B^\alpha p(\mathbb{R}^{n-1})} \to 0, \quad j = 1, \ldots, n,
\]

as \( m \to \infty \), where as usual the endpoint cases \( \alpha = 0 \) and \( \alpha = 1 \) have the meaning of Definition 1.3.

It is well-known that Definitions 1.3 and 1.4 are equivalent for \( p < \infty \). Since the simple proof seems to be lacking in the literature we take the opportunity to supply one even though we will mainly be working with Definition 1.3. One direction of the equivalence is immediate and the other is provided by the following lemma.

**Proposition 1.5.** Suppose \( 1 \leq p \leq \infty \) and \( f \in WA^p_{\alpha + \frac{1}{2}}(\partial \Omega) \). Then, given any coordinate chart \((Z, \varphi)\), there exists a sequence \( \{F_m\} \), where \( F_m \in C^\infty_0(\mathbb{R}^n) \), such that the array \( (F_m|_{\partial \Omega \cap \{Z, \varphi\}}, D_j F_m|_{\partial \Omega \cap \{Z, \varphi\}}) \) converges to \( f \) in \( WA^p_{\alpha + \frac{1}{2}}(\partial \Omega) \).

**Proof.** We will treat here only the cases \( \alpha = 0 \) and \( \alpha = 1 \) directly. The case \( 0 < \alpha < 1 \) can be derived in a similar manner but is also a consequence of the equality of the array spaces and the Besov spaces on the boundary, our result below, Lemma 1.7, together with the extension theorem for the boundary Besov spaces.

**Step 1.** Suppose \( \partial \Omega = (x, \varphi(x)) \), i.e. \( \partial \Omega \) is a graph. Let \( \eta \) be an approximate identity. Put

\[
F_\delta(x, t) = f_\delta(\cdot, \varphi(\cdot)) * \eta_\delta(x) + t(f_\delta(\cdot, \varphi(\cdot)) * \eta_\delta(\cdot))(x) - (\varphi(\cdot) f_\delta(\cdot, \varphi(\cdot)) * \eta_\delta)(x).
\]

We show that \( F_\delta \to f \).

**Case \( \alpha = 0 \).** We have to show that

1. \( F_\delta(x, \varphi(x)) \to f_\delta(x, \varphi(x)) \) in \( L^p_\delta \),
2. \( (D_j F_\delta)(x, \varphi(x)) \to f_j(x, \varphi(x)) \) in \( L^p \) for \( j = 1, \ldots, n \).

Clearly, \( F_\delta(x, \varphi(x)) \to f_\delta(x, \varphi(x)) \) in \( L^p \). For \( 1 \leq j \leq n - 1 \) a simple computation shows that
\[ D_j(F_j(x, \varphi(x))) = \frac{\partial}{\partial x_j} (f_d(\cdot, \varphi(\cdot))) \ast \eta_d(x) + \frac{\partial \varphi}{\partial x_j} (x) f_d(\cdot, \varphi(\cdot)) \ast \eta_d(x) \]

\[ + \varphi(x) \left( f_d(\cdot, \varphi(\cdot)) \ast \frac{\partial \eta_d}{\partial x_j}(x) - (\varphi(\cdot) f_d(\cdot, \varphi(\cdot))) \ast \frac{\partial \eta_d}{\partial x_j}(x) \right) \]

\[ = \frac{\partial}{\partial x_j} (f_d(\cdot, \varphi(\cdot))) \ast \eta_d(x) + \frac{\partial \varphi}{\partial x_j} (x) (f_d(\cdot, \varphi(\cdot)) \ast \eta_d(x)) \]

\[ - \frac{\varphi(x) - \varphi(x')}{x_j - x'_j} f_d(x', \varphi(x')) \Theta_d(x - x') \, dx'. \]

where \( \Theta_d(x) = -x_j (\partial \eta_d/\partial x_j)(x) \) is an approximate identity. The last expression converges a.e. to \(- (\partial \varphi/\partial x_j) f_d\) and has \( L^p \) norm bounded by \( \| \nabla \varphi \|_{\infty} \| f_d \|_{L^p} \), for all \( \varepsilon \); so dominated convergence proves (1). For (2) we note, using the compatibility conditions, that for \( j = 1, \ldots, n - 1 \) we have

\[ (D_j F_j)(x, \varphi(x)) \]

\[ = \left( f_d(\cdot, \varphi(\cdot)) + \frac{\partial \varphi}{\partial x_j} (\cdot) f_d(\cdot, \varphi(\cdot)) \right) \ast \eta_d(x) \]

\[ + \varphi(x) \left( f_d(\cdot, \varphi(\cdot)) \ast \frac{\partial \eta_d}{\partial x_j}(x) - (\varphi(\cdot) f_d(\cdot, \varphi(\cdot))) \ast \frac{\partial \eta_d}{\partial x_j}(x) \right), \quad (*) \]

with the last two terms the same as the ones treated above. Thus the result follows in a similar manner. The case \( j = n \) is immediate.

Case \( x = 1 \). In this situation (1) is the same and (2) is strengthened to be convergence in \( L^p \). Consider \( j, k = 1, \ldots, n - 1 \). Since \( f_j \in L^p \), \( j = 1, \ldots, n \), moving the derivatives off of \( \eta_d \) in \( (*) \) gives that

\[ D_k [(D_j F_j)(x, \varphi(x)) - f_j(x, \varphi(x))] \]

\[ = \left( \frac{\partial}{\partial x_k} (f_d(\cdot, \varphi(\cdot))) \ast \eta_d(x) - \frac{\partial}{\partial x_k} (f_j(x, \varphi(x))) \right) \]

\[ + \left( \frac{\partial \varphi}{\partial x_k} (x) \left( \frac{\partial}{\partial x_j} (f_d(\cdot, \varphi(\cdot))) \ast \eta_d(x) \right) \right) \]

\[ + \varphi(x) \left( \frac{\partial}{\partial x_k} (f_d(\cdot, \varphi(\cdot))) \ast \frac{\partial \eta_d}{\partial x_k}(x) \right) \]

\[ - \left( \varphi(\cdot) \frac{\partial}{\partial x_j} (f_d(\cdot, \varphi(\cdot))) \ast \frac{\partial \eta_d}{\partial x_k}(x) \right). \]
The expression within the curly brackets converges to 0 in $L^p$ by arguments similar to those used above in the $x=0$ case. The behavior of the remaining expression is immediate, as is the case $j=n$.

Step 2. The bounded domain case. Given $f \in W^{s,p}_\infty(\partial \Omega)$ we let $\{\psi_k\}$ be a partition of unity subordinate to a covering of $\partial \Omega$ by coordinate charts with associated Lipschitz functions. Put $g^k = (g^k_0, g^k_1)$ where $g^k_0 = f_0 \psi_k$, $g^k_1 = f_0 \psi_k + f_0(\partial \psi_k)(\partial \chi_k)$ for $j = 1, ..., n$. Then $g \in W^{s,p}_\infty(\partial \Omega)$ and $\sum_k g^k_j = f_0$. Define $G^k_t(x, t)$ in a manner analogous to the definition of $F_\infty(x, t)$. Then $G^k_t \to g^k$ in $W^{s,p}_\infty(\partial \Omega)$.

This finishes the proof of the proposition.

Another view of Besov spaces on the boundary is by conditions on the remainder term in the Taylor-expansion, i.e., the space is given as a Local Polynomial Approximation space.

**Definition 1.6 [JW].** Let $\alpha > 0$ and $1 \leq p, q \leq \infty$. Let $k$ be a non-negative integer such that $k < \alpha \leq k + 1$. The collection $\{f_j\}_{|j| \leq k}$ belongs to the Besov space $B^\alpha_{p,q}(\partial \Omega)$ if there is a sequence of families $\{f_{j,v}\}_{|j| \leq \langle x \rangle}$, $v = 0, 1, 2, ...$ with $f_{j,v} \in L^p(\partial \Omega)$ such that for $v = 0, 1, 2, ...$ and certain numbers $a_v$ satisfying $(\sum a_v^q)^{1/q} < \infty$, we have that

(a) $\|f_j - f_{j,v}\|_{L^p(\partial \Omega)} \leq 2^{-\nu(|j| - |\langle x \rangle|)} a_v$, $|j| \leq k$,

(b) $\|f_{j,v} - f_{j,v+1}\|_{L^p(\partial \Omega)} \leq 2^{-\nu(|j| - |\langle x \rangle|)} a_v$ if $\alpha = |j| = k + 1$,

(c) $(2^{\nu(|j| - 1)} \sum_{|\nu - j| \leq 2^{\nu - 1}} |R_{\nu,v}(x, y)|^p d(x) d(y))^{1/p} \leq 2^{-\nu(|j| - |\langle x \rangle|)} a_v$,

and

(d) $\|f_{j,v}\|_{L^p(\partial \Omega)} \leq a_v$, $|j| = \langle x \rangle$.

Here $j$ is an multiindex and $\langle x \rangle$ denotes the integer part. The norm of $\{f_j\}_{|j| \leq k}$ in $B^\alpha_{p,q}(\partial \Omega)$ is $\inf(\sum a_v^q)^{1/q}$ where the infimum is taken over all possible sequences $\{a_v\}$. Further, $R_{\nu,v}$ (or $R_{\nu,v}$), is the remainder in the formal Taylor series expansion of $f_{j,v}$, i.e. $R_{\nu,v}(x, y) = f_\nu(x) - P_\nu(x, y)$, where the Taylor polynomial is given by $P_\nu(x, y) = \sum_{|j| \leq k} (f_j(x)/j!) (x - y)^j$.

We refer to [JW] for further details. Note that $B^\alpha_{p,q} = Lip(x, p, q, \partial \Omega)$ for non-integer $\alpha$ and, thus, for $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$: the space $B^\alpha_{p,q}(\partial \Omega)$ just defined is equivalent to the space obtained in the definition given previously, so there is no ambiguity in the definitions given. We will again use $B^\alpha_{p,q}(d\Omega)$ to denote $B^\alpha_{p,q}(d\Omega)$.
Lemma 1.7. For $1 \leq p \leq \infty$ and $0 < \alpha < 1$ we have that

$$\text{WA}_p^{\alpha,+}(\partial \Omega) = B_p^{\alpha,+}(\partial \Omega),$$

with equivalence of norms.

Proof. We first consider the case $p < \infty$. We will prove the continuous inclusion $B_p^{\alpha,+}(\partial \Omega) \subset \text{WA}_p^{\alpha,+}(\partial \Omega)$. Suppose $f = \{f_j\}_{j \in \mathbb{N}} \in B_p^{\alpha,+}(\partial \Omega)$, where $j$ is a multiindex in $\mathbb{R}^{n+1}$. We can identify $f$ and $f = (f_0, f_1, ..., f_n)$. From the extension theorem, [JW], there is an $F \in B_p^{\alpha,+1/\beta}(\mathbb{R}^n)$ extending $f = \{f_j^{(0)}\}_{j \leq 1}$. Thus, $D_j F \in B_p^{\alpha+1/\beta}(\mathbb{R}^n)$ and from the restriction theorem it is clear that $f_j \in B_p^{\alpha}(\partial \Omega)$, $j = 1, ..., n$. Furthermore,

$$\| f_j | B_p^{\alpha}(\partial \Omega) \| \leq C \| D_j F | B_p^{\alpha+1/\beta}(\mathbb{R}^n) \| \leq C \| F | B_p^{\alpha+1/\beta}(\mathbb{R}^n) \|.$$

We next show that $\hat{f} \in C(\overline{\Omega})$, which implies that $f_0 \in L_p(\partial \Omega)$. Since $p < \infty$ there exists $F_\ell \in \mathscr{S}(\mathbb{R}^n)$, the Schwartz class, such that $\| F - F_\ell \| B_p^{\alpha+1/\beta}(\mathbb{R}^n) \| \to 0$ as $\ell \to \infty$. Define $f_\ell = f_\ell = (f_{\ell,0}, f_{\ell,1}, ..., f_{\ell,n})$ where $f_{\ell,j} \equiv D_j F_\ell |_{\partial \Omega}$. Since $1 + \alpha$ is non-integer we have that $B_p^{\alpha}(\partial \Omega) = \text{Lip}(1 + \alpha, p, \partial \Omega)$ and therefore

$$\| f - f_\ell | L_p(\partial \Omega) \| \leq C \| f - f_\ell | B_p^{\alpha}(\partial \Omega) \| \leq C \| F - F_\ell | B_p^{\alpha+1/\beta}(\mathbb{R}^n) \|,$$

for $| j | \leq 1$. Let $(Z, \varphi)$ be a coordinate chart. Then

$$\frac{\partial}{\partial x_j} (f_{0,\ell}(x, \varphi(x))) = \frac{\partial}{\partial x_j} (F_\ell(x, \varphi(x)))$$

$$= f_{\ell,j}(x, \varphi(x)) + \frac{\partial \varphi}{\partial x_j} (x, f_{\ell,j}(x, \varphi(x))) + \frac{\partial \varphi}{\partial x_j} f_n$$

in $L_p$, so that $\hat{f} \in C(\overline{\Omega})$. Moreover,

$$\| \hat{f} | \text{WA}_p^{\alpha,+}(\partial \Omega) \| = \| f_0 | L_p^{1}(\partial \Omega) \| + \sum_{j=1}^{n} \| f_j | B_p^{\alpha}(\partial \Omega) \|$$

$$\leq C \| F | B_p^{\alpha+1/\beta}(\mathbb{R}^n) \| \leq C \| f | B_p^{\alpha+1/\beta}(\mathbb{R}^n) \|,$$

so that the inclusion is continuous.

For the reversed inclusion let $f \in \text{WA}_p^{\alpha,+}(\partial \Omega)$. We identify the array $\hat{f}$ with $f = \{f_j^{(0)}\}_{j \leq 1}$ as above. Since $B_p^{\alpha}(\partial \Omega) = \text{Lip}(1 + \alpha, p, \partial \Omega)$, see [JW] for the appropriate definition, we have to show that for $\nu = 0, 1, 2, ...$

$$\left(2^{\nu(n-1)} \right) \int_{|P-Q| < 2^{-\nu}} |R_j(P, Q)|^p \; d\sigma(P) \; d\sigma(Q) \right)^{1/p} \leq 2^{-\nu(1 + \alpha - |j|)} d_\nu,$$  \hspace{1cm} (1.3)
for \(|j| \leq 1\) and some sequence \((a_\nu)_{\nu \geq 0}\) with \(\sum_{\nu \geq 0} a_\nu^p < \infty\). The norm of \(f\) in \(\text{Lip}(1 + \alpha, p, p, \partial \Omega)\) is

\[
\sum_{|j| \leq 1} \|f_j\|_{L^p(\partial \Omega)} + \inf \left( \sum_{\nu \geq 0} a_\nu^p \right)^{1/p},
\]

where the infimum is taken over all such sequences. Put for \(c_1 > 0\)

\[
a_{j, \nu} = \left( 2^{2\nu} 2^{(n-1)} \int_{|P - Q| < c_1 2^{-\nu}} |f_j(P) - f_j(Q)|^p \, d\sigma(P) \, d\sigma(Q) \right)^{1/p};
\]

then \(\|f_j\|_{B^p_\infty(\partial \Omega)} \sim \|f_j|_{L^p(\partial \Omega)}\) and \((\sum_{\nu \geq 0} a_\nu^p)^{1/p}\), see for example [JW, p. 114]. Let now \((Z_k, \varphi_k)_{k=1}^N\) be an atlas on \(\partial \Omega\). Let \(\psi_k \in C_0^\infty(\mathbb{R}^n)\) be a partition of unity subordinate to this cover so that \(\sum_{k=1}^N \psi_k \equiv 1\) on \(\partial \Omega\). Let \(\psi_k f\) be the array given by

\[
\psi_k f = \left( \psi_k f_0, \psi_k f_1 + f_0 \partial \psi_k / \partial x_j \right).
\]

Then \(\psi_k f \in \text{WA}_{+1}(\partial \Omega)\) and

\[
\sum_{k=1}^N \psi_k f = \left( \sum_{k=1}^N \psi_k f \right)_{|j| \leq 1} = f.
\]

Hence, \(f \in \text{Lip}(1 + \alpha, p, p, \partial \Omega)\) if the same is true for \(\psi_k f\), each \(k\). Suppressing the index \(k\) in the following we can assume that \(P, Q \in \partial \Omega\) are contained in the same coordinate chart. We first deal with the case \(|j| = 1\). Then,

\[
R_j(P, Q) = (\psi f)^{(1)}(P) - (\psi f)^{(1)}(Q)
\]

\[= \psi(P)(f_j(P) - f_j(Q)) + f_j(Q)(\psi(P) - \psi(Q)) + \partial \psi / \partial x_j (P)(f_0(P) - f_0(Q)) + f_0(Q) \left( \partial \psi / \partial x_j (P) - \partial \psi / \partial x_j (Q) \right).
\]

Hence,

\[
2^{(n-1)} \int_{|P - Q| < c_1 2^{-\nu}} |R_j(P, Q)|^p \, d\sigma(P) \, d\sigma(Q) \right)^{1/p}
\]

\[
\leq C 2^{-n} \left[ a_{j, \nu} + a_{\alpha, \nu} + 2^{-n(1-n)} \|f_j|_{L^p(\partial \Omega)}\| + ||f_0|_{L^p(\partial \Omega)}\| \right].
\]

In the case \(|j| = 0\) we have for \(P, Q \in \partial \Omega \cap (Z, \varphi)\) with \(P = (x, \varphi(x))\), \(Q = (y, \varphi(y))\), \(\psi(x) = \psi(x, \varphi(x))\), \((\partial \psi / \partial \varphi_j)(x) = (\partial \varphi_j / \partial x_j)(x, \varphi(x))\) and \(f_j(x) = f_j(x, \varphi(x))\), that
\[ R_P^0 (P, Q) = \psi(x) f_0(x) - \psi(y) f_0(y) \]
\[ - \sum_{j=1}^{n-1} \left( \psi(x) f_0(y) + f_0(y) \frac{\partial \psi}{\partial x_j} (y) \right) (x_j - y_j) \]
\[ - \left( \psi(y) f_0(x) + f_0(x) \frac{\partial \psi}{\partial x_n} (x) \right) (\varphi(x) - \varphi(y)) \].

Writing a function as the integral of its derivative,
\[ \psi(x) f_0(x) - \psi(y) f_0(y) = \int_0^1 \frac{d}{dt} ( \psi(y + t(x - y)) f_0(y + t(x - y)) ) \, dt, \]
putting \( \tilde{y} = y + t(x - y) \), using the chainrule and the compatibility condition for \( f \), we get
\[ R_P^0 (P, Q) = \sum_{j=1}^{n-1} \int_0^1 \left( f_j(\tilde{y}) \psi(\tilde{y}) - f_j(y) \psi(y) \right) (x_j - y_j) \, dt \]
\[ + \sum_{j=1}^{n-1} \int_0^1 \left( f_0(\tilde{y}) \frac{\partial \psi}{\partial x_j} (\tilde{y}) - f_0(y) \frac{\partial \psi}{\partial x_j} (y) \right) (x_j - y_j) \, dt \]
\[ + \left\{ \sum_{j=1}^{n-1} \int_0^1 \frac{\partial \varphi}{\partial x_j} (\tilde{y}) (x_j - y_j) \, dt - (\varphi(x) - \varphi(y)) \right\} (x_j - y_j) \]
\[ + \sum_{j=1}^{n-1} \int_0^1 \left( f_0(\tilde{y}) \frac{\partial \psi}{\partial x_n} (\tilde{y}) - f_0(y) \frac{\partial \psi}{\partial x_n} (y) \right) \frac{\partial \varphi}{\partial x_j} (\tilde{y}) (x_j - y_j) \, dt \]
\[ + \left\{ \sum_{j=1}^{n-1} \int_0^1 \frac{\partial \varphi}{\partial x_n} (\tilde{y}) (x_j - y_j) \, dt - (\varphi(x) - \varphi(y)) \right\} \frac{\partial \varphi}{\partial x_n} (y) f_0(y). \]

The expression within the curled brackets is zero. Thus,
\[ (2^{n-1}) \int_{\|P-Q\| < 2^{-l}} |R_P^0 (P, Q)|^p \, d\sigma(P) \, d\sigma(Q) \] \[ \leq C 2^{-n(1 + a)} \sum_{j=1}^{n} (b_{j^+} + c_{j^-}), \quad (1.5) \]
where
\[ b_{j^+} = \left( 2^{np} 2^{n-1} \right) \int_{|x-y| < 2^{-l}} \left| f_j(\tilde{y}) \psi(\tilde{y}) - f_j(y) \psi(y) \right|^p \, dx \, dy \] \[ \left( 1/p \right) \]
and

\[ h_{\lambda_j} = \left( 2^{n+2^{n-1}} \int_{|x-y| < 2^{-t}} \left| f_\lambda(y) \frac{\partial \psi}{\partial x_j}(\tilde{y}) - f_\lambda(y) \frac{\partial \psi}{\partial x_j}(y) \right|^p \, dt \, dx \, dy \right)^{1/p}, \]

for \( j = 1, \ldots, n \). The series \( \sum_{r \geq 0} h_{r \lambda_j} \) and \( \sum_{r \geq 0} c_{r \lambda_j} \) are of the same type and we will only give the estimate for one of them, the other being similar. Using Fubini we get

\[ \sum_{r \geq 0} h_{r \lambda_j} = \int_0^1 \left( \sum_{r \geq 0} 2^{n+2^{n-1}} \right) \times \int_{|x-y| < 2^{-t}} \left| f_\lambda(x) \psi(x) - f_\lambda(y) \psi(y) \right|^p \, dx \, dy \, dt. \]

Now, with \( g_j(x) = f_\lambda(x) \psi(x) \) and \( A(k, t, x, y) = \{ 2^{-(k+1)t} < |x-y| < 2^{-kt} \}, \)

\[ \sum_{r \geq 0} 2^{n+2^{n-1}} \int_{|x-y| < 2^{-t}} \left| f_\lambda(x) \psi(x) - f_\lambda(y) \psi(y) \right|^p \, dx \, dy = \sum_{r \geq 0} 2^{n+2^{n-1}} \sum_{k \geq r} \int_{A(k, t, x, y)} \left| g_j(x) - g_j(y) \right|^p \, dx \, dy, \]

which by Hardy's inequality is less than

\[ C \sum_{r \geq 0} 2^{n+2^{n-1}} \int_{d(r, t, x, y)} \left| g_j(x) - g_j(y) \right|^p \, dx \, dy \]

\[ \leq C \sum_{r \geq 0} 2^{n+2^{n-1}} \int_{d(r, t, x, y)} \left| g_j(x) - g_j(y) \right|^p \, dx \, dy \]

\[ = C t^{n+2^{n-1}} \int_{|x-y| < t} \left| g_j(x) - g_j(y) \right|^p \, dx \, dy \]

and using that \( t < 1 \) gives

\[ \leq C \sum_{r \geq 0} 2^{n+2^{n-1}} \int_{d(r, 1, x, y)} \left| g_j(x) - g_j(y) \right|^p \, dx \, dy \]

\[ \leq C t^{n+2^{n-1}} \sum_{r \geq 0} 2^{(n+2^{n-1})} \int_{|P-Q| < C(M) 2^{-s}} \left| f_\lambda(P) \psi(P) - f_\lambda(Q) \psi(Q) \right|^p \, d\sigma(P) \, d\sigma(Q) \]

\[ \leq C t^{n+2^{n-1}} \left\| f_\lambda \psi \right\|_{B^p_2(\partial \Omega)}^p \leq C t^{n+2^{n-1}} \left\| f_\lambda \right\|_{B^p_2(\partial \Omega)}^p, \]
where we again have used the equivalent norm for $B^s_p$, [JW, p. 114] and, for the last inequality, e.g., the extension/restriction theorem for $B^s_p(\partial \Omega)$.

Further, $C(M)$ is a constant depending only on $M$, the Lipschitz constant for the function giving the boundary. Consequently,

$$
\sum_{r \geq 0} b_r \leq C \int_0^1 t^{\alpha + s - 1} \frac{dt}{\mu - 1} \| f_j \| B^s_p(\partial \Omega) \| ^p \leq C \| f_j \| B^s_p(\partial \Omega) \| ^p,
$$

and in the same way,

$$
\sum_{r \geq 0} c_r \leq C \| f_0 \| B^s_p(\partial \Omega) \| ^p \leq C \| f_0 \| L^p(\partial \Omega) \| ^p.
$$

From (4)–(7) we obtain for $|j| \leq 1$

$$
\left( \sum_{r \geq 0} a_r \right)^{1/p} \leq C \| f \| W^{s/p}_p(\partial \Omega),
$$

where

$$
a_r = \sum_{j=1}^n \left[ a_{0,r} + a_{1,r} + a_{2,r} + c_{0,r} + 2 - n (1 + s - |j|) \right].
$$

so that

$$
\left( \sum_{r \geq 0} a_r \right)^{1/p} \leq C \| f \| W^{s/p}_p(\partial \Omega),
$$

which gives the desired continuous inclusion. The constant in the imbedding depends on the Lipschitz character of the domain.

It remains to treat the case $p = \infty$. The same type of arguments just used will show that $W^{s/p}_p(\partial \Omega) \subset B^s_p(\partial \Omega)$. Here of course the norms used should be interpreted in the natural limiting way. For example, the expression in (1.3) should be replaced by

$$
\sup_{|Q - P| \leq 2^{-r}} | R_j(P, Q) | \leq 2^n (1 + s - |j|) a_r, \quad |j| \leq 1,
$$

where the supremum is with respect to $d\sigma$ and $(a_r)_{r \geq 0}$ is a sequence such that $\sup_{r \geq 0} a_r < \infty$. To see the reversed imbedding we note first that by the extension/restriction theorem of [JW] we have that $B^s_p(\partial \Omega) = B^s_p(\Omega^*) |_{\partial \Omega}$ and that $B^s_p(\Omega^*)$ is the ordinary Besov space on $\Omega^*$ which coincides with the Lipschitz type space $A^1_\infty(\Omega^*)$, which in turn coincides, for non-integer values of the smoothness index, with the Lipschitz space $Lip(1 + s, \Omega^*)$. This space consists of $C^1(\Omega^*)$ functions such that the func-
tion and its first derivatives are bounded on $\mathbb{R}^n$, and the first derivatives are Hölder continuous of order $\alpha$. Thus we get that $W^{\alpha,p}_{1+a}(\partial \Omega) \subset B^{\alpha,p}_{1+a}(\partial \Omega)$.

**Theorem 1.8.** For $0 < \alpha < 1$, and $1 < p < \infty$ we have the real interpolation scale:

$$(W^{\alpha}_1(\partial \Omega), W^{\alpha}_2(\partial \Omega))_{s,p} = W^{\alpha,p}_{1+a}(\partial \Omega).$$

**Proof.** For notation conventions and further information in connection with the standard interpolation techniques, we refer to [BL]. We first show that the interpolation space is contained in $W^{\alpha,p}_{1+a}(\partial \Omega)$. Note that since $p < \infty$ we have that $W^{\alpha}_p(\partial \Omega) \subset W^{\alpha,p}_{1+a}(\partial \Omega)$, the normed space of intersection, is dense in the interpolation space. Let now $f \in W^{\alpha}_p(\partial \Omega) \subset W^{\alpha,p}_{1+a}(\partial \Omega)$ be such that $f = g + h$ where $g \in W^{\alpha}_1(\partial \Omega)$ and $h \in W^{\alpha}_2(\partial \Omega)$. Let $(Z_l, \varphi_l)_{l=1}^N$ be an atlas on $\partial \Omega$ and take a partition of unity subordinate to this covering. Suppressing the index $l$ as above and modifying the array to

$$(\hat{\psi} f) = (\psi f_0, \psi f_j + f_0 \frac{\partial \psi}{\partial x_j}) \in W^{\alpha,p}_{1+a}(\partial \Omega),$$

we see that $f$ is a sum of such arrays and that $\hat{\psi} f = \hat{\psi} g + \hat{\psi} h$. Here $\psi \in C^\infty_c(\mathbb{R}^n)$. Letting $\Phi: \mathbb{R}^{n-1} \to \partial \Omega$ be the mapping $\Phi(x) = (x, \varphi(x))$ for $x$ in the appropriate chart, we have that

$$||\hat{\psi} f||_{W^{\alpha,p}_{1+a}(\partial \Omega)} \leq C \left( ||(\psi f)_0||_{L^p(\partial \Omega)} + \sum_{j=1}^n \left( ||\hat{\psi} * (\psi f)_j \circ \Phi||_{L^p(\mathbb{R}^{n-1})} + \sum_{k \geq 1} (2^{k\alpha} ||\tilde{\phi}_k * ((\psi f)_j \circ \Phi)||_{L^p(\mathbb{R}^{n-1})})^p \right) \right). \quad (1.8)$$

Here $\hat{\psi}$ and $\tilde{\phi}_k$ denote the functions in the definition of the Besov spaces, see [BL, p. 139]. The K-functional is given by

$$K(t, f) = \inf_{\tilde{g} + \tilde{h} = f} \left( ||\tilde{g}||_{W^{\alpha}_1(\partial \Omega)} + t ||\tilde{h}||_{W^{\alpha}_2(\partial \Omega)} \right).$$

**Claim.** For $j = 1, \ldots, n$ and $k \geq 1$ we have that

1. $||\tilde{\phi}_k * ((\psi f)_j \circ \Phi)||_{L^p(\mathbb{R}^{n-1})} \leq CK(2^{-k}, f)$,
2. $||\hat{\psi} * (\psi f)_j \circ \Phi||_{L^p(\mathbb{R}^{n-1})} \leq CK(1, f)$,
3. $||\psi f_0||_{L^p(\partial \Omega)} \leq K(1, f)$.
To see the claim we note that
\[
\| \tilde{\phi}_k * ((\psi f)_j \cdot \Phi) \|_{L^p(\mathbb{R}^{n-1})} \\
\leq C \left( \| \psi g \|_{L^p(\mathbb{R}^{n-1})} + 2^{-k} \| J^1(\psi h)_j \cdot \Phi \|_{L^p(\mathbb{R}^{n-1})} \right) \\
\leq C \left( \| \psi g \|_{L^p(\partial \Omega)} + 2^{-k} \| (\psi h)_j \|_{L^p(\partial \Omega)} \right) \\
\leq C \left( \| \psi g \|_{\text{WA}_{1}^p(\partial \Omega)} + 2^{-k} \| \psi h \|_{\text{WA}_{1}^p(\partial \Omega)} \right),
\]
where $J^1$, the Bessel potential of order $-1$, acts on $f$ as $\mathcal{F}^{-1}\{(1+|\cdot|^2)^{-1/2}f\}$. Taking infimum gives (i), (ii) is proved in a similar manner, and (iii) is immediate. Using the claim it follows for the interpolation norm, $\| \cdot \|_{p,p}$, that
\[
\left\| \chi \right\|_{p,p} \geq \left( \sum_{k \geq 1} \int_{2^{-k-1}}^{2^{-k}} (2^k K(t, \chi)) \frac{dt}{t} \right)^{1/p} \geq C \left( \sum_{k \geq 1} (2^k K(2^{-k}, \chi)) \right)^{1/p} \\
\geq C \left( \sum_{k \geq 1} (2^k \| \tilde{\phi}_k * ((\psi f)_j \cdot \Phi) \|_{L^p(\mathbb{R}^{n-1})}) \right)^{1/p}.
\]
This together with similar estimates for the remaining terms in (1.8) give that
\[
\| \phi \|_{\text{WA}_{1}^p(\partial \Omega)} \leq C \| \phi \|_{p,p},
\]
where the constant depends on the Lipschitz character of the domain.

For the reverse estimate we will use the $J$-functional, the extension/restriction theorem and an endpoint trace formula. Suppose $f \in \text{WA}_{1}^p(\partial \Omega)$. The interpolation norm is given by
\[
\| f \|_{p,p} = \inf_{u} \left( \int_0^\infty \frac{(1-\theta J(t, \dot{u}(t))) \frac{dt}{t} \right)^{1/p},
\]
where $J$ is the $J$-functional, the infimum is taken over all $\dot{u}$ such that the integral is finite and
\[
\dot{f} = \int_0^\infty \dot{u}(t) \frac{dt}{t},
\]
with $\dot{u}$ having values in $\text{WA}_{2}^p(\partial \Omega) = M(\text{WA}_{1}^p(\partial \Omega), \text{WA}_{2}^p(\partial \Omega))$. Now, by Lemma 1.7 and the extension/restriction theorem of [JW], there exist an extension $F \in B_{1+x+1/p}^p(\mathbb{R}^n)$ of $f$ such that
\[
\| F \|_{B_{1+x+1/p}^p(\mathbb{R}^n)} \leq C \| \dot{f} \|_{\text{WA}_{1+x}^p(\partial \Omega)},
\]
and \( D_j F \rvert_{\partial \Omega} = f_j \), for \( j = 0, 1, ..., n \). Let as before \( \tilde{\varphi}_k \) and \( \tilde{\psi} \) be functions in the definition of the Besov spaces on \( \mathbb{R}^n \). Then

\[
F = \tilde{\psi} \ast F + \sum_{k \geq 1} \tilde{\varphi}_k \ast F,
\]
in \( B^s_{\infty, 1}(\mathbb{R}^n) \) and

\[
D_j(F) = \tilde{\psi} \ast D_j(F) + \sum_{k \geq 1} \tilde{\varphi}_k \ast D_j(F),
\]
in \( B^s_{\infty, 1}(\mathbb{R}^n) \). We thus have that

\[
f_j = D_j(F) \rvert_{\partial \Omega} = (\tilde{\psi} \ast D_j(F)) \rvert_{\partial \Omega} + \sum_{k \geq 1} (\tilde{\varphi}_k \ast D_j(F)) \rvert_{\partial \Omega},
\]
for \( j = 0, 1, ..., n \). Define for \( 0 < t < \infty \), \( \bar{u}(t) = (u_0(t), u_1(t), ..., u_n(t)) \) by

\[
\bar{u}_j(t) = \begin{cases} 
(\ln 2)^{-1} (\tilde{\psi} \ast D_j(F)) \rvert_{\partial \Omega} & \text{for } 1/2 < t \leq 1, \\
(\ln 2)^{-1} (\tilde{\varphi}_k \ast D_j(F)) \rvert_{\partial \Omega} & \text{for } 2^{-(k+1)} \leq t \leq 2^{-k}, \\
0 & \text{if } t > 1.
\end{cases}
\]
Then,

\[
\int_0^\infty \bar{u}_j(t) \frac{dt}{t} = (\tilde{\psi} \ast D_j(F)) \rvert_{\partial \Omega} + \sum_{k \geq 1} (\tilde{\varphi}_k \ast D_j(F)) \rvert_{\partial \Omega} = f_j,
\]
with convergence in \( \sum (WA^1(\partial \Omega), WA^2(\partial \Omega)) = WA^2(\partial \Omega) \). Since \( u(t) \in C^\infty, \bar{u}(t) \in A(WA^1(\partial \Omega), WA^2(\partial \Omega)) = WA^2(\partial \Omega) \). Hence, we have to estimate the various norms appearing in the last expression. This is the content of the next lemma.
Lemma 1.9. (i) $\| (\tilde{\phi}_k \ast D_j F) \|_{L^p (\partial \Omega)} \leq C 2^{k(1 + 1/p)} \| \tilde{\phi}_k \ast F \|_{L^p (\mathbb{R}^n)}$, 
(ii) $\| (\tilde{\phi}_k \ast D_j F) \|_{L^p (\partial \Omega)} \leq C 2^{k(2 + 1/p)} \| \tilde{\phi}_k \ast F \|_{L^p (\mathbb{R}^n)}$, 
(iii) $\| (\tilde{\psi} \ast D_j F) \|_{L^p (\partial \Omega)} \leq C \| \tilde{\psi} \ast F \|_{L^p (\mathbb{R}^n)}$, 
(iv) $\| (\tilde{\psi} \ast D_j F) \|_{L^p (\partial \Omega)} \leq C \| \tilde{\psi} \ast F \|_{L^p (\mathbb{R}^n)}$.

The lemma will be proved below. Taking the lemma for granted for a moment it is easy to see that,

$$
\| \tilde{\psi} \ast F \|_{L^p (\mathbb{R}^n)} + \left( \sum_{k \geq 1} (2^k (1 + 1/p)) (\| \tilde{\phi}_k \ast F \|_{L^p (\mathbb{R}^n)})^p \right)^{1/p} 
\leq C \| F \|_{L^p (\mathbb{R}^n)} + \| F \|_{W^{1, \frac{p}{p-1}} (\partial \Omega)},
$$

by choosing $\theta = \alpha$. This ends the proof of the theorem.

It remains to prove the last lemma. We need a variation of an endpoint result first due to Agmon and Hörmander, [AH], in the case $p = 2$ and Peetre, [P], and Polking, [Po], in the general case.

Lemma 1.10. The trace mapping $\text{tr}: B^1_0 (\mathbb{R}^n) \to L^p (\partial \Omega)$ is a bounded mapping, where $\partial \Omega$ is a Lipschitz graph and $1 < p < \infty$.

Proof. When $D = \mathbb{R}^*_+$, the result is given in the works just mentioned. Put $F(x, t) = H(x, \varphi (x) + t)$. Then, from the $\mathbb{R}^*_+$ case we have that

$$
\| H(\cdot, \varphi (\cdot)) \|_{L^p (\mathbb{R}^{n-1})} = \| F(\cdot, 0) \|_{L^p (\mathbb{R}^{n-1})} \leq C \| F \|_{B^1_0 (\mathbb{R}^n)}.
$$

We need to bound the right-hand side by the same norm of $H$. To do this we need an equivalence of norms, [JW, p. 121], which together with a change of variables shows that

$$
\| F \|_{B^1_0 (\mathbb{R}^n)} \sim \| F \|_{L^p (\mathbb{R}^n)} + \sum_{r \geq 0} 2^{r/p} \left( \int_{|h| < 2^r} \int \int_{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}} |H(x + h', \varphi (x + h') + t + h_a) - H(x, \varphi (x) + t)|^p \, dx \, dt \, dh \right)^{1/p},
$$

where $h = (h', h_a)$. Using Fubini's theorem and changes of variables the desired result easily follows.
**Proof of Lemma 1.9.** We will give the estimate of $\| (\tilde{\phi}_k \ast D_j F) \|_{\partial \Omega} | L^p(\partial \Omega)]$, the other estimates being similar. Using a partition of unity argument, the previous lemma and various standard estimates, we get that

$$
\left\| (\tilde{\phi}_k \ast D_j F) \right\|_{\partial \Omega} | L^p(\partial \Omega)] \\
\leq C \sum_{i=0}^{n} \| D_i (\tilde{\phi}_k \ast D_j F) \|_{L^p(\partial \Omega)} \\
\leq C \sum_{i=0}^{n} \| D_i (\tilde{\phi}_k \ast D_j F) \|_{B^{1,1}_{\tilde{p}}(\mathbb{R}^n)} \\
= C \sum_{i=0}^{n} \left[ \left\| \tilde{\psi}_k \ast D_i (\tilde{\phi}_k \ast D_j F) \right\|_{L^p(\mathbb{R}^n)} \right] \\
+ \left( \sum_{\ell \geq n+1} \left( 2^{\ell/p} \| \tilde{\phi}_\ell \ast D_i (\tilde{\phi}_k \ast D_j F) \|_{L^p(\mathbb{R}^n)} \right) \right) \\
\leq C \left( \left\| \tilde{\psi}_k \ast D_j F \right\|_{L^p(\mathbb{R}^n)} + \sum_{\ell \geq n+1} 2^{\ell/p} \| \tilde{\phi}_\ell \ast D_j F \ast \tilde{\phi}_k \|_{L^p(\mathbb{R}^n)} \right) \\
\leq C \left[ 2^k \left\| \tilde{\psi}_k \ast D_j F \ast \tilde{\phi}_k \right\|_{L^p(\mathbb{R}^n)} \right] \\
+ \sum_{\ell \geq n+1} 2^{\ell/p} 2^k \| \tilde{\phi}_\ell \ast D_j F \ast \tilde{\phi}_k \|_{L^p(\mathbb{R}^n)} \right) \\
\leq C \left[ 2^{2k} \left\| \tilde{\psi}_k \ast F \ast \tilde{\phi}_k \right\|_{L^p(\mathbb{R}^n)} \right] + \sum_{\ell \geq n+1} 2^{\ell/p} 2^{2k} \| \tilde{\phi}_\ell \ast F \ast \tilde{\phi}_k \|_{L^p(\mathbb{R}^n)} \right) \\
\leq C \left[ 2^{2k} \left\| F \ast \tilde{\phi}_k \right\|_{L^p(\mathbb{R}^n)} \right] + \sum_{\ell \geq n+1} 2^{\ell/p} 2^{2k} \| F \ast \tilde{\phi}_k \|_{L^p(\mathbb{R}^n)} \right) \\
\leq C 2^{(1+1/p)k} \left\| F \ast \tilde{\phi}_k \right\|_{L^p(\mathbb{R}^n)} \right),
$$

and by the support properties of the functions $\tilde{\phi}_\ell$.

As is well known, functions with a certain degree of regularity in a domain will have a trace on the boundary of “1/p less regularity.” In this direction there are restriction and extension theorems for domains of a quite general nature. We have used above special cases of more general results of Jonsson–Wallin. See [JW], pp. 141, 182 and 205–209. The restriction of functions to the sets considered there are in the sense of strictly defined functions. We know from the restriction theorem of [JW] that $L^p(\mathbb{R}^N)_{\partial \Omega} = B_{\tilde{p}}^{1,1}(\tilde{\partial} \Omega), 1 < p < \infty, \beta - 1/p > 0$, where the restriction is in the sense of strictly defined functions. Let $E$ be the linear extension.
operator of Stein, \([S]\). Then, \(E: W^t_p(\Omega) \rightarrow W^s_p(\mathbb{R}^n)\) is bounded for every non-negative integer \(k\) and every \(1 \leq p \leq \infty\). So \(E\) extends, via interpolation and the fact that \(L^t_p = W^t_p\) for \(1 < p < \infty\), to a mapping \(E: L^t_p(\Omega) \rightarrow L^s_p(\mathbb{R}^n)\) for all \(\beta \geq 0, 1 < p < \infty\). For \(u \in L^t_p(\Omega)\) we define by an abuse of notation, \(\text{tr}(u) = (\text{tr}(Eu), \text{tr}(\nabla Eu))\), if \(\beta > 1 + 1/p\) and \(\text{tr}\) is in the sense of strictly defined functions. It should be clear from the smoothness degree when \(\text{tr}\) denotes the restriction of the function and when it also includes the gradient of the function. Otherwise this will be explicitly stated. Also, for \(u \in C^1(\Omega)\) we, again by an abuse of notation, define \(\text{Tr}(u) = (\text{Tr}(u), \text{Tr}(\nabla u))\), to be the pointwise restriction. For \(\psi \in C^\infty(\mathbb{R}^n)\), \(\text{tr}(\psi)\) coincides with the pointwise restriction. If \(u \in L^t_p(\Omega)\), with \(1 + 1/p < \beta\), then \(\text{Tr}\) extends by continuity, and we have the following fact.

**Proposition 1.11.** Suppose \(u \in L^t_p(\Omega)\) where \(1 < p < \infty, 1 + 1/p < \beta < 2 + 1/p\). Then \(\text{tr} u \equiv (\text{tr}(Eu), \text{tr}(\nabla Eu)) = (\text{Tr}(u), \text{Tr}(\nabla u)) \equiv \text{Tr} u\).

Similar considerations apply to the Besov spaces. Combining the results of [JW], in the special case of a Lipschitz domain, and the above notes, we get:

**Theorem 1.12.** Suppose that \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^n\), that \(1 + 1/p < \pi < 2 + 1/p\), and that \(s = \pi - 1/p\). If \(1 < p < \infty\) then the mapping \(\text{Tr}\), initially defined on \(C^\infty(\mathbb{R}^n)\) as the restriction to \(\partial \Omega\) of the function and its gradient, extends to a bounded linear operator from \(L^t_p(\Omega)\) to \(WA^s_p(\partial \Omega)\). If \(1 \leq p \leq \infty\), then the mapping \(\text{Tr}\) extends to a bounded linear operator from \(B^s_p(\Omega)\) to \(WA^s_p(\partial \Omega)\). Moreover, there is a linear extension operator \(\mathcal{E}\) that maps \(WA^s_p(\partial \Omega)\) to \(B^s_p(\Omega)\) for all \(0 < s - 1 < 1\) and all \(p, 1 < p < \infty\). In addition, \(\mathcal{E}\) maps \(WA^s_p(\partial \Omega)\) to \(B^s_p\) for \(p = 1\) and \(\infty\), and \(\text{Tr} \circ \mathcal{E}\) is the identity operator on \(WA^s_p(\partial \Omega)\).

The other trace that we would like to consider appears naturally in connection with solutions to boundary value problems and is the notion of non-tangential limits, which we denote by \(\text{Tr}_n\) and

\[
\text{Tr}_n u(P) = \lim_{x \to P, x \in C(P)} (u(x), \nabla u(x)),
\]

where \(C(P)\) is the associated cone at \(P \in \partial \Omega\) contained in the Lipschitz domain \(\Omega\). It is not true of course that any function defined in \(\Omega\) has a non-tangential trace. For solutions however this property appears naturally. We will prove that our solutions are in fact in certain Sobolev and Besov spaces; thus they have a classical trace. We will defer a discussion of non-tangential traces for solutions and their relation to the classical traces until after we have the results stating the Sobolev and Besov space properties of solutions.
We note that in [JW] the trace of $L^p_\varphi$ is characterized for all values of $\varphi$, but the boundary values for integer values of $\varphi - 1/p$ are very different from the spaces obtained in the non-integer case and the extension operator depends on $\varphi$ and is non-linear for $\varphi - 1/p$ being a positive integer. The relation between these trace spaces, $B^p_\varphi(\partial \Omega)$, and Sobolev spaces on the boundary are not at all straightforward in the case of non-regular domains. An example of G. David shows that in the case of a Lipschitz domain the functions in the trace space of $L^p_\varphi$ need not have a tangential derivative in $L^p_\varphi(\partial \Omega)$, i.e., the trace is not in $L^1_\varphi(\partial \Omega)$. This is contrary to the smooth case where it is well known that the trace of $L^p_\varphi$ to $\partial \Omega$ is the space $B^p_\varphi(\partial \Omega)$ and for $1 < p \leq 2$, $B^p_\varphi(\partial \Omega) \subset L^1_\varphi(\partial \Omega)$.

In fact, in the example of G. David, see [JK, p. 14], a function $g(x, y) = 3(x, y) f(y)$ is constructed, with $3$ a $C_\infty$ cut-off function and $f \in L^p_\varphi(R^2)$, but on a sawtooth region $\Omega$, 

$$\|\nabla g\|_{L^p(\partial \Omega)} \to \infty,$$

as $\varepsilon \to 0$. It is easy to see that this example also shows that the trace of $L^p_\varphi$ to $\partial \Omega$ is not in general contained in $W^1_\varphi(\partial \Omega)$, for $\Omega$ a Lipschitz domain. Put

$$u(x, y) = \psi(x, y) \int_{-\infty}^{t} \Theta(x, y) f(y) \, dy,$$

where $\psi$ is again a $C_\infty$ cut-off function. Then clearly, on a piece of the boundary where $\psi$ is equal to one, we have $D_t u = f(x, t)$. Hence, $(u|_{\partial \Omega}, Vu|_{\partial \Omega})$ will be an array whose $W^1_\varphi(\partial \Omega)$ norm tends to $\infty$ as $\varepsilon \to 0$. Also, $u \in L^p_{\varphi-1/p}(R^2)$ and $Vu \in L^p_{\varphi-1/p}(R^2)$. In [JK] David’s example is modified to extend to $C^1$ domains and such a modification also applies in our situation.

**Definition 1.13.** Let $-\infty < \varphi < \infty$ and $1 < p < \infty$. Define $L^\varphi_{p,0}(\Omega)$ as the space of all distributions $f \in L^\varphi_p(R^n)$ with support in $\Omega$. The norm is

$$\|f\|_{L^\varphi_{p,0}(\Omega)} = \|f\|_{L^\varphi_p(R^n)}.$$

Note that $C_\infty^\varphi(\Omega)$ is dense in $L^\varphi_{p,0}(\Omega)$. The following characterization is useful in uniqueness questions.

**Proposition 1.14.** Suppose that $1 + 1/p < \varphi < 2 + 1/p$, $1 < p < \infty$, and that $\Omega$ is a bounded Lipschitz domain in $R^n$. Then $L^\varphi_{p,0}(\Omega)$ is the space of all functions $u$ in $L^\varphi_p(\Omega)$ whose boundary trace,

$$\text{Tr } u = (\text{Tr } u, \text{Tr } Vu),$$

is zero.
Proof. One direction is immediate. Calderón’s extension operator, $E_1^f(x) = K_1 \ast (\chi_{\Omega} Vf)$ with $K_1$ being the kernel of a singular integral operator. The results of [JK, Prop. 3.3] show that while $\text{Tr} f = 0$ we have that $\chi_{\Omega} Vf \in L^p_{\Omega,-1}$ and

$$\|\chi_{\Omega} Vf\|_{L^p_{\Omega,-1}} \leq \|Vf\|_{L^p_{\Omega,-1}(\Omega)}.$$  

Thus, since $\text{Tr} f = 0$ implies that $E_1^f(x) = \chi_{\Omega} f$ we have that

$$V(\chi_{\Omega} f) = V K_1 \ast \chi_{\Omega} Vf \in L^p_{\Omega,-1},$$

and

$$\|V(\chi_{\Omega} f)\|_{L^p_{\Omega,-1}} \leq C \|Vf\|_{L^p_{\Omega,-1}(\Omega)}.$$  

Therefore,

$$\|\chi_{\Omega} f\|_{L^p_{\Omega}} \leq C(\|f\|_{L^p(\Omega)} + \|V(\chi_{\Omega} f)\|_{L^p_{\Omega,-1}})$$

$$\leq C(\|f\|_{L^p(\Omega)} + \|Vf\|_{L^p_{\Omega,-1}(\Omega)}) \leq C \|f\|_{L^p_{\Omega}}.$$  

This concludes the proof in the remaining direction.

**Corollary 1.15.** Let $1 < p < \infty$, $1 + 1/p < \alpha < 2 + 1/p$ and $\Omega$ a bounded Lipschitz domain in $\mathbb{R}^n$. Then $C^\infty_0(\Omega)$ is dense in the space of all functions in $L^p_{\Omega}(\Omega)$ with boundary trace zero.

The corresponding result for Besov spaces is:

**Proposition 1.16.** Suppose that $1 \leq p < \infty$, $1 + 1/p < \alpha < 2 + 1/p$ and that $\Omega$ is a bounded Lipschitz domain. Then $C^\infty_0(\Omega)$ is dense in the space of all functions $u$ in $B^\alpha_{p,\infty}(\Omega)$ whose boundary trace, $\text{Tr} u = (\text{Tr} u, \text{Tr} Vu)$, is zero.

A proof of this result follows along the same lines as the corresponding result for the range $1/p < \alpha < 1 + 1/p$, see [JK, p. 18].

2. MAIN RESULTS

**Theorem 2.1.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Consider the inhomogeneous Dirichlet problem

$$\begin{cases}
\Delta u = f & \text{in } \Omega, \\
\text{Tr}(u, Vu) = 0 & \text{on } \partial \Omega,
\end{cases}$$
where $\mathrm{Tr}(u, Vu)$ is understood in the sense of Section 1. Put $1/p_0 = (1 + \varepsilon)/2$ and $1/p'_0 = (1 - \varepsilon)/2$. Then there exists an $0 < \varepsilon = \varepsilon(\text{Lipschitz character of } \Omega)$ such that the problem can be solved uniquely in $L^p_{\alpha + 1/2}(\Omega)$ with the estimate

$$\|u\|_{L^p_{\alpha + 1/2}(\Omega)} \leq C \|f\|_{L^p(\Omega)},$$

under the following conditions on $p, \alpha$ and the dimension $n$:

(i) for $p_0 \leq p \leq p'_0$, and $1/p < 3 + \alpha < 1 + 1/p$,

(i') if $\Omega$ is a $C^1$ domain we may take $1 < p < \infty$ and $1/p < 3 + \alpha < 1 + 1/p$,

(ii) for $n = 3, 1 < p < p_0$, and $(3/p) - 1 - \varepsilon < 3 + \alpha < 1 + 1/p$,

(iii) for $n = 3, p'_0 < p < \infty$, and $1/p < 3 + \alpha < 3/p + \varepsilon$.

Theorem 2.1 will follow from the results of Section 1 and the following theorem on the homogeneous Dirichlet problem.

**Theorem 2.2.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 3$. Consider the homogeneous Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega,$$

$$\mathrm{Tr}(u, Vu) = f \in W^{1,p}_0(\partial \Omega),$$

$$u \in L^p_{\alpha + 1/2}(\Omega).$$

Put $1/p_0 = (1 + \varepsilon)/2$ and $1/p'_0 = (1 - \varepsilon)/2$. Then there exists an $0 < \varepsilon = \varepsilon(\text{Lipschitz character of } \Omega)$ such that there exists a unique $u$ solving $(*)$ under the following conditions on $p, \alpha$ and $n$:

(i) for $p_0 \leq p \leq p'_0$, and $0 < \alpha < 1$,

(i') if $\Omega$ is a $C^1$ domain we may take $1 < p < \infty$ and $0 < \alpha < 1$,

(ii) for $n = 3, 1 < p < p_0$, and $2/p - 1 - \varepsilon < \alpha < 1$,

(iii) for $n = 3, p'_0 < p < \infty$, and $0 < \alpha < 2/p + \varepsilon$.

The main previous results that we will need are as follows. We will denote the non-tangential maximal function by $N(\cdot)$ and $(\cdot)^*$ alternately.

**Theorem D(p), the Dirichlet Problem in $L^p$.** (i) [DKV]; Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, with connected boundary. Then there exist $\varepsilon > 0$, which depends only on $p$ and the Lipschitz character of $\Omega$, so that for any $2 - \varepsilon \leq p \leq 2 + \varepsilon$, and $f \in L^p(\partial \Omega), g \in L^p(\partial \Omega)$, there exists a unique $u$ in $\Omega$, such that

(a) $\Delta^2 u = 0$ in $\Omega$,

(b) $\lim_{x \to Q, x \not\in \partial \Omega} u(x) = f(Q) \ a.e. \ (d\sigma)$,

(c) $\lim_{x \to Q, x \not\in \partial \Omega} N\cdot \nabla u(x) = g(Q) \ a.e. \ (d\sigma)$,

(d) $\|N(\nabla u)\|_{L^p(\partial \Omega)} < \infty$. 

**AN INHOMOGENEOUS DIRICHLET PROBLEM**
In fact,
\[
\|N(u)\|_{L^p(\partial \Omega)} + \|N(\nabla u)\|_{L^p(\partial \Omega)} \leq C\{ \|f\|_{L^p_0(\partial \Omega)} + \|g\|_{L^p_0(\partial \Omega)} \},
\]
where C depends only on p and the Lipschitz character of \( \Omega \).

(ii) [V1]: If \( \Omega \) is a bounded \( C^1 \) domain in \( \mathbb{R}^n \), then the theorem in i) is true in the range \( 1 < p < \infty \).

(iii) [PV1]: If \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^3 \) the theorem is true also in the range \( 2 - \varepsilon < p < \infty \), for some \( \varepsilon > 0 \) depending only on the Lipschitz character of \( \Omega \).

**Theorem R(p), the Regularity Problem in \( L^p \).** (i) [V2]: Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) with connected boundary. Then there exist \( \varepsilon > 0 \), which depends only on \( p \) and the Lipschitz character of \( \Omega \), so that for any \( 2 - \varepsilon < p < 2 + \varepsilon \), and \( f = (f_1, f_2, \ldots, f_n) \in W^{1,p}_0(\partial \Omega) \), there exists a unique \( u \) in \( \Omega \), such that

(a) \( \Delta^2 u = 0 \) in \( \Omega \),
(b) \( \lim_{X \to Q, X \in \Gamma_1(\partial \Omega)} u(X) = f_0(Q) \) a.e. \((d\sigma)\),
(c) \( \lim_{X \to Q, X \in \Gamma_1(\partial \Omega)} \partial N/\partial \nu = \sum_{j=1}^n N_j f_j \) a.e. \((d\sigma)\),
(d) \( \|N(\nabla u)\|_{L^p(\partial \Omega)} < \infty \).

In fact,
\[
\|N(\nabla u)\|_{L^p(\partial \Omega)} \leq C \sum_{j=1}^n \|\nabla f_j\|_{L^p(\partial \Omega)},
\]
where C depends only on \( p \) and the Lipschitz character of \( \Omega \).

(ii) [V1]: If \( \Omega \) is a bounded \( C^1 \) domain in \( \mathbb{R}^n \), then the theorem in (i) is true in the range \( 1 < p < \infty \).

(iii) [PV1]: If \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^3 \) the theorem is true in the range \( 1 < p < 2 + \varepsilon \), for some \( \varepsilon > 0 \) depending only on the Lipschitz character of \( \Omega \).

We will have use for the corresponding endpoint results of the last theorems. The appropriate replacement of \( L^1 \) is well known to be the Hardy space \( H^1 \). We will use its atomic formulation and also smoother versions of these atomic spaces. Recall that for \( 1 < q \leq \infty \), a \((1, q)\)-atom is a function supported in a surface ball \( B = B(Q, r) = \{ P \in \partial \Omega : |P - Q| \leq r \} \), such that
\[
\|a\|_{L^q(\partial \Omega)} \leq (\sigma(B))^{(1/q) - 1},
\]
and \( \{ \sigma _{a} \} d \sigma = 0 \). Then \( H_{0}^{1}(\partial \Omega ) = \{ f \in L^{1}(\partial \Omega ) : f = \sum _{j} \lambda _{j} a_{j} \) where \( a_{j} \) are \((1, q)\)-atoms and \( \sum _{j} |\lambda _{j}| < \infty \)\}. Equivalent norms for \( H_{0}^{1}(\partial \Omega ) \) are given by

\[
\| f \| _{H_{0}^{1}(\partial \Omega )} = \inf \left\{ \sum _{j} |\lambda _{j}| : f = \sum _{j} \lambda _{j} a_{j} \text{ and } a_{j} \text{ are } (1, q)\text{-atoms} \right\}.
\]

The following spaces were defined and studied in [DK]. Let \( \Lambda \) be the graph of a Lipschitz function \( \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \). We say that \( f \) is an \( H_{1, a}^{1}(\Lambda ) - L^{q} \) atom \( q > 1 \), if \( f \in L^{q}(\Lambda ) \) with \( \nabla _{\varphi } f(x, \varphi (x)) \) supported in a ball \( B \) and each \( \partial (\partial T_{j}) f, 1 \leq j \leq n-1 \), which automatically has mean-value zero, is a \((1, q)\)-atom with

\[
\left\| \frac{\partial }{\partial T_{j}} f \right\| _{L^{q}(\Lambda )} \leq \sigma (B)^{1/q-1}.
\]

We then say that \( f \in H_{1, a}^{1}(\Lambda ) \) if there are \( H_{1, a}^{1}(\Lambda ) - L^{2} \) atoms \( f_{k} \) so that

\[
\frac{\partial }{\partial T_{j}} f = \sum _{k \geq 1} \lambda _{k} \frac{\partial }{\partial T_{j}} f_{k},
\]

for \( 1 \leq j \leq n \) and \( \sum _{k \geq 1} |\lambda _{k}| < \infty \). Putting \( \| f \| _{H_{1, a}^{1}(\Lambda )} = \inf \sum _{k \geq 1} |\lambda _{k}| \) makes \( H_{1, a}^{1}(\Lambda ) \) into a Banach space modulo constants. We could also consider spaces using \( H_{1, a}^{1}(\Lambda ) - L^{q} \) atoms with \( q \neq 2 \). The norms obtained for different \( q > 1 \) are equivalent. Now \( H_{1, a}^{1}(\partial \Omega ) - L^{q} \) atoms are defined analogously but are required to be supported in a coordinate cylinder, and \( H_{1, a}^{1}(\partial \Omega ) \) is defined to be built up from such atoms in the natural way. Finally, one can define, as in [PV1], a smoother Hardy space \( H_{2, a}^{1}(\partial \Omega ) \) analogously to the Whitney array space \( WA_{2}(\partial \Omega ) \) but with \( H_{1, a}^{1}(\partial \Omega ) \) replacing \( L^{2}(\partial \Omega ) \). We have

\[
\| f \| _{H_{2, a}^{1}(\partial \Omega )} = \sum _{j = 0}^{n} \| f_{j} \| _{H_{1, a}^{1}(\partial \Omega )}.
\]

Remember that \( \hat{f} = (f_{0}, f_{1}, \ldots , f_{n}) \) where for notational convenience we occasionally put \( \hat{f} = (f_{1}, \ldots , f_{n}) \).

**Theorem 2.3**, the Atomic Regularity Problem. (i) [PV1]: Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^{3} \), (or \( \mathbb{R}^{2} \)) with connected boundary. Given \( f \in H_{2, a}^{1}(\partial \Omega ) \), there exists a unique function \( u \) in \( \Omega \) such that

(a) \( \Lambda ^{2} u = 0 \) in \( Q \),

(b) \( \lim _{x \to Q, x \in \mathbb{R}^{1}Q} u(X) = f_{d}(Q) \) a.e. (\( d\sigma \)),

(c) \( \lim _{x \to Q, x \in \mathbb{R}^{1}Q} \nabla u(X) = \hat{f}(Q) \) a.e. (\( d\sigma \)),

(d) \( \| \nabla u \| _{L^{2}(\partial \Omega )} < \infty \).
In fact,
\[ \|N(\nabla u)\|_{L^p(\partial \Omega)} \leq C \sum_{j=1}^{n} \| \nabla T f_j \|_{H^1(\partial \Omega)}, \]
where \( C \) depends only on the Lipschitz character of \( \Omega \).

(ii) [V1]: If \( \Omega \) is a bounded \( C^1 \) domain in \( \mathbb{R}^n \), then the theorem in (i) is also true.

The area integral or square function of \( v \) at \( Q \in \partial \Omega \) is given by
\[ S_v(Q) = \left\{ \left( \int_{\Gamma(Q)} |\nabla v(X)|^2 \, d(X)^{2-n} \, dX \right)^{1/2}, \right. \]
where \( d(X) \) denotes \( \text{dist}(X, \partial \Omega) \), and \( \{ \Gamma(Q) \} \) is a regular family of cones associated to \( \Omega \) (see [D2, p. 298]).

**Theorem 2.4**, the Area Integral Estimate, [PV2]. Let \( 0 < p < \infty \). Suppose \( D \) is a Lipschitz domain in \( \mathbb{R}^n \), with \( d_\Omega \) the surface measure on \( \partial D \). Fix \( P_0 \in D \) and let \( \{ \Gamma(Q) \} \) be a regular family of cones for \( Q \in \partial D \). If \( u \) is biharmonic in \( D \) and \( \nabla u(P_0) = 0 \) then there exist constants \( C_1, C_2 \) such that
\[ \|N(\nabla u)\|_{L^p(\partial D, d_\Omega)} \leq C_1 \|S(\nabla u)\|_{L^p(\partial D, d_\Omega)} \leq C_2 \|N(\nabla u)\|_{L^p(\partial D, d_\Omega)}, \]
where \( C_1 \) and \( C_2 \) depend on \( P_0 \), on the aperture of the cones \( \{ \Gamma(Q) \} \), and on the Lipschitz character of \( D \).

**Theorem 2.5**, the Maximum Principle, [PV3].

(i) Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), (or \( \mathbb{R}^2 \)). Let \( u \) be the unique solution of \( D(2) \), the Dirichlet problem with data \( f, g \in L^1_\Omega, L^2 \). If \( |u| \in L^p(\partial \Omega) \) then
\[ \|\nabla u\|_{L^p(\partial \Omega)} \leq C \|\nabla u\|_{L^p(\partial \Omega)}, \]
for some \( C \) depending only on the Lipschitz character of \( \Omega \).

(ii) If \( \Omega \) is a bounded \( C^1 \) domain in \( \mathbb{R}^n \), then the statement in (i) is also true.

We will also need various equivalent characterizations of Sobolev and Besov spaces of solutions.

**Proposition S.** (i) Let \( \delta(x) \) be the distance of \( x \) to the boundary. If \( \nabla^2 u = 0 \) in \( \Omega \), \( 0 < \alpha < 1 \), \( k \) a non-negative integer and \( 1 \leq p \leq \infty \), then the following are equivalent,
(a) \( u \in B_{k+1}^\varphi(\Omega) \),

(b) \( \int_\Omega |\partial(x)^{1-s} \nabla^{k+1}u(x)|^p + |\nabla \partial u|^p + |u|^p \, dx < \infty \).

(ii) If \( A^2 u = 0 \) in \( \Omega \), \( 0 \leq a \leq 1 \), \( k \) a non-negative integer and \( 1 < p < \infty \), then the following are equivalent,

(a) \( u \in L_{k+1}^\varphi(\Omega) \),

(b) \( \int_\Omega |\partial(x)^{1-s} \nabla^{k+1}u(x)|^p + |\nabla \partial u|^p + |u|^p \, dx < \infty \).

Proof. The argument is nearly identical to the harmonic case. We give a brief sketch. It is clear that it is sufficient to consider the case \( k = 0 \). For (i) the proof is identical to the proof in [JK] of their Theorem 4.1, with interior estimates, see, e.g., [PV4] and [J], substituted for use of the mean value property of harmonic functions.

For (ii) we first prove that \( u \in L_2^\varphi(\Omega) \) implies the estimate in (b). This is clear for \( \alpha = 1 \). For \( \alpha = 0 \), we claim that

\[
\int_Q (\delta |\nabla u|)^p \, dx \leq C \int_{2Q} |u|^p \, dx,
\]

for any cube \( Q \) such that \( 2Q \subset \Omega \) and diameter comparable to \( \text{dist}(Q, \partial \Omega) \), by interior estimates for solutions. There is, nothing special about the 2 appearing in the left hand side. The statement of the claim is valid for Whitney cubes with the relation between the radii of the left and right hand sides chosen in such a way that the bigger cubes have the finite intersection property and cover \( \Omega \). Thus, the claim above implies

\[
\int_\Omega (\delta |\nabla u|)^p \, dx \leq C \int_\Omega |u|^p \, dx.
\]

The claim follows from Lemma 1 below.

Consequently, putting \( \mathcal{B} = \{ u; A^2 u = 0 \ \text{in} \ \Omega \} \), the map \( u \to \nabla u \) is continuous \( \mathcal{B} \cap L^p(\Omega) \to L^p(\Omega, \delta(x)^p \, dx) \) and \( \mathcal{B} \cap L_1^p(\Omega) \to L_1^p(\Omega, \delta(x)^p \, dx) \). Complex interpolation shows that the map is continuous

\[ \left[ \mathcal{B} \cap L^p(\Omega), \mathcal{B} \cap L_1^p(\Omega) \right] \to \left[ L_1^p(\Omega, \delta(x)^p \, dx), L^p(\Omega, \delta(x)^p \, dx) \right]. \]

The change of measure theorem of Stein–Weiss, [SW], implies that the right hand side equals \( L_1^p(\Omega, \delta(x)^p \, dx) \). Now the desired result is

\[ \left[ \mathcal{B} \cap L^p(\Omega), \mathcal{B} \cap W_1^p(\Omega) \right] = \mathcal{B} \cap L_1^p(\Omega), \]
where $\alpha = 0$. This interpolation result was proved in [JK] with $H$ replaced by $\mathcal{H} = \{ u : \Delta u = 0 \text{ in } \Omega \}$. The argument for our result is identical to the one in [JK], with the fundamental solution of the bi-Laplacian substituted for the fundamental solution of the Laplacian.

It remains to check that the estimate of (b) implies that $u \in L^p_2(\Omega)$ for solutions. The proof of this proceeds along exactly the same steps as in the corresponding harmonic situation, [JK]. This finishes the proof of Proposition S.

To prove the claim we will without loss of generality use balls instead of cubes. We note that using dilations it suffices to consider the case $r = 1$. We put $B_s = B(x_c, s)$, the ball centered at $x_c$ with radius $s$.

**Lemma I.** Let $B_{1+s} \subset \Omega$ and $p > 0$. Then

$$
\int_{B_{1+s}} |u|^p \, dx \leq C(z, p) \int_{B_1} |u|^p \, dx.
$$

This follows from Lemmas II–IV below.

**Lemma II.** Suppose $B_1 \subset \Omega$, $0 < s < t \leq 1$, and $\Delta^2 u = 0$ in $\Omega$. Then,

$$
\int_{B_t} |\nabla u|^2 \, dx \leq C/(t-s)^2 \int_{B_s} |u|^2 \, dx.
$$

**Proof.** Let $\eta$ be a smooth cutoff function $0 \leq \eta \leq 1$ such that $\eta = 1$ in $B_s$ and supp $\eta \subset B_t$. Consider first a harmonic function $v$. Then, using the summation convention,

$$
\int_{B_t} |D_k v|^2 \eta^6 \, dx = -\int_{B_t} \eta D_k v D_k \eta^6 \, dx - \int_{B_t} \eta D_k v 6\eta^5 D_k \eta \, dx
\leq \frac{C}{(t-s)^2} \left( \int_{B_t} |\nabla v|^2 \eta^4 \, dx \right)^{1/2} \left( \int_{B_t} |D_k v|^2 \eta^6 \, dx \right)^{1/2}.
$$

Hence,

$$
\int_{B_t} |D_k v|^2 \eta^6 \, dx \leq \frac{C}{(t-s)^2} \int_{B_t} |\nabla v|^2 \eta^4 \, dx. \quad (a)
$$

Consider now bi-harmonic $u$. We put

$$
I = \int_{B_t} |\Delta u|^2 \eta^4 \, dx.
$$
From the divergence theorem follows
\[
I = -\int_{B_t} 2u D_j u D_j x - \int_{B_t} u(D_j D_j u) \eta^2 dx \\
\leq \frac{C}{(t-s)} \left( \int_{B_t} u^2 dx \right)^{1/2} I^{1/2} + C \left( \int_{B_t} u^2 dx \right)^{1/2} \left( \int_{B_t} |D_j D_j u|^2 \eta^4 dx \right)^{1/2}.
\]
(b)

Put
\[
II = \int_{B_t} |D_j D_k u|^2 \eta^4 dx \geq \int_{B_t} |D_j D_k u|^2 \eta^4 dx.
\]
(c)

Again using the divergence theorem, we find that
\[
II = -\int_{B_t} D_k u(D_j D_j D_k u) \eta^4 dx - \int_{B_t} D_k u(D_j D_k u) 4\eta^3 D_j \eta dx \\
\leq \left( \int_{B_t} |D_k u|^2 \eta^2 dx \right)^{1/2} \left( \int_{B_t} |D_k A u|^2 \eta^6 dx \right)^{1/2} \\
+ \frac{C}{(t-s)} \left( \int_{B_t} |D_k u|^2 \eta^2 dx \right)^{1/2} \left( \int_{B_t} |D_j D_k u|^2 \eta^4 dx \right)^{1/2},
\]
and using (a) for harmonic functions, this is
\[
\leq \frac{C}{(t-s)} \left[ \left( \int_{B_t} |A u|^2 \eta^4 dx \right)^{1/2} + \left( \int_{B_t} |D_j D_k u|^2 \eta^4 dx \right)^{1/2} \right] \\
\times \left( \int_{B_t} |D_k u|^2 \eta^2 dx \right)^{1/2}
\leq \frac{C}{t-s} II^{1/2} I^{1/2},
\]
so that
\[
II^{1/2} \leq \frac{C}{t-s} I^{1/2}.
\]
This together with (b) and (c) implies the lemma.

Lemma III. Suppose \( B_{t+s} \subset \Omega \) for some \( \alpha > 0 \), small. If \( u \) is bi-harmonic in \( \Omega \) and \( 0 < s < t < 1 \) we have that
\[
\| \nabla u \|_{L^\infty(B_t)} \leq \frac{C(\alpha)}{t-s} \left( \int_{B_t} |u|^2 dx \right)^{1/2}.
\]
Proof. We note, see [PV4] and [J], that for $0 < z < 1$, $|u(\nu)| \geq k$ where $1 \leq k \leq 3$, we have from interior estimates that

$$|D^\gamma u(x)| \leq C \delta^{\gamma - |\alpha|} |\nabla u(y)| dy. \quad (\dagger)$$

Suppose now that $x \in B_s$. We have that $\delta(x) \geq (t-s) + \alpha \geq \alpha$. Furthermore, $\delta(x) \leq\! \text{diam}(\Omega)$. Choose $z$ such that $z\text{ diam}(\Omega) \leq (t-s)/2$. Consequently $B(x, z\delta(x)) \subset B_{s+((t-s)/2)}$ for $x \in B_s$. Since $0 < z < 1$, choosing $k = 1$ in $(\dagger)$ above gives

$$\|\nabla u\|_{L^\infty(B_s)} \leq \frac{C}{|B_s|} \int_{B_{s+((t-s)/2)}} \|\nabla u(y)| dy \leq C \left( \frac{|B_{s+((t-s)/2)}|}{|B_s|} \right)^{1/2} \left( \int_{B_{s+((t-s)/2)}} |\nabla u|^2 dx \right)^{1/2} \leq C(z) \left( \frac{1}{(t-(s+(t-s)/2))^2} \right)^{1/2} \left( \int_{B_s} u^2 dx \right)^{1/2},$$

by Lemma II since $0 < s + (t-s)/2 < t \leq 1$. This proves the lemma.

Lemma IV. Suppose $B_{1+s} \subset \Omega$ for some $\alpha > 0$, small. If $u$ is bi-harmonic in $\Omega$ and $\frac{1}{2} \leq s < t \leq 1$ we have that

$$\|u\|_{L^\infty(B_s)} \leq \frac{C(z)}{1-s} \left( \int_{B_s} u^2 dx \right)^{1/2}.$$

Proof. Let $x \in B_s$. We have that,

$$|u(x)| \leq \int_{B_s} |u(x) - u(y)| dy + \int_{B_s} |u| dy. \quad (\dagger)$$

It is clear that

$$\int_{B_s} |u| dy \leq C \int_{B_s} |u| dy,$$

since $s \geq 1/2$. Since for $y \in B_s$, $|u(x) - u(y)|$ is estimated by $2s \|\nabla u\|_{L^\infty(B_s)}$ we get the desired result using Lemma III.

Proof of Lemma I. For the range $p \geq 2$ the result of the lemma follows from Lemma III. For $p < 2$ the result follows from the reverse Hölder inequality of Lemma IV, the technique of B. Dahlberg and C. Kenig.
presented in [FS, p. 1004], the result for \( p \geq 2 \) and straightforward applications of Hölder’s inequality. The proof is complete.

**Remark R1.** We would here like to observe a minor reformulation of the Dirichlet problem, \((D)_p\), which is useful for interpolation. The data for this problem is usually expressed:

\[
\begin{cases}
\frac{\partial u}{\partial N} = G \in L^p, \\
u|_{\partial \Omega} = F \in L^q(\partial \Omega).
\end{cases}
\]

It is easy to see that this is equivalent to the formulation

\[
\begin{cases}
\frac{\partial u}{\partial N} = \sum_{j=1}^{n} N_j f_j, \\
u|_{\partial \Omega} = f_0,
\end{cases}
\]

where \( \vec{f} = (f_0, f_1, \ldots, f_n) \in W^q(\partial \Omega) \). One defines

\[
f_n = N^n \left( G - \sum_{j=1}^{n-1} \frac{\partial F}{\partial T_j} N_j \right),
\]

\[
f_j = \frac{\partial F}{\partial T_j} + \frac{N_j}{N^n} f_n, \quad j = 1, \ldots, n-1,
\]

and \( f_0 = F \), where, as before, \( \partial / \partial T_j = -N_j D_n + N^n D_j \). Using these definitions, if one restricts to a coordinate chart, one sees that the required compatibility conditions are obtained and it is easy to show that \( G = \sum_{k=1}^n N^n f_k \).

The reduction of Theorem 2.1 to Theorem 2.2 is as follows. Let \( f \in L^p_0(\Omega) \) and let \( f \in L^p_0(\mathbb{R}^n) \), compactly supported, such that \( \vec{f} = f \) in \( \Omega \). Let \( w = \Gamma(f) \), where \( \Gamma(x, y) = c_n |x - y|^{n-\alpha} \) is the fundamental solution of \( \Delta^2 \) in \( \mathbb{R}^n \). (Here \( n = 3 \) or \( n \geq 5 \); when \( n = 4 \) the fundamental solution has a logarithmic term.) Thus \( \Delta^2 w = \vec{f} \) in \( \mathbb{R}^n \) and \( w \in L^p_0(\mathbb{R}^n) \). Now let \( \vec{g} \) be the trace of \( w \) on \( \partial \Omega \), which is well defined, in the sense of [JW], if \( 3 + \beta - 1/p > 0 \). Now, if in addition \( 3 + \beta - 1/p \) is integer, then this trace operator coincides with the classical trace operator. From the discussion following Theorem 1.12 we have that \( \text{Tr}(L^p_{3\alpha,1/p}(\mathbb{R}^n)) \) on \( \partial \Omega \) will not be contained in the array space \( \mathbb{W}^q(\partial \Omega) \). Thus we restrict the study to the range \( 0 < 3 + \beta - 1/p < 1 \), and for such \( x \), \( \vec{g} \in \mathbb{W}^q_{3\beta-1/p}(\partial \Omega) \), by the restriction theorem and Lemma 1.7.

The proof of Theorem 2.2 will follow by real and complex interpolation between several endpoint results, some of which are known and others which are new in this context. The main estimates needed to prove the first
Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Then there exists an $\varepsilon > 0$ depending on the Lipschitz character of $\Omega$, such that

(a) If $2 \leq p < 2 + \varepsilon$ and $g \in \text{WA}_f(\partial \Omega)$, then there exists a unique biharmonic function $u \in L^p_{\varepsilon, 1/p}(\Omega)$ such that $(\nabla u)^* \in L^p(\partial \Omega)$ and $(u, \nabla u)$ converges non-tangentially to $g$ and, Tr$(u) = g_0$. If the dimension $n = 3$ or $\Omega$ is $C^1$, then this is true in the range $2 \leq p < \infty$.

(a') When $2 - \varepsilon < p \leq 2$ and $g \in \text{WA}_f(\partial \Omega)$, then there exists a unique biharmonic function $u \in B^p_{\varepsilon, 2}(\Omega)$ with $(\nabla u)^* \in L^p(\partial \Omega)$ and $(u, \nabla u)$ converging non-tangentially to $g$, and Tr$(u, \nabla u) = g$. If $\Omega$ is $C^1$, then this is true in the range $1 < p \leq 2$.

(b) If $2 - \varepsilon < p \leq 2$ and $g \in \text{WA}_f(\partial \Omega)$, then there exists a unique biharmonic function $u \in L^p_{\varepsilon, 1/p}(\Omega)$ with $(\nabla u)^* \in L^p(\partial \Omega)$, $(u, \nabla u)$ converging non-tangentially to $g$, and Tr$(u, \nabla u) = g$. If the dimension $n = 3$ or $\Omega$ is $C^1$, then this is true in the range $1 < p \leq 2$.

Proof. The statements about the traces all follow from the approximation property of the spaces $\text{WA}_f(\partial \Omega)$ and $\text{WA}_j(\partial \Omega)$, Proposition 1.5, together with the arguments in the proof of (the following) Proposition 2.7. Specifically, in the first term in $(*)$ in the proof of Proposition 2.7, the norm of the traces has to be taken in a $B^p_1(\partial \Omega)$ space with $\beta < 1$ and $\beta < 2$ respectively, in order to use the extension theorems. This explains the trace result appearing in (a) and (a') only stating equality for the first component, $g_0$, of the data $g \in \text{WA}_f(\partial \Omega)$.

We first consider (a). From the remark R1 the existence of a unique biharmonic $u$ such that $(\nabla u)^* + u^* \in L^p(\partial \Omega)$ and $(u, \nabla u) \rightarrow g$ non-tangentially, follows from Theorem D(p), (i). We want to see that $u \in L^p_{\varepsilon, 1/p}(\Omega)$. For $p = 2$ the argument proceeds by a simple estimate of the area integral.

$$\int_{g\Omega} |S(\nabla u)(Q)|^2 dQ = \int_{g\Omega} \int_{\Gamma \cap Q} |\nabla u(x)|^2 \delta(x)^{n-\epsilon} dx dQ$$

$$= \int_{\Omega} |\nabla u(x)|^2 \delta(x)^{n-\epsilon} \int_{\varepsilon \Omega} \chi_{\Gamma \cap Q}(x) dQ dx$$

$$\approx \int_{\Omega} |\nabla u(x)|^2 \delta^2 \delta^{n-1}(x) dx.$$
The result now follows from Theorem 2.4. and Proposition S. For \( p \geq 2 \) the argument is a bit more lengthy. It is clearly, by the characterization of Sobolev spaces, sufficient to show that \( \mathcal{V} u \in L^p_{\ast} (\Omega) \). Using a partition of unity, \( \{ \theta_j \}_{j=0}^\infty \) on \( \Omega \) where supp \( \theta_0 \subset \subset \Omega \) and \( \theta_j, j \geq 1 \) are supported in appropriate coordinate patches, we have to show that \( \theta_j \mathcal{V} u \in L^p_{\ast} (\Omega) \). We can suppress the \( \theta_j \)'s and assume that \( \mathcal{V} u \) is supported in a coordinate patch. Well-known arguments show that, (see \([S, \text{pp. 214–216}]\)),

\[
\| S_2(\mathcal{V} u) \|_{L^p(\partial \Omega)} \leq C \| S(\mathcal{V} u) \|_{L^p(\partial \Omega)}\]

(i)

where, using summation convention,

\[
S_2(\mathcal{V} u)^2 (\Omega) = \int_{\Omega} \delta^{4-s}(x) | D_j D_k D_l u(x) |^2 \, dx,
\]

and \( S \) is taken over a slightly larger cone, and

\[
\int_0^h y^3 | D_j D_k D_l u(x, \varphi(x) + y) |^2 \, dy \leq C S_2(\mathcal{V} u)^2 (Q).\]

(ii)

for \( Q = (x, \varphi(x)) \in \partial \Omega \) and \( h \) being the height of the coordinate patch. The one-variable inequality

\[
\left( \int_0^h y^p \left( \int_0^y h(s) \, ds \right)^{\frac{p}{2}} \, dy \right)^{\frac{1}{p}} \leq C \left( \int_0^h y^3 h^2(y) \, dy \right)^{\frac{1}{2}}\]

(iii)

follows by interpolation between the endpoint cases \( p = 2 \) and \( p = \infty \) obtained from Hardy's inequality and Cauchy–Schwarz respectively. Next,

\[
| \nabla \mathcal{V} u(x, \varphi(x) + s) | \leq \int_0^h | D_j D_k D_l u(x, \varphi(x) + y) | \, dy + \sup_K | D_j D_k D_l u |,
\]

where \( K \subset \subset \Omega \). From interior estimates, see, e.g., \([PV4]\), we get

\[
\sup_K | D_j D_k D_l u | \leq C \| (\nabla u)^* \|_{L^p(\partial \Omega)}.
\]

From (iii) follows

\[
\left( \int_0^h y^p \left( \int_0^y | D_j D_k D_l u(x, \varphi(x) + s) | \, ds \right)^{\frac{p}{2}} \, dy \right)^{\frac{1}{p}} \leq C \left( \int_0^h y^3 | D_j D_k D_l u(x, \varphi(x) + y) |^2 \, dy \right)^{\frac{1}{2}}.
\]
Thus,
\[ \int_{\Omega} (\delta^{1-1/p}(x)|\nabla u(x)|^{p} \, dx \]
\[ \leq \int \left( \int_{0}^{h} y^{p} |\nabla u(x, \varphi(x) + y)|^{p} \frac{dy}{y} \right) \, dx \]
\[ \leq \int \left( \int_{0}^{h} y^{p} \left( \int_{y}^{h} |D_{1} D_{2} u(x, \varphi(x) + s)| \, ds \right) \frac{dy}{y} \right) \, dx \]
\[ + C \| (\nabla u)^{*} \|_{L^p(\partial \Omega)}. \]

The first term can now be estimated by
\[ C \int \left( \int_{\partial \Omega} |D_{1} D_{2} D_{1} u(x, \varphi(x) + y)|^{2} \, dy \right)^{p/2} \, dx \]
\[ \leq C \int_{\partial \Omega} (S_{\partial \Omega}(|\nabla u|))^{p} \, dQ \leq C \int_{\partial \Omega} (S(|\nabla u|))^{p} \, dQ \]
\[ \leq C \| (\nabla u)^{*} \|_{L^p(\partial \Omega)}. \]

That \( u \in L^{1/p}_{p+1}(\Omega) \) is now again a consequence of Theorem 2.4. and Proposition 5.

The existence of a solution with \((\nabla u)^{*} \in L^{p}(\partial \Omega)\) and non-tangential convergence in case \( n = 3, 2 \leq p < \infty \), and \( C^{1} \) domains in \( \mathbb{R}^{n} \), is given in Theorem D(p), (ii) and (iii). The solution again belongs to \( L^{1/p+1}_{p}(\Omega) \) by the argument given above.

For (a'), assume now that \( g \in A^{1}(\partial \Omega) \). Then for \( 2 - \varepsilon < p \leq 2 \), \( \Omega \subset \mathbb{R}^{n} \) and Lipschitz, the existence of a unique biharmonic function with \((\nabla u)^{*} + u^{*} \in L^{p}(\partial \Omega)\) is given by Theorem D(p), (i). For \( C^{1} \) domains in \( \mathbb{R}^{n} \) and \( 1 < p < 2 \) this follows from Theorem D(p), (ii). We next wish to conclude that \( u \in B^{1/p+1}_{p+1}(\Omega) \). This is, by (1.2), equivalent to \( \nabla u \in B^{1/p+1}_{p+1}(\Omega) \) and \( u \in L^{p}(\Omega) \), where the last statement is obvious. That \( \nabla u \in B^{1/p+1}_{p+1}(\Omega) \) follows as in [JK, Theorem 5.15], from the fact that
\[ ||S(\nabla u)||_{p} \leq ||(\nabla u)^{*}||_{p} \leq ||g||_{A^{1}(\partial \Omega)}. \]

For (b), by Theorem R(p), (i) and applying the proof in part (a) to the gradient, the only issue to consider here is \( \text{Tr}(u, \nabla u) = g \) and this last fact follows as indicated in the beginning of the proof. In the case of a \( C^{1} \) domain the same results follows using Theorem R(p), (ii).

Finally, the argument for (b') is nearly a repetition of those above.
Proof of Theorem 2.2. We will first prove the existence. The statement about the traces again follows from Proposition 2.7.

Given the known results for data \( g \in \text{WA}_1^p(\partial \Omega) \) and \( g \in \text{WA}_2^p(\partial \Omega) \), we need only record the following interpolation identities:

\[
\begin{align*}
[B_{p + 1 + 1/p}^p(\Omega), B_{p + 1 + 1/p}^p(\Omega)]_{s+p} &= B_{p + 1 + 1/p}^p(\Omega), \\
[L_{p + 1 + 1/p}^p(\Omega), L_{p + 1 + 1/p}^p(\Omega)]_{s+p} &= B_{p + 1 + 1/p}^p(\Omega),
\end{align*}
\]

(3.4) (3.5)

see, e.g., [BL], and

\[
\text{WA}_1^p(\partial \Omega), \text{WA}_2^p(\partial \Omega) = \text{WA}_s^p(\partial \Omega),
\]

(3.6)

by Theorem 1.8.

Thus let \( g \in \text{WA}_{s_0}^p(\partial \Omega), p_0 \leq p \leq p_0 \) and so \( g \in [\text{WA}_1^p(\partial \Omega), \text{WA}_2^p(\partial \Omega)]_{s+p} \)

by (3.6). By Theorem 2.6 and the real interpolation results (3.4) and (3.5), there is a unique biharmonic solution \( u \in B_{p + 1 + 1/p}^p(\Omega) \) when \( 0 < s < 1 \). The uniqueness follows from uniqueness in the Dirichlet problem \( D(p) \), i.e. the endpoint case with data in \( \text{WA}_{s_0}^p(\partial \Omega) \subset \text{WA}_1^p(\partial \Omega) \). If in addition, \( 1 + s + 1/p \neq 2 \), then \( u \in L_{p + 1 + 1/p}^p(\Omega) \), by Proposition 2.6. It remains to treat the case \( 1 + s + 1/p = 2 \).

From Lemma 1.7, the trace and extension theorems, [JW], and the corresponding result for Besov spaces in \( \mathbb{R}^n \), we have the complex interpolation result

\[
\text{WA}_{s_0}^p(\partial \Omega), \text{WA}_{s_1}^p(\partial \Omega) = \text{WA}_{s + \theta s_1}^p(\partial \Omega),
\]

(3.7)

where \( 0 < s_0, s_1 < 1, s = (1 - \theta)s_0 + \theta s_1 \). (It is important to note that the interpolation is being done for \( s \) strictly between two integer points.) Now, in the case under consideration \( g \in \text{WA}_{s_0}^p(\partial \Omega) = \text{WA}_{s_1}^p(\partial \Omega) \) and \( 1 < 2 - 1/p < 2 \). So we can choose \( 0 < s, s_1 < 1 \) such that \( 1 + s_0 < 2 - 1/p < 1 + s_1 \) and

\[
g \in \text{WA}_{s_0}^p(\partial \Omega) = [\text{WA}_{s_0}^p(\partial \Omega), \text{WA}_{s_1}^p(\partial \Omega)]_{\theta},
\]

for some \( 0 < \theta < 1 \) such that \( 2 - 1/p = 1 + (1 - \theta)s_0 + \theta s_1 \). This now completes the proof of part (i) since

\[
[L_{p + 1 + 1/p}^p(\Omega), L_{p + 1 + 1/p}^p(\Omega)]_{\theta} = L_{s + \theta s_1}^p(\Omega).
\]

In the range \( 1 < p < p_0 \), for Lipschitz domains in \( \mathbb{R}^n \) or \( C^1 \) domains in \( \mathbb{R}^n \), our results follow by complex interpolation between \( \text{WA}_{s + \theta s_1}^p(\partial \Omega) \) and \( \text{WA}_{s + \theta s_1}^p(\partial \Omega) \). The complex interpolation result

\[
[\text{WA}_{s_0}^p(\partial \Omega), \text{WA}_{s_1}^p(\partial \Omega)]_{\theta} = \text{WA}_{s + \theta s_1}^p(\partial \Omega),
\]
where \( 0 < \varepsilon_1 < \delta, 0 < \varepsilon_2 < 1, 1/p = \theta + (1 - \theta)/p_0 \) and \( 0 < \varepsilon = \theta \varepsilon_1 + (1 - \theta) \varepsilon_2, \) follows as in the case of (3.7) above. From Theorem WA_1\( \mathcal{D} \), see the next section, together with Proposition S, and case (i) we get a unique solution

\[
\begin{aligned}
    u &\in [B_{1-\varepsilon_1}^1(\mathcal{D}), B_{1/p_0 + 1-\varepsilon_2}^0(\mathcal{D})]_\theta \\
    &= B_{2 + (1-\varepsilon)(\theta + (1-\theta)/(1/p_0 - \varepsilon_2))}^0(\mathcal{D}) = B_{2 - \varepsilon + 1/p(\mathcal{D})}^0(\mathcal{D}).
\end{aligned}
\]

It remains to determine the range of \( 1 - \varepsilon \). We have that \( \theta = p_0/(p_0 - 1) \) \((1/p - 1/p_0)\) and \( \varepsilon < \delta \theta (1 - \theta) = 1 + \theta (\delta - 1) \). Thus \( 1 > 1 - \varepsilon > (p_0/(1 - \delta))/(p_0 - 1)(1/p - 1/p_0) = (1 - \delta)/(1 - \varepsilon)(2/p - 1 - \varepsilon) > 2/p - 1 - \varepsilon, \) by choosing \( \delta \) in a suitable manner. The same type of arguments as in part (i) now shows that \( u \in L_{1-\varepsilon + 1 + 1/p}^\infty(\mathcal{D}) \). This finishes part (ii).

For the range \( p_0 < p < \infty, \) we have, by Theorem 3.8 of [PV3], solvability of the array problem:

\[
\begin{cases}
    A^2 u = 0 & \text{in } \Omega \\
    (u, \nabla u) = f & \text{on } \partial \Omega,
\end{cases}
\]

for \( f \in \mathcal{WA}_{1 + s}^\infty(\partial \Omega) \) and hence the analog of the classical estimate of Kellogg:

\[
\| \nabla u \|_{\mathcal{W}^s_{1 + s}(\partial \Omega)} + \sup_\Omega \| \delta(x)^{1-s} |\nabla \nabla u(x)| \| \leq C \| f \|_{\mathcal{WA}_{1 + s}^\infty(\partial \Omega)},
\]

for \( 0 < \alpha \) sufficiently small. This is true for \( \Omega \subset R^3, \) Lipschitz or \( \Omega \subset R^n, C^1. \)

Again by complex interpolation the array data can be taken on in

\[
[\mathcal{WA}_{1 + s}^{p_0}(\partial \Omega), \mathcal{WA}_{1 + s}^{p_0}(\partial \Omega)]_\theta = \mathcal{WA}_{1 + s}^{p}(\partial \Omega),
\]

and \( \nabla u \) belongs to

\[
[B_{1 + s}^{p_0}(\Omega), B_{p}(\Omega)]_\theta = B_{1 + s}^{p}(\Omega),
\]

where \( 1/p = \theta/p_0 \) and \( \beta = \theta s + (1 - \theta) \alpha = (p_0'/p)s + (1 - (p_0'/p)\alpha < (p_0'/p) \)

\(+ (1 - (p_0'/p)\alpha = 2/p((1 - \alpha)/(1 - \varepsilon)) + \alpha, \) and the result of (iii) follows by choosing \( \alpha \) suitably and using the same type of arguments as in the previous cases.

Observe that we have proved the existence by exhibiting solutions also having the appropriate estimates of the non-tangential maximal functions of the solution and its gradient. Solutions within this class are clearly unique as noted in the proof. Uniqueness in the theorem, i.e., without this additional information, is proven as follows, [JK, Proposition 5.17].

Consider \( u \) biharmonic in \( \Omega \) with \( u \in L_{2 + 1/p}^\infty(\Omega) \) and \( \text{Tr}(u, \nabla u) = 0. \) We
have to show that \( u \) is identically zero. Since \( u \in L^{p,1+1/p}_{\text{loc}}(\Omega) \) there exist functions \( \phi_j \in C_0^\infty(\Omega) \) such that \( \phi_j \) tends to \( u \) in \( L^{p,1+1/p}_{\text{loc}}(\Omega) \). Choose smooth domains \( \Omega_j \) such that \( \Omega_j \subset \Omega \), \( \phi_j \in C_0^\infty(\Omega_j) \), and \( \partial\Omega_j \) are uniformly Lipschitz and tend to \( \partial\Omega \). Letting \( \text{Tr}_j \) denote the restriction to \( \partial\Omega_j \), then as \( j \to \infty \),

\[
\| \text{Tr}_j u \|_{p_{\text{loc}}(\partial\Omega)} = \| \text{Tr}_j (\phi_j - u) \|_{p_{\text{loc}}(\partial\Omega)} \leq C \| \phi_j - u \|_{L^{p,1+1/p}(\Omega)} \to 0.
\]

Furthermore, from the hypo-ellipticity of the bi-Laplacian we know that \( u \in C^\infty \) in \( \Omega \). It is thus immediate that \( u \) has the appropriate non-tangential maximal function estimates on \( \partial\Omega \). Thus, by continuity, \( \text{Tr}_j(u, Vu) = \text{Tr}_{\text{nt}}(u, Vu) \). Consequently, by the regularity estimate proved in the existence part we have that

\[
\| u \|_{L^{p,1+1/p}(\Omega)} \leq C \| \text{Tr}_j u \|_{p_{\text{loc}}(\partial\Omega)},
\]

which proves that \( u \) is identically zero on each compact in \( \Omega \). This finishes the proof of the theorem.

The result about the traces being the same as the non-tangential traces follows in certain ranges of \( \alpha \) and \( p \) from the continuity of the trace operator and the solution operator. The lack of an appropriate maximum principle, \([PV3]\), complicates the matter for other ranges of indices and is the reason for including the following result.

**Proposition 2.7.** Suppose that \( \alpha \) and \( p \) with \( 1 < 4 + \alpha - 1/p < 2 \), \( 1 < p < \infty \) are such that for each \( g \in B_{p-\alpha+\alpha-1/p}^{1/p}(\partial\Omega) \) there exists a unique solution \( u \) such that \( (Vu)^* \in L^p(\partial\Omega) \), to

\[
\begin{align*}
\Delta^2 u &= 0 & \text{in } \Omega, \\
\text{Tr}_{\text{nt}} u &= g & \text{on } \partial\Omega,
\end{align*}
\]

and moreover, \( \|(Vu)^*\|_{L^p(\partial\Omega)} \leq C \|g\|_{p_{\text{loc}}(\partial\Omega)} \). Suppose also that \( u \in L^{p,\alpha}_{\text{loc}}(\Omega) \) and that \( |u| \leq \|p_{\text{loc}}(\partial\Omega) \leq C \|g\|_{p_{\text{loc}}(\partial\Omega)} \), with \( C \) depending only on the Lipschitz character of \( \Omega \). Then \( \text{Tr}_j u = \text{Tr}_{\text{nt}} u = g \).

**Proof.** We can assume that \( \Omega \) is a bounded domain and it follows from the extension/restriction theorem, \([JW]\), that there exist functions \( w_j \in C_0^\infty(\mathbb{R}^n) \) such that

\[
\|g - \text{Tr}_j w_j \|_{B_{p-\alpha+\alpha-1/p}(\partial\Omega)} \to 0.
\]

**Claim.** There are functions \( v_j \in L^{p,\alpha}_{\text{loc}}(\Omega) \) such that \( \Delta^2 v_j = \Delta^2 w_j \), the non-tangential maximal function of \( \nabla v_j \) is bounded in \( L^p(\partial\Omega) \) and consequently the non-tangential limits of \( v_j \) and \( \nabla v_j \) exist. Furthermore, these non-tangential limits are zero.
As a consequence $w_j - v_j$ is the unique solution, (in the class of functions $h$ with $(\forall h^* \in L^p(\partial \Omega))$, of

\[
\begin{align*}
\mathcal{A} (w_j - v_j) &= 0 & \text{in } \Omega \\
\text{Tr}_m (w_j - v_j) &= (w_j|_{\partial \Omega}, v_j|_{\partial \Omega}),
\end{align*}
\]

and hence, \( \| \nabla (w_j - v_j) \|_{L^p(\partial \Omega)} \leq C \| w \|_{B^{r_q}_{p,q+1,p}(\partial \Omega)} \). We now get from Proposition 1.11 that

\[
\| \text{Tr} u - \text{Tr} w_j | B^{x}_{p,q+1,q}(\partial \Omega) \| = \| \text{Tr} u - \text{Tr}(w_j - v_j) | B^{x}_{p,q+1,q}(\partial \Omega) \|
\]

\[
\leq \| E u - E (w_j - v_j) | L^p_q(\Omega) \|
\]

\[
\leq C \| u - (w_j - v_j) | L^p_q(\Omega) \|
\]

\[
\leq C \| \text{Tr}_m (u - (w_j - v_j)) | B^{x}_{p,q+1,q}(\partial \Omega) \|
\]

\[
= C \| \text{Tr}_m u - \text{Tr} w_j | B^{x}_{p,q+1,q}(\partial \Omega) \| ,
\]

(*)

by the estimates for solutions.

There remains to prove the Claim. Let $\Omega_{\ell} \supset \Omega$, $\Omega_{\ell} \in C^\infty$ and such that $\Omega_{\ell}$ are uniformly Lipschitz. Suppressing the index $j$, put $\psi = A^2 w_j$. Let $\Gamma$ be the fundamental solution and $\Gamma(\psi)$ its potential of $\psi$. Let $w_{\ell}$ solve

\[
\begin{align*}
\mathcal{A}^2 w_{\ell} &= 0 & \text{in } \Omega_{\ell} \\
\text{Tr}_m w_{\ell} &= \text{Tr} w_{\ell} = \Gamma(\psi) & \text{on } \partial \Omega_{\ell},
\end{align*}
\]

with $(\nabla w_{\ell})^* \in L^p(\partial \Omega)$. Then, by the assumptions of the proposition, we have that

\[
\| w_{\ell} \|_{L^p_{q+1}(\Omega_{\ell})} \leq C \| \text{Tr}_m w_{\ell} \|_{B^{x}_{p,q+1,q}(\partial \Omega_{\ell})} \leq C,
\]

uniformly in $\ell$. Let $v_{\ell} = \Gamma(\psi) - w_{\ell}$, and $\bar{v}_{\ell}$ the zero extension of $v_{\ell}$ to $\Omega$. Then by Proposition 1.14, $\bar{v}_{\ell} \in L^p_{q+1}(\Omega)$ since $1 < q+1 < 2$. Furthermore,

\[
\| \bar{v}_{\ell} \|_{L^p_{q+1}(\Omega)} \leq C,
\]

independently of $\ell$. Consequently, passing to a subsequence we can assume that

\[
\bar{v}_{\ell} \rightharpoonup v \quad \text{weakly in } L^p_{q+1}(\Omega).
\]

From the convexity of the subspace $L^p_{q+1}(\Omega)$ we get that $v \in L^p_{q+1}(\Omega)$, since the weak closure equals the strong closure for a convex set in a locally
convex space, see, e.g., [R]. Moreover, again passing to a subsequence we can assume
\[ \tilde{v}_\varepsilon(x) \to v(x) \quad \text{a.e. in } \Omega, \]
by the compact imbedding of Sobolev or Besov spaces and the uniqueness of weak limits. If \( K \subset \subset \Omega \), and \( \varepsilon > \varepsilon(K) \), then \( \{w_j\} \) and also \( \{D^n w_j\} \) are equicontinuous on \( K \), (by interior estimates, see e.g. Lemma III above). Thus we can get a uniformly convergent subsequence on \( K \). So let \( \{K_j\} \) be an exhaustion of \( \Omega \) by compact sets, \( K_j \subset K_{j+1} \). Let \( w^{(1)} = \lim_{\varepsilon \to 0} (K_\varepsilon) w_\varepsilon \) be the uniform limit on \( K_1 \). Then \( w^{(1)} \) is a solution and \( D^n w_j \to D^n w^{(1)} \). By taking a further subsequence of the \( \{w_{\varepsilon_j}\} \) for \( K_1 \), get \( w^{(2)} \) as a uniform limit of \( w_j \)'s on \( K_2 \), which in fact equals \( w^{(1)} \) on \( K_1 \). Thus, obtain a sequence \( \{w^{(n)}\} \) such that \( D^n w^{(n)} = 0 \). Let \( w(x) = \lim_{n \to \infty} w^{(n)}(x) \). Then \( D^n w = 0 \) in \( \Omega \) and \( v(x) = f(q)(x) - w(x) \) a.e. in \( \Omega \).

Next we will show that \( w \) so defined has non-tangential limits a.e. on \( \partial \Omega \). Choose \( \delta > 0 \), and consider, for \( Q \in \partial \Omega \),
\[ N_d(\tilde{v}_\varepsilon)(Q) = \sup_{x \in \Gamma(Q), \text{dist}(x, \partial \Omega) > \delta} |\tilde{v}_\varepsilon(x)|. \]
We claim that then \( |N_d(\tilde{v}_\varepsilon)|_{Q} \leq C \), with \( C \) independent of \( \varepsilon \). To see this fix \( \delta > 0 \). Then \( \{x \in \Omega: \text{dist}(x, \partial \Omega) > \delta\} \subset K_n \) for some \( n \), and in \( K_n \), \( w = w^{(n)} \). Then for \( \eta > 0 \) we have
\[ |\tilde{v}_\varepsilon^{(n)}(x)| = |\tilde{v}_\varepsilon^{(n)}(x) - \tilde{v}_\varepsilon(x)| + |\tilde{v}_\varepsilon(x)| < \eta + |\tilde{v}_\varepsilon(x)|, \]
if \( \varepsilon \) is sufficiently large independent of \( x \in \{x \in \Omega: \text{dist}(x, \partial \Omega) > \delta\} \). Thus,
\[ N_d(\tilde{v}_\varepsilon)(Q) \leq \eta + N_d(\tilde{v}_\varepsilon)(Q), \]
for \( \varepsilon > \varepsilon_0 \), independent of \( Q \). If \( x \in \Gamma(Q) \) and dist(\( x, \partial \Omega \)) > \( \delta \), then choose \( \varepsilon \) so that dist(\( \partial \Omega, \partial \Omega \)) \leq \( \delta/10 \). Then
\[ |\tilde{v}_\varepsilon(x)| \leq N(\tilde{v}_\varepsilon)(Q), \]
for any \( Q' \in \partial \Omega \) such that \( x \in \Gamma(Q') \). We can assume that the boundaries \( \partial \Omega \) and \( \partial \Omega_\varepsilon \) are given by graphs in a local coordinate system and we let \( Q_\varepsilon \in \partial \Omega_\varepsilon \) be the projection of \( Q \in \partial \Omega \) onto \( \partial \Omega_\varepsilon \) along the \( x_\varepsilon \)-axis. Letting \( A = \{Q' \in \partial \Omega_\varepsilon: x \in \Gamma(Q') \} \) we have that \( 2A \) contains \( Q \). Thus, averaging we get
\[ N_d(\tilde{v}_\varepsilon)(Q) \leq M(N(\tilde{v}_\varepsilon))(Q), \]
where \( M \) is the Hardy-Littlewood maximal function. Consequently,
\[ ||N_d(\tilde{v}_\varepsilon)||_{L^\infty} \leq C, \]
independent of \( l \), since \( \|N(\nabla w_l)\|_{L^p(\partial\Omega)} \leq C \), for all \( l \). This shows the last claim. Taking the limit \( \delta \to 0 \), proves the \( L^p \) boundedness of the non-tangential maximal function for \( \nabla w \). Thus we get the existence of \( \text{Tr}_{\Omega} \), i.e., the non-tangential limits of \( w \) and \( \nabla w \) a.e. on \( \partial\Omega \).

Next we show that the non-tangential trace equals the trace of \( I(\psi) \) on \( \partial\Omega \). Let \( w(Q) \) denote the non-tangential trace of \( w \) on \( \partial\Omega \). We choose any surface ball \( A \subset \partial\Omega \), \( A = B \cap \partial\Omega \) where \( B \) is a ball centered at a point on \( \partial\Omega \) and such that \( B \cap \partial\Omega \) for all \( l \)'s are given as the graphs of functions in the same coordinate chart. Then, to see that \( w(Q) = I(\psi)(Q) \), \( Q \in \partial\Omega \), it suffices to check that \( \|w - I(\psi)(\cdot, Q)\|_{L^p(A)} \) is arbitrarily small for some \( p \). To this end, let \( \varepsilon > 0 \) be arbitrarily small and fix \( K \subset \partial\Omega \), \( \partial K \cap B \equiv A_K \), such that \( \text{dist}(k, \partial\Omega) < \varepsilon \) for all \( k \in \partial K \). Then,

\[
\int_A |w(Q) - I(\psi)(Q)|^p \, d\sigma(Q) \leq C \int_A |w(Q) - w(\tilde{Q})|^p + |w(\tilde{Q}) - I(\psi)(\tilde{Q})|^p + |I(\psi)(\tilde{Q}) - |I(\psi)(Q)|^p \, d\sigma(Q),
\]

where \( \tilde{Q} = Q + r \mathbf{n} \), in the local coordinate system, with \( r \) chosen so that \( \tilde{Q} \) lies on \( \partial K \). Now \( |w(Q) - w(\tilde{Q})| \leq N(\nabla w)(Q) \, |Q - \tilde{Q}| \), so the first integral is bounded by

\[
\varepsilon^p \int_{\partial\Omega} |N(\nabla w)|^p \, d\sigma \leq C \varepsilon^p.
\]

The third term in the integral is also small. It remains to consider

\[
\int_A |w(P) - I(\psi)(P)|^p \, d\sigma(P) \sim \int_{A_K} |w(P) - I(\psi)(P)|^p \, d\sigma(P).
\]

For any \( l \) such that \( K \subset \Omega_l \subset \Omega \),

\[
\int_{A_K} |w(P) - I(\psi)(P)|^p \, d\sigma(P)
\leq \int_{A_K} |w(P) - w(P_l)|^p \, d\sigma(P) + \int_{A_K} |w(P_l) - I(\psi)(P)|^p \, d\sigma(P),
\]

where \( P_l \) is the projection of \( P \) onto \( \partial\Omega_l \) along the \( l \)th coordinate direction. The first term is bounded by

\[
\|N(\nabla w_l)\|_{L^p(\partial\Omega_l)} \cdot |\text{dist}(K, \partial\Omega)|^p \leq C \varepsilon^p,
\]

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independently of \( \ell \); the second term is small, independently of \( \ell \), since 
\[
|w(P) - w_r(P)|^p \, d(P)
\]
is small if \( \ell \) is chosen sufficiently large, which is true since \( w \) is the uniform limit of the \( w_r \)'s on \( K \). The case of \( \nabla w(Q) \) follows in an analogous manner. This finishes the proof of the claim.

3. THE BIHARMONIC WA\(_{1-\delta}\)(\( \partial \Omega \)) PROBLEM

We will first treat the problem in a domain \( D \) above a graph. Then we will apply a localization procedure to get the result for bounded Lipschitz domains in \( \mathbb{R}^3 \) or \( C^1 \) domains in \( \mathbb{R}^n \).

**Definition 3.1.** We let for \( 0 < \alpha < 1, f \in \text{WA}_{1-\delta}(\partial D) \equiv \text{WA}_{1-\gamma}(\partial D) \) so that in this case \( f_j \in B_{1-\delta}(\partial D) \) and \( f_0 \in L(\partial D) \) and compactly supported.

**Definition 3.2.** We put
\[
B_{1-\delta}(\partial D) = \{ f \in \sum_{k \geq 1} \lambda_k a_k \},
\]
where the sum converges in the sense of distributions, \( a_k \) is an atom on \( \partial D \), i.e. \( \text{supp} \, a_k \subset A_k, \| a_k \|_\infty \leq |A_k|^{-\gamma(n-1)-1} \text{ and } A_k \text{ is a surface ball} \). Further, we put
\[
\| f \|_{B^\alpha_{1-\delta}} = \inf \left\{ \sum \| \lambda_k \| \right\},
\]
where the infimum is taken over all such decompositions.

**Lemma 3.3.** If \( F \in B_{1-\delta}(\partial D), 0 < \alpha < 1 \), then \( \partial F/\partial T_j \in B_{1-\delta}(\partial D) \).

**Proof.** We use the fact that \( B_{1-\delta}(\partial D) \) has an atomic decomposition, [FJ]. Namely, we say that \( a \) is an \( (1-\alpha, 1) \)-atom on \( \partial D \) if \( \text{supp} \, a \subset A, \subset \partial D \), \( |a| \leq r^{-\gamma(n-1)-\gamma} \) and \( |\partial a| \leq r^{-1-\gamma} \). Then if \( f \in B_{1-\delta}(\partial D) \), we have that \( F = \sum \lambda_k \beta_k \) where \( \beta_k \) is a \( (1-\alpha, 1) \)-atom and \( \sum |\lambda_k| \leq C \| F \|_{B^\alpha_{1-\delta}(\partial D)} \). Thus,
\[
\frac{\partial F}{\partial T_j} = \sum \lambda_k \frac{\partial \beta_k}{\partial T_j},
\]
and each \( \partial \beta_k/\partial T_j \) is a \( B_{1-\delta}(\partial D) \)-atom as defined above.
We shall carry out the proof of the following theorem for a Lipschitz domain in $\mathbb{R}^3$, and then make some remarks about the $C^1$ case at the end of the proof.

**Theorem WA$^1_{1,\underline{\omega}}(\partial D)$.** Suppose $D$ is a domain above a graph and that either $D$ is a Lipschitz domain in $\mathbb{R}^3$ or a $C^1$ domain in $\mathbb{R}^n$. For sufficiently small $\omega$ there exist a unique $u$ such that

\[
\begin{aligned}
  \Delta^2 u &= 0 \quad \text{in } D, \\
  (u, \nabla u) &= f \in WA^1_{1,\omega}(\partial D),
\end{aligned}
\]

with

\[
\int_D \delta(x)^\omega |\nabla^3 u(x)| \, dx < \infty,
\]

and moreover,

\[
\int_D \delta(x)^\omega |\nabla^3 u(x)| \, dx \leq \| f \|_{WA^1_{1,\omega}(\partial D)}
\]

\[
= C \left( \| f_0 \|_{L^1(\partial D)} + \sum_{j \geq 1} \| f_j \|_{B^1_{1,\omega}(\partial D)} \right)
\]

The localized version is:

**Theorem WA$^1_{1,\omega}(\partial \Omega)$.** The above theorem is true for a bounded Lipschitz domain in $\mathbb{R}^3$ or a $C^1$ domain in $\mathbb{R}^n$.

For domains $D$ above a graph we can formulate an alternative boundary value problem:

\[
\begin{aligned}
  \Delta^2 u &= 0 \quad \text{in } D \\
  D_n u &= f \in B^1_{1,\omega}(\partial D), \\
  \sum_{\partial D_j} \frac{\partial}{\partial T_j} D_{j, u} &= g \in B^1_{1,\omega}(\partial D)
\end{aligned}
\]

with

\[
\int_D \delta(x)^\omega |\nabla^3 v(x)| \, dx < \infty,
\]

and where the equality on the boundary is in the sense of non-tangential convergence.
Suppose that we can solve \((\ast)_{1-\epsilon}\). Then this gives a solution to the boundary value problem of Theorem WA\(_{1-\epsilon}\). For take any \(f \in WA_{1-\epsilon}(\partial D)\) and set \(f = f_0\) and \(g = \sum_{j=1}^{\epsilon-1} (\partial \partial T_j) f_j\). By the previous lemma, \(g \in B_{1-\epsilon}(\partial D)\). Thus, a solution to \((\ast)_{1-\epsilon}\) for this \(f\) and \(g\), with appropriate estimates, proves the theorem. It is convenient to recast the \(WA_{1-\epsilon}(\partial D)\) problem in the form \((\ast)_{1-\epsilon}\) above in order to reduce the problem to an “atomic” problem. Our first reduction will be to show that one may take \(f = 0\) in \((\ast)_{1-\epsilon}\). To see this, suppose that \(f \neq B_{1-\epsilon}(\partial D)\) and consider the problem

\[
\begin{aligned}
\begin{cases}
  \mathcal{A}u = 0 & \text{in } D, \\
  v = f & \text{on } \partial D
\end{cases}
\end{aligned}
\]

with

\[
\int_D \delta(x)^\ast |\nabla \nabla u(x)| \, dx < \infty.
\]

In [JK], \((D)_{\epsilon}^{-1}\) was shown to have, for sufficiently small \(\epsilon\), a solution \(v\) such that

\[
\int_D \delta(x)^\ast |\nabla \nabla v(x)| \, dx \leq C \|f\|_{B_{1-\epsilon}(\partial D)}.
\]

In fact \((D)_{\epsilon}^{-1}(\partial D)\) was solved for sufficiently small \(\epsilon\) in [JK] in the case of \(\Omega\) a bounded Lipschitz domain, with the estimate

\[
\int_\Omega \delta(x)^\ast |\nabla \nabla v(x)| + |v(x)| + |v(x)| \, dx < \infty.
\]

Here the estimates on the lower order terms \(|\nabla v|\) and \(|v|\) depend on \(|\Omega|\), but it is not hard to see from the proof that \(\delta(x)^\ast |\nabla v| \in L^1\) even if \(\Omega\) is the infinite domain above a graph. To this end, set, in the infinite domain \(D\),

\[
H(x, t) = \int_0^t v(x, s) \, ds + \int_0^\infty v(x, s) - v(x_0, s) - (x - x_0) \nabla_s v(x_0, s) \, ds,
\]

where \(\bar{0} \notin \partial D\), \(X^0 \equiv (x_0, s_0) \notin \partial D\) is fixed, and \(t_0 = \max \{ \varphi(x) : x \in \mathbb{R}^{n-1} \}\), \(\partial D = \{ (x, \varphi(x)) \}\), so that the above integral is convergent and

\[
\begin{aligned}
&A H = 0 & \text{in } D, \\
&\partial_n H(x, t) = v(x, t).
\end{aligned}
\]

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Thus,

$$
\int_D \delta^2(x) \, |\nabla D_h H| \, dx < \infty,
$$

and is bounded by a constant times $\|f\|_{B^{-1}_1(\partial D)}$. Since $H$ is harmonic, in fact

$$
\int_D \delta^2(x) \, |\nabla \nabla H| \, dx < \infty,
$$

by applying the arguments in [S, p. 213].

**Proof of Theorem WA**. From the preceding discussion we may assume that $f = 0$ in the alternate formulation $(\ast)_{1-a}$. Suppose now that $g \in B^{-1}_1(\partial D)$ and that $g$ is an atom. The desired estimate

$$
\int_D \delta^2(x) \, |\nabla \nabla u| \, dx \leq C,
$$

rescales so that we may assume that $g$ is a unit size atom. Let $A$ be the surface ball centered, say, at $0$, of radius 3 containing $\text{supp} \ g$. We have, then, $\int g \, d\sigma = 0$ and $\|g\|_{\infty} \leq 1$. By Theorem 2.3, a biharmonic solution to this atomic problem exists with $(\nabla \nabla u)^n \in L^1(\partial D)$. Define $R_k = D \cap U_{k+3} \setminus U_k$ for $k \geq 4$ where $U_k = \{ (x, y) \in \mathbb{R}^n : |x| < 2^k, \ y < 2M \cdot 2^k \}$, $\varphi(0) = 0$ and $M \geq \|\varphi\|_{\infty}$. Put $\delta_{\varphi}(x) = \text{dist}(x, \partial R_k)$ for $x \in R_k$ and $\delta_{\varphi}(x) = 0$ outside $R_k$.

Let $R_3 = D \cap U_3$. We first provide an estimate on $R_3$:

$$
\int_{R_3} \delta^2(x) \, |\nabla \nabla u(x)| \, dx \leq \left( \int_{D \cap U_3} \delta_{\varphi}(x) \, |\nabla \nabla u(x)|^2 \, dx \right)^{1/2} \left( \int_{D \cap U_3} \delta_{\varphi}^2(x) \, dx \right)^{1/2},
$$

where $\delta_{\varphi}(x) = \text{dist}(x, \partial (D \cap U_3))$. The second factor in the RHS above is bounded by a constant, and the first integral is exactly

$$
\left( \int_{(D \cap U_3)} \delta_{\varphi}^2(\nabla u) \, d\sigma \right)^{1/2},
$$

where $\delta_{\varphi}$ is the square function with respect to the region $D \cap U_3$. Hence, by the equivalence in $L^2$ (Theorem 2.4), between $S_{\varphi}$ and the non-tangential maximal function we have
\[ \begin{align*}
\int_{R_k} \delta^3(x) |\nabla \nabla u(x)| \, dx & \leq \left( \int_{\partial D \cap U_k} ((\nabla u)^* (P))^2 \, d\sigma(P) \right)^{1/2} \leq \left( \int_{\partial D} ((\nabla u)^* (P))^2 \, d\sigma(P) \right)^{1/2} \\
& \leq \left( \int_{\partial D} |\nabla^2 u|^2 \, d\sigma \right)^{1/2} \leq \left( \int_{\partial D} |g|^2 \, d\sigma \right)^{1/2} \leq 1,
\end{align*} \]

by the solvability of the $L^2$ regularity problem. Similarly,
\[ \int_{R_k} \delta^3(x) |\nabla \nabla u(x)| \, dx \leq C \left( \int_{R_k} \delta^3(x) |\nabla \nabla u(x)|^2 \, dx \right)^{1/2} \cdot 2^{k(1+\eta)}, \]
since the dimension is 3. By the equivalence of square function estimates and non-tangential maximal estimates, together with solvability of the $L^2$ regularity problem, we obtain as before,
\[ \left( \int_{R_k} \delta^3(x) |\nabla^3 u(x)|^2 \, dx \right)^{1/2} \leq \left( \int_{R_k} |\nabla^2 u(P)|^2 \, d\sigma(P) \right)^{1/2} . \]

For any $h$, define a truncated maximal function at $Q \in \partial D$, $M_2(h)(Q)$, by
\[ M_2 h(Q) = \sup_{y \in \Gamma(Q) \cap \partial B_j(Q)} |h(y)|, \]
where $r = \text{dist}(Q, \tilde{O})$. (The aperture of $\Gamma(Q)$ is fixed and depends on the Lipschitz constant $M$.) Thus, for some fixed constant $M' \geq 2$,
\[ \left( \int_{R_k} |\nabla^2 u(P)|^2 \, d\sigma(P) \right)^{1/2} \leq \left( \int_{A(M', k)} |M(\nabla^2 u(P))|^2 \, d\sigma(P) \right)^{1/2} , \]
where $A(M', k) = \{ Q \in \partial D : 2^k M' \leq \text{dist}(Q, \tilde{O}) \leq 2^k \}$. Define
\[ \|f\|_{A, p} = \left( \frac{1}{\sigma(A)} \int_A |f|^p \, d\sigma \right)^{1/p} . \]

The main estimate of [PV1], (see also [PV5]), is:
\[ \|M(\nabla^2 u)\|_{A(M', k), 2} \leq (2^k)^{-1/2 - \eta}(\|M(\nabla^2 u)\|_{A(M', 2^k + 1), 2})^{3/4} + (2^k)^{-2 - \eta}, \]
for some small $\eta > 0$ depending on the Lipschitz constant of $D$. We shall see that the $\alpha$ for which the WA$_{1-\alpha}(\partial D)$ problem is solvable, is any $\alpha < \eta$. AN INHOMOGENEOUS DIRICHLET PROBLEM 181
Define
\[ b_j = 2^{j(1+\varepsilon)} \left( \int_{\mathbb{R}^d} |M(\nabla^2 u)|^2 \, dx \right)^{1/2} = 2^{j(1+\varepsilon)} \| M(\nabla^2 u) \|_{L^2(\mathbb{R}^d, 2^{'})}. \]

By (\#),
\[
\begin{align*}
\sum_{k=4}^{\infty} \int_{R_0} \delta_n^a (x) |\nabla^2 u(x)| \, dx & \\
& \leq \sum_{k=4}^{\infty} \frac{b_k}{\sum_{k=4}^{\infty} \frac{b_k}{2^{k(1+\varepsilon)} + 2^{(1+\varepsilon)(k+b_k+1+b_k+2)} + C2^{j(x-a)}}},
\end{align*}
\]
and assuming that \( \alpha < \eta \) the left hand side is bounded by \( C_{\alpha, \eta} \) from Hölder’s inequality and a bootstrapping argument. This finishes the proof in the case of a Lipschitz domain in \( \mathbb{R}^3 \).

In case \( D \) is a \( C^1 \) domain in \( \mathbb{R}^n \) we view this locally as a Lipschitz domain with small Lipschitz constant and in this case the problem is solvable for any \( 1 < p < \infty \) and the main estimate (\#) can be achieved with \( p = 2 \) changed to \( p \) such that \( 1/p = 1 - ((n-2+\varepsilon)/n - 1) \), see [PV5, p. 408] and [PV1]. The argument then proceeds along the same lines.

Next we will do the necessary localization to obtain the \( WA_{\frac{1}{2}}^1 (\partial \Omega) \)-theorem in the case of \( \Omega \) a bounded domain. Again we will only explicitly treat the case of a Lipschitz domain in \( \mathbb{R}^3 \). The \( C^1 \) case follows along the same lines and we will only comment briefly on the necessary modifications.

**Proof of Theorem** \( WA_{\frac{1}{2}}^1 (\partial \Omega) \). Assume that \( \hat{f} \in WA_{\frac{1}{2}}^1 (\partial \Omega) \), i.e., \( f_j \in B_{1}^1 (\partial \Omega) \) and \( f_0 \in L^1 (\partial \Omega) \). From the embedding theorems for Sobolev and Besov spaces, [BL], it follows that \( f_j \in L^{p_0} (\partial \Omega) \), where \( p_0 = 2/1 + \varepsilon \). Hence, if \( \alpha = \gamma (\text{Lip. character of } \Omega) \) is sufficiently small the array \( \hat{f} \) is in \( WA_{\frac{1}{2}}^1 (\partial \Omega) \) for which there exists a unique solution in \( \Omega \) to the Dirichlet problem with this data. Moreover,
\[
\| N(\nabla u) \|_{L^p (\partial \Omega)} \leq C \| \hat{f} \|_{WA_{\frac{1}{2}}^1 (\partial \Omega)},
\]
by the estimates for solutions, together with the imbedding result. We want
to show that
\[
\int_{\Omega} \delta(x)^s |\nabla^3 u(x)| \, dx + \int_{\Omega} |\nabla^2 u(x)| + |\nabla u(x)| + |u(x)| \, dx \leq C \| \hat{f} \|_{W^{1,4}(\partial \Omega)}.
\]
It is clear that for any $K \subset \subset \Omega$ there are constants so that

(i) $\int_{\Omega} |\nabla u(x)| \, dx \leq C(1) \| \hat{f} \|_{W^{1,4}(\partial \Omega)}$,

(ii) $\int_{K} |\nabla^j u(x)| \, dx \leq C(K) \| \hat{f} \|_{W^{1,4}(\partial \Omega)}$, $j = 0, 1, 2, \ldots$

Here (i) follows from the non-tangential maximal function estimate for
solutions, and (ii) follows from (i) and interior estimates.

We can assume that
\[
\int_{\Omega} |\nabla u(x)| \, dx
\]
is a priori finite by standard arguments involving approximating $\Omega$ by
smooth domains, see, e.g., [PV1]. Let $Z$ be a coordinate cylinder “beveled”
in a small neighborhood of the corners to make it a $C^\omega$ domain. Let $pZ$
denote its concentric dilations. Using translations and rotations we can
assume that there is a coordinate system in which $\Omega$ intersected with $pZ$
is given as the set above a Lipschitz graph intersected with $pZ$ for $1/2 \leq p \leq 2$,
that the boundary of $\Omega$ is given accordingly and is simply connected for the
given range of $p$. Further we can assume that in these coordinates, the $n$th
coordinate axis coincides with the symmetry axis of $Z$, and putting for
$1/2 \leq p \leq 2$, $D_p = \Omega \cap pZ$ we can assume that the origin in the new coor-
dinates is on the flat top of $D_p \subset \mathbb{R}^{n-1}$ for fixed $p$.

Then the domain $\bar{D}_p = \{ X \in \mathbb{R}^n: (X/|X|)^2 \in D_p \}$ is the domain above the
graph of a compactly supported Lipschitz function.

We will show, using the graph case result, that
\[
\int_{\Omega \cap (1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx \leq C \| \hat{f} \|_{W^{1,4}(\partial \Omega)} + C \int_{\Omega \cap (3/2)Z} |\nabla^2 u(x)| \, dx.
\]

Adding these inequalities over all coordinate charts gives
\[
\int_{\Omega} \delta(x)^s |\nabla^3 u(x)| \, dx \leq C \| \hat{f} \|_{W^{1,4}(\partial \Omega)} + C \int_{\Omega} |\nabla^2 u(x)| \, dx.
\]

From standard interpolation inequalities, (see [BL]), one has for any $\epsilon > 0$,
\[
\int_{\Omega} |\nabla^2 u(x)| \, dx \leq \epsilon \int_{\Omega} \delta(x)^s |\nabla^3 u(x)| \, dx + C_{\epsilon} \int_{\Omega} \delta(x)^{-s} |u(x)| \, dx.
\]
Furthermore,

\[ \int_\Omega \delta(x)^{-s} |\nabla^2 u(x)| \, dx \leq C(\Omega) \| f \|_{W^{1,s}_{\text{loc}}(\Omega)}, \]

by (i). This finishes the proof of the result. It remains to consider the proof of \((\ast)\). To this end we let \( \Omega_h = \{ x \in \Omega : \text{dist}(\partial \Omega, x) \leq h \} \) and we split the integral

\[ \int_{\Omega \cap (0,1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx \]

into two parts

\[ \int_{\Omega_h \cap (0,1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx \quad \text{and} \quad \int_{(\partial \Omega_h) \cap (0,1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx. \]

The second integral can be estimated by \( C_h \| f \|_{W^{1,s}_{\text{loc}}(\Omega)} \), using (ii). To estimate the first integral we will estimate

\[ \int_{\Omega_h \cap (0,1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx, \]

for \( 1 \leq \rho \leq 3/2 \). We let \( \delta_{\rho}(x) \) be the distance from \( x \) to \( \partial(\Omega \cap \rho Z) \). Then there is a \( h \) such that for \( x \in \Omega \cap (\rho/2)Z \) we have \( \delta(x) \leq \delta_{\rho}(x) \). By introducing a \( C^0(\Omega) \) cut-off function we may think of the Dirichlet data of \( u \mid_{\Omega \cap \rho Z} \) as consisting of a part supported away from \( \partial \Omega \) and a part supported in \( \Omega_h \). Letting \( v \) be the classical solution to the biharmonic Dirichlet problem with the first kind of data, in all of the smooth domain \( \rho Z \). We now estimate

\[ \int_{\Omega_h \cap (0,1/2)Z} \delta(x)^s |\nabla^3 u(x)| \, dx \leq C \int_{\Omega_h \cap (0,1/2)Z} \delta_{\rho}(x)^s |\nabla^3 u(x) - v(x)| \, dx + C \int_{\Omega_h \cap (0,1/2)Z} |\nabla^3 v(x)| \, dx. \]

\((\ast\ast)\)

It follows from interior estimates that the last integral on the right hand side is bounded by a constant independent of \( \rho \) times \( \| u \|_{W^{1,s}_{\text{loc}}(\Omega \cap (\rho,2\rho))} \). This can in turn be estimated by \( C_h \| f \|_{W^{1,s}_{\text{loc}}(\Omega)} \), again from interior estimates and the solvability of the Dirichlet problem.

It remains to treat the first integral on the right hand side of \((\ast\ast)\). Introduce a change of variables \( x = y |y|^2 \). The domain \( D_\rho = \rho Z \cap \Omega \) is mapped to the domain above a compactly supported Lipschitz graph,
which we denote $\bar{D}_\rho$. Furthermore, if $Q \in (\rho/2) Z \cap \partial \Omega$, then $|Q|$ is bounded from above and below. Let

$$w(x) = |x| \left\{ u \left( \frac{x}{|x|^2} \right) - v \left( \frac{x}{|x|^2} \right) \right\}.$$  

Then $w$ is biharmonic in $\bar{D}_\rho \subset \mathbb{R}^3$, and

$$|\nabla^3 w(x)| \sim \frac{1}{|x|^3} \left| \nabla^3 (u - v) \left( \frac{x}{|x|^2} \right) \right|$$

+ terms containing lower order differentiation of $u - v$.

We need to find how the integral under consideration transforms under this change of variables. The Jacobian is of the order of $1/|y|^3$, (in dimension 3), and $|\nabla^3 (u - v)(y/|y|^2)|$ is approximately $|y|^5 |\nabla^3 w(y)|$, modulo lower order terms. The distance function $\delta_{\rho}(x)$ is transformed to $\text{dist}(y, \partial \bar{D}_\rho)$ when $x$ belongs to $(\rho/2) Z \cap \Omega$. Hence,

$$\int_{\delta_{\rho}(x) \sim 1/2Z} \delta_{\rho}(x)^n |\nabla^3 w(x) - v(x)| \, dx$$

$$\leq C \int \text{dist}(y, \partial \bar{D}_\rho)^n |\nabla^3 w(y)| \frac{dy}{|y|}$$

+ lower order terms.

The lower order terms can be absorbed into an error term, so we concentrate only on the last integral above. By the theorem above a graph, the integral is bounded by $|\bar{w}|_{W^{3,1}(-\delta \bar{D}_\rho)}$, which after transforming back to the domain $\rho Z \cap \Omega$ is bounded by $|\bar{u} - \bar{v}|_{W^{3,1}((\rho Z \cap \Omega) \cap \partial \rho Z)}$. On this boundary part we can estimate $|\bar{u}|_{W^{3,1}(-\delta \bar{D}_\rho)}$ by the values on the boundary of $\rho Z$ by the maximum principle, since $\rho Z$ is a $C^\infty$ domain, ([$A$, ADN]). Note that the constant in these estimates will be uniform in $\rho$. Note also that the boundary values for $v$ on $\rho Z$ are non-zero only in $0 < \rho < \rho Z$. Thus, by interior estimates for $u$ we can estimate $|\bar{u}|_{W^{3,1}(-\rho Z \cap \partial \rho Z)}$. Consider now $|\bar{u}|_{W^{3,1}((\rho Z \cap \Omega) \cap \partial \rho Z)}$. On that part of the boundary which coincides with $\partial \Omega$ we recover $|\bar{u}|_{W^{3,1}(\partial \rho Z)}$. On the remaining part we estimate the norm by

$$\int_{\partial (\rho Z \cap \rho Z)} |\nabla u(x)| \, d\sigma(x),$$

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which after averaging in $\rho$ can be estimated by the error term
\[ \int_{(3/2)Z} |\mathcal{V}u(x)| \, dx. \]

This finishes the proof in the Lipschitz case.

For the $C^1$ case we note that the imbedding theorems used above for the Lipschitz case, will give boundary values for a Dirichlet problem in $L^{p_0}$ where $p_0$ is not necessarily close to 2; but for the $C^1$ situation we have the solvability of the Dirichlet problem in the full range $1 < p$. ([V1]).

4. AN APPLICATION TO HARMONIC BERGMAN SPACES

For a domain $\Omega$ in $\mathbb{R}^n$ we denote by $M^p(\Omega)$ the closure in $L^p(\Omega)$ of \{ $A\varphi, \varphi \in C^\infty_c(\Omega)$ \}. Further, we put $L^p_0(\Omega) = \{ f \in L^p(\Omega); Af = 0 \}$. The harmonic projection $B: L^2(\Omega) \to L^2_b(\Omega)$ is the operator defined by $Bf(x) = \int_{\Omega} K(x, z) f(z) \, dz$, where $K$ is the reproducing kernel for the space $L^2_b(\Omega)$.

The following questions have been studied in the literature; see e.g. [CC] for further details and references to the problems treated in this section. For what domains do we have, for some $1 < p < \infty$, that

1. $(L^p_b(\Omega))^* = L^p(\Omega)$ where $p$ and $p'$ are dual exponents,
2. $L^p(\Omega) = L^p_0(\Omega) \oplus M^p(\Omega)$, direct sum,
3. $B: L^p(\Omega) \to L^p_b(\Omega)$ is bounded?

It is straightforward to see that (1) and (2) are equivalent for all $p$ and all $\Omega$, and that all the statements are true for $p = 2$. In [CC] it was shown that question (1) can be reduced to a biharmonic problem
\[
\begin{cases}
A^2 u = w & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial N} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with $u \in W^{2, p}(\Omega)$. We briefly recall the arguments. For all $1 < p < \infty$, $L^p_b(\Omega) \subset (L^p_b(\Omega))^*$, i.e., every element $\varphi \in L^p_b(\Omega)$ gives rise to a bounded linear functional
\[
\lambda_\varphi(f) = \int_\Omega \varphi f \, dx.
\]

Denote by $i_p$ the inclusion map in this case. Question: When is $i_p$ one to one and onto? Fact: For $p > 2$, $L^p_b(\Omega) \subset L^p_b(\Omega) = (L^p_b(\Omega))^*$, which implies that $i_p$ is injective when $p > 2$. Furthermore, $(i_p)^* = i_{p'}$, $(p, p') = 1$, so that
Claim \( i_p \) is injective if \( i_{p'} \) is onto for \( p' > 2 \). Hence, in order to show that the duality property (1) above holds, it suffices to show that \( i_p \) is surjective for \( p > 2 \). Let \( \hat{\lambda} \in (L^p_w(\Omega))^* \), i.e. there exist a \( \varphi \in L^p(\Omega) \) such that

\[
\hat{\lambda}(u) = \iint_{\Omega} \varphi u \, dx, \quad \forall u \in L^p_w(\Omega). \tag{4.1}
\]

We want to find a \( \varphi \in L^p(\Omega) \) satisfying (4.1) and such that \( A\varphi = 0 \). To this end, find \( w \) such that \( Aw = \varphi \). If \( w \) is biharmonic we are done. If not, fix a \( \psi \) and consider

\[
\hat{\lambda}(u) = \iint_{\Omega} \varphi u \, dx = \iint_{\partial \Omega} (Aw - \psi) \, u \, dx + \iint_{\Omega} A\psi \, u \, dx.
\]

If

\[
\begin{cases}
A^2 v = 0 & \text{in } \Omega \\
\left. \begin{array}{l}
v, \frac{\partial v}{\partial N} \\
\frac{\partial}{\partial N}
\end{array} \right| \left. \begin{array}{l}
w, \frac{\partial w}{\partial N} \\
\frac{\partial}{\partial N}
\end{array} \right| = 0 & \text{on } \partial \Omega \\
\text{and } A\psi \in L^p(\Omega),
\end{cases}
\tag{4.2}
\]

then

\[
\hat{\lambda}(u) = \iint_{\Omega} A\psi \, u \, dx.
\]

Since \( A^2 v = 0 \) we are done. To solve (4.2), we need to find a \( \Phi \), where we think of \( \Phi \) as \( v - w \), such that

\[
\begin{cases}
A^2 \Phi = A^2 w \\
\left. \begin{array}{l}
\Phi, \frac{\partial \Phi}{\partial N} \\
\frac{\partial}{\partial N}
\end{array} \right| = 0
\end{cases}
\tag{4.3}
\]

and \( A\Phi \in L^p(\Omega). \)

Now, \( Aw = \varphi \in L^p(\Omega) \) so \( A^2 \psi \in W^{2,p}(\Omega). \) Therefore, what is needed is to solve (4.3) with data in \( W^{2,p}(\Omega) \). According to Theorem 2.1 with \( \alpha = -2 \):

**Theorem 4.4.** If \( \Omega \subset \mathbb{R}^3 \) is a Lipschitz domain the duality property \((L^p_w(\Omega))^* = L^p_w(\Omega)\) holds in the range \( \frac{1}{2} - \delta_0 < p < 3 + \delta' \), where \( \delta_0, \delta' \) depend on the Lipschitz character of the domain. If \( \Omega \subset \mathbb{R}^n \) is a \( C^1 \) domain, then the duality holds for all \( 1 < p < \infty \). The range \( \frac{1}{2} - \delta_0 < p < 3 + \delta' \) is sharp, as will be demonstrated in the following section.
In this short section we will sketch some rather simple counterexamples. As is well known, the Green function for the biharmonic operator may change sign, see, e.g., [H] for these properties, a lucid historical account of the results on the sign of the Green function and further references. In [JK], the counterexamples rely heavily on conformal mappings of the Green function and use the positivity of that function in showing the sharpness of the results obtained in certain ranges. A study along the lines of [JK] seems to be a difficult task in light of the lack of positivity of the Green function for the biharmonic operator. The sharpness in connection with these questions remains here an open problem. Nevertheless, certain of the results can be seen to be sharp from examples of biharmonic functions in the exterior of a cone.

Let \[ \Omega_\varepsilon, \varepsilon \geq 0, \] be the open subset of the unit sphere in \( \mathbb{R}^n \) with complement the spherical cap of radius \( \varepsilon \) about the south pole. Put \( \Gamma_\varepsilon = \Omega_\varepsilon \times R_+ \subset \mathbb{R}^n \).

From the techniques of [PV1, Lemma 10.6] it is readily seen that there exists a function \( u \) in \( \Gamma_\varepsilon \) with \( A^2 u = 0 \), \( u \), \( \partial u / \partial N = 0 \) on the boundary \( \partial \Gamma_\varepsilon \), and of the form

\[
u(x) = |x|^{-1} \varphi(x |x|),\]

where \( \varphi(x) \downarrow 1 \) as \( \varepsilon \to 0 \). In a neighborhood \( N \) of \( \overline{0} \), \( \nabla u \notin L^p_0(N) \) for \( p > 2 \) and \( \alpha > 3/p + \delta \), where \( \delta = \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). To see this, consider \( u_R \) where \( R \) is the ray \( \{ 0 < x_3 < \infty \} \). Then \( |\nabla u| \sim |x|^\beta \) where \( \beta \to 0 \), but

\[
\int_R |\nabla u|^p |x|^{-(3/(2p))} \ dx = \infty.
\]

Thus, since \( \nabla u \mid_{\partial R} = 0 \), the Sobolev embedding theorem implies that \( \nabla u \notin L^2_0 \).

This shows that the range \( 3 + \alpha < 3/p + \varepsilon, p > 2 \), in (iii) of Theorem 2.1 is sharp. Indeed, the lower bound on \( \alpha \) is necessary even in smooth domains, since the homogeneous problem is not uniquely solvable below that range. These results thus imply that the range \( \frac{3}{2} \leq p \leq 3 \) for \( n = 3 \) is sharp for the duality of Bergman spaces as in Theorem 4.4.

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