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Classification of a class of crossed product C^*-algebras associated with residually finite groups

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Abstract

A residually finite group acts on a profinite completion by left translation. We consider the corresponding crossed product C^*-algebra for discrete countable groups that are central extensions of finitely generated abelian groups by finitely generated abelian groups (these are automatically residually finite). We prove that all such crossed products are classifiable by K-theoretic invariants using techniques from the classification theory for nuclear C^*-algebras.

Keywords: Classification of C^*-algebras; Crossed product; Residually finite group; $C(X)$-algebra; Decomposition rank; Generalized Bunce–Deddens algebra

1. Introduction

In [16] Orfanos introduced a class of C^*-algebras generalizing the classical Bunce–Deddens algebras [4]. These generalized Bunce–Deddens algebras can be constructed starting with any discrete, countable, amenable and residually finite group—the construction yields the classical Bunce–Deddens algebras when starting with $\mathbb{Z}$. They are C^*-algebras of the form $C(\tilde{G}) \rtimes G$, where G acts on a profinite completion $\tilde{G}$ by left translation. From the point of view of the classification theory for nuclear C^*-algebras [10,23], these C^*-algebras enjoy many desirable properties: they are simple, separable, nuclear and quasidiagonal; they have real rank zero, stable rank one, unique trace and comparability of projections (see [16]).

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In this note we prove that the generalized Bunce–Deddens algebras associated with a discrete, countable and residually finite group G that is a central extension of a finitely generated abelian group by a finitely generated abelian group have finite decomposition rank (Proposition 2.1). This result covers all the profinite completions of G, see Section 2. For this, we will use that the decomposition rank of a $C(X)$-algebra is finite as long as X has finite covering dimension and the decomposition rank of the fibers is uniformly bounded (Lemma 3.1). We are then free to use powerful theorems of Lin and Winter [14,25] to observe that the generalized Bunce–Deddens algebras associated with such groups G are classified by K-theory. The above class of groups contains some examples of interest, such as the integer Heisenberg group (of any dimension). It is also shown that the decomposition rank of $C^*(G)$ is finite for all such groups (Theorem 2.2).

The classical Bunce–Deddens algebras were classified by Bunce and Deddens along the same lines as the Uniformly Hyperfinite (UHF) algebras. Tensor products of Bunce–Deddens algebras were classified by Pasnicu in [18]. In both cases the algebras in question are inductive limits of circle algebras—in fact, of algebras of the form $C(\mathbb{T}) \otimes M_r$. The trivial observation that these algebras have slow dimension growth (owing to the fact that $\mathbb{T}$ has covering dimension equal to 1) can be used to subsume these results under subsequent and very general classification theorems ([8,5], see also [23, Theorem 3.3.1]).

In the present case, the algebras in question are inductive limits of C^*-algebras of the form $C^*(L) \otimes M_r$ for certain (usually non-abelian) subgroups $L \subset G$. In the place of covering dimension we use decomposition rank, a noncommutative analog of covering dimension introduced by Kirchberg and Winter [13].

Let $\mathcal{C}$ be the class of all unital, separable and simple C^*-algebras A with real rank zero and finite decomposition rank that satisfy the Universal Coefficient Theorem, and such that $\partial_e T(A)$, the extreme boundary of the tracial state space, is compact and zero-dimensional. The Elliott invariant for such algebras is their ordered K-theory:

$$\mathrm{Ell}(A) = \big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A)\big).$$

(See [10] for the general version of the Elliott invariant.) A combination of results of Lin and Winter gives:

Theorem 1.1. *(See [25, Corollary 6.5].) Let $A, B \in \mathcal{C}$. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

In the early 1990s Elliott [7] initiated a program to classify nuclear C^*-algebras and conjectured such a result (for a larger class and with a refined invariant, including for example the space of tracial states). One aim of this paper is to show that the class of generalized Bunce–Deddens algebras associated with a large collection of groups is a natural example where the program succeeds.

This paper has three additional sections. In Section 2 we recall the relevant definitions, state our main result (Theorem 2.3) and the key technical result behind its proof (Proposition 2.1). In Section 3 we recall the definition of decomposition rank and prove a lemma concerned with the decomposition rank of $C(X)$-algebras. Section 4 contains the proof of Proposition 2.1. The proof uses a continuous field structure of certain group C^*-algebras due to Packer and Raeburn [17]. The decomposition rank such a group C^*-algebra is estimated in terms of the ranks of the normal subgroup and its quotient (Theorem 2.2). For this, we use a result of Poguntke [20] concerning the structure of twisted group algebras of abelian groups. The decomposition rank of these twisted group algebras is then estimated using the decomposition rank of noncommutative tori, for which we provide an estimate in Lemma 4.4.

2. Main results

We first recall Orfanos' definition of a generalized Bunce–Deddens algebra. Let G be a discrete, countable and amenable group. Assume also that G is residually finite, so that there is a nested sequence of finite index, normal subgroups of G with trivial intersection. Let $G \supset L_1 \supset L_2 \supset \cdots$ be such a sequence. The *profinite completion* of G with respect to this sequence is the inverse limit

$$\tilde{G} = \varprojlim G/L_i$$

with connecting maps $xL_{i+1} \mapsto xL_i$. This is a compact, Hausdorff and totally disconnected group. The group G acts by left multiplication. The corresponding crossed product

$$C(\tilde{G}) \rtimes G$$

is called a *generalized Bunce–Deddens algebra* associated with G.

For example, let $q = (q_i)$ be a sequence of positive integers with $q_i | q_{i+1}$ for every i. With $G = \mathbb{Z}$ and $L_i = q_i\mathbb{Z}$, the above construction yields the usual Bunce–Deddens algebra of type q.

Let $\mathcal{G}$ be the collection of all (discrete) groups G with the following property: there exist finitely generated abelian groups N and Q such that G is a central extension

$$1 \to N \to G \to Q \to 1 \tag{1}$$

(where by "central" we mean that the image of N lies in the center of G). A celebrated theorem of Hall [22, 15.4.1] states that a finitely generated group that is an extension of an abelian group by a nilpotent group is residually finite. The groups in $\mathcal{G}$ are therefore residually finite. They are also amenable, as an extension of amenable groups is itself amenable. As noted in the introduction, $\mathcal{G}$ contains the integer Heisenberg groups of all dimensions (see Example 2.4 below).

Proposition 2.1. *If $G \in \mathcal{G}$, then any generalized Bunce–Deddens algebra $C(\tilde{G}) \rtimes G$ associated with G has finite decomposition rank.* (*This covers all profinite completions $\tilde{G}$ of G.*)

See Section 3 for the definition of decomposition rank and Section 4 for the proof. One ingredient of its proof is the following result, also proved in Section 4.

Theorem 2.2. *Let G be a countable, discrete group that is a central extension*

$$1 \to N \to G \to Q \to 1$$

where N and Q are finitely generated abelian groups of ranks n and m, respectively. Then

$$\operatorname{dr} C^*(G) \leqslant (n+1)\big(2m^2 + 4m + 1\big) - 1.$$

Our main result can now be stated as:

Theorem 2.3. *The generalized Bunce–Deddens algebras $C(\tilde{G}) \rtimes G$ associated with groups $G \in \mathcal{G}$ are classified by their Elliott invariant.*

Proof. Corollary 7 and Theorem 12 of [16] show that all generalized Bunce–Deddens algebras are unital, separable, simple, and that they have real rank zero and unique trace. By Proposition 2.1 the ones associated with groups in $\mathcal{G}$ have finite decomposition rank, so they satisfy the hypothesis of the classification Theorem 1.1. □

Example 2.4. Fix a positive integer m. Define $\mathbb{H}_{2m+1}$ as the group of all square matrices of size $m+2$ of the form

$$\begin{pmatrix} 1 & x^{\mathrm{t}} & z \\ 0 & 1_m & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y \in \mathbb{Z}^m$, $z \in \mathbb{Z}$, and 1_m is the identity matrix in M_m. This is a discrete, countable and residually finite group. Indeed, let (q_i) be a sequence of positive integers such that $q_i | q_{i+1}$ for all i. For each i let L_i be the subgroup of $\mathbb{H}_{2m+1}$ consisting of those matrices with $x, y \in (q_i\mathbb{Z})^m$ and $z \in q_i\mathbb{Z}$. This provides a nested sequence of finite index, normal subgroups with trivial intersection. Moreover, $\mathbb{H}_{2m+1}$ is a central extension

$$1 \to \mathbb{Z} \to \mathbb{H}_{2m+1} \to \mathbb{Z}^{2m} \to 1,$$

as one can check (notice that the center of $\mathbb{H}_{2m+1}$ is generated by the matrix with $x = y = 0$ and $z = 1$). Thus $\mathbb{H}_{2m+1} \in \mathcal{G}$.

It is well known that $C^*(\mathbb{H}_3)$ may be regarded as the C^*-algebra of sections of a continuous field of (irrational and rational) rotation algebras over $\mathbb{T}$ [1]. Similarly, one can prove that the C^*-algebra of $\mathbb{H}_{2m+1}$ may be regarded as the C^*-algebra of a continuous field of noncommutative tori (in fact, of m-th tensor powers of rotation algebras) over $\mathbb{T}$. A theorem of Packer and Raeburn [17] shows that this is an instance of a more general phenomenon, a fact we use in Section 4.

3. The decomposition rank of $C(X)$-algebras

For any two C^*-algebras A and B, we have

$$\mathrm{dr}(A \otimes B) \leqslant (\mathrm{dr}\, A - 1)(\mathrm{dr}\, B - 1) + 1,$$

see Remark 3.2 of [13]. In this section we prove an analogous estimate for $C(X)$-algebras. Let us first recall some of the relevant definitions.

Let A and B be C^*-algebras. A completely positive (c.p.) map $\phi : F \to A$ is said to have *order zero* if $\phi(a)\phi(b) = 0$ for all $a, b \in A_+$ with $ab = 0$. We refer the reader to [26] for a detailed treatment of order zero maps. A c.p. map $\phi : \bigoplus_{i=1}^{s} M_{r_i} \to A$ is *k-decomposable* if there is a partition $\bigsqcup_{j=0}^{k} I_j = \{1, \ldots, s\}$ such that ϕ restricted to $\bigoplus_{i \in I_j} M_{r_i}$ has order zero for all $j \in \{0, \ldots, k\}$.

In [13], Kirchberg and Winter introduced a noncommutative analog of topological covering dimension called decomposition rank. A separable C^*-algebra A has *decomposition rank k*,

$\operatorname{dr} A = k$ for short, if k is the least integer such that the following holds: for any finite subset $\mathcal{F}$ of A and any $\epsilon > 0$, there are a finite dimensional C^*-algebra F and c.p. contractions $\psi : A \to F$, $\phi : F \to A$, with ϕ k-decomposable, satisfying

$$\|a - \phi\psi(a)\| < \epsilon$$

for all a in $\mathcal{F}$. (That is, (F, ψ, ϕ) is a c.p. approximation of $\mathcal{F}$ to within ϵ.) We have, for example, that $\operatorname{dr} C(X) = \dim X$ when X is a compact metric space (see [13, Proposition 3.3]).

A $C(X)$-*algebra* is a C^*-algebra A endowed with a nondegenerate $*$-homomorphism of $C(X)$ to the center of the multiplier algebra of A. (Nondegeneracy means that $C(X)A$ is dense in A.) Consult for example [3, §2.1] for an introduction to $C(X)$-algebras. Such algebras were introduced by Kasparov in [12] and generalize continuous fields of C^*-algebras; they may be regarded as "upper semicontinuous" fields of C^*-algebras.

Let A be a $C(X)$-algebra. For a closed subset $F \subset A$ we write $A|_F$ for the quotient of A by the (closed) ideal $C_0(X \setminus F)A$. If F consists of a single point x we write $A(x)$ instead of $A|_{\{x\}}$ and call $A(x)$ the *fiber* of A at x. The image of an element $a \in A$ under the quotient map of A to $A(x)$ is written $a(x)$.

Lemma 3.1. *Let X be a compact metric space and A a (separable) $C(X)$-algebra. If* $\dim X \leqslant l$ *and* $\operatorname{dr} A(x) \leqslant k$ *for every $x \in X$, then* $\operatorname{dr} A \leqslant (l+1)(k+1) - 1$.

Proof. Let a finite subset $\mathcal{F}$ of A and $\epsilon > 0$ be given. We claim that for every $x \in X$ there are a finite dimensional C^*-algebra F_x and c.p. contractions

$$A \xrightarrow{\psi_x} F_x \xrightarrow{\phi_x} A,$$

with ϕ_x k-decomposable, such that

$$\|(\phi_x\psi_x - \mathrm{id}_A)(a)(x)\| < \epsilon \tag{2}$$

for all a in $\mathcal{F}$.

Fix $x \in X$. Because $\operatorname{dr} A(x) \leqslant k$, there is a c.p. approximation (F_x, ψ^x, ϕ^x) of $\{a(x)\colon a \in \mathcal{F}\}$ to within $\epsilon/2$, with ϕ^x k-decomposable. Let ψ_x be the composition of ψ^x with the quotient map $A \to A(x)$. By Remark 2.4 of [13], ϕ^x lifts to a c.p. map $\Phi^x : F_x \to A$ that is k-decomposable, but not necessarily contractive.

We may assume that $\Phi^x(1)(x) = \phi^x(1) = \|\phi^x\| \neq 0$ (we abbreviate 1_{F_x} to 1). Recall from [3] that for every $a \in A$ the function $N(a) : y \mapsto \|a(y)\|_{A(y)}$ is upper semicontinuous on X and satisfies $N(fa) = |f|N(a)$ for every $f \in C_b(X)$. Then there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $N(\Phi^x(1))(y) \leqslant 1 + \epsilon/2$.

Let $h \in C(X)_+$ be of norm $1/(1+\epsilon/2)$, taking on its maximum value at x and vanishing on $\{y\colon d(x,y) \geqslant \delta\}$. Then

$$\|h\Phi^x(1)\| = \sup_{y \in X} N\big(h\Phi^x(1)\big)(y) = \sup_{\{y\colon d(y,x)<\delta\}} h(y)N\big(\Phi^x(1)\big)(y) \leqslant 1.$$

This proves that the k-decomposable c.p. map $\phi_x : f \mapsto h\Phi^x(f)$, $f \in F_x$, is contractive. Moreover,

$$\left\|\phi_x(f)(x) - \phi^x(f)\right\| < \epsilon/2,$$

and therefore ψ_x and ϕ_x satisfy (2). This proves the claim.

Since $\dim X = l$, we may apply Proposition 1.5 of [13] to obtain $x_1, \dots, x_s \in X$, an open cover $\{U_i\}_{i=1}^s$ of X, and a partition $\bigsqcup_{j=0}^l I_j = \{1, \dots, s\}$ such that for all $i \in \{1, \dots, s\}$, $j \in \{0, \dots, l\}$, $a \in \mathcal{F}$, and $y \in U_i$, we have

$$U, U' \in I_j \quad \Rightarrow \quad U \cap U' = \emptyset, \quad \text{and} \quad \left\|(\phi_{x_i}\psi_{x_i} - \mathrm{id}_A)(a)(y)\right\| < \epsilon.$$

Let $\{h_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define

- $F := \bigoplus_{i=1}^s F_{x_i}$,
- $\psi : A \to F$ by $\psi(a) = \psi_{x_1}(a) \oplus \dots \oplus \psi_{x_s}(a)$, and
- $\phi : F \to A$ by $\phi(f_1 \oplus \dots \oplus f_s) = \sum h_i \phi_{x_i}(f_i)$.

Then ψ and ϕ are c.p. contractions with ϕ an $((l+1)(k+1)-1)$-decomposable map. A standard estimate gives $\|\phi\psi(a) - a\| < \epsilon$ for all a in $\mathcal{F}$, showing that $\mathrm{dr}\, A \leqslant (l+1)(k+1) - 1$. □

4. Continuous fields of twisted group algebras

We begin by briefly reviewing the definition of a twisted group C^*-algebra. We restrict ourselves to discrete groups. For a more general treatment, see for example [6].

Let G be a discrete group. A *multiplier* (or *normalized 2-cocycle* with values in $\mathbb{T}$) on G is a map $\omega : G \times G \to \mathbb{T}$ satisfying

$$\omega(s,1) = \omega(1,s) = 1 \quad \text{and} \quad \omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r)$$

for all $s, t, r \in G$. If $f : G \to \mathbb{T}$, then we refer to

$$\partial f(s,t) := f(s)f(t)\overline{f(st)}$$

as a coboundary. Two multipliers ω and ω' are *cohomologous* if they differ by a coboundary. The group of all cohomology classes $[\omega]$ is written $H^2(G, \mathbb{T})$, and is in fact isomorphic to the (usual) second group cohomology of G with coefficients in $\mathbb{T}$ (because G is discrete).

Let ω be a multiplier on G. Define an ω-twisted convolution product and an ω-twisted involution on $C_c(G)$ by

$$\left(\sum_s \lambda_s s\right) *_\omega \left(\sum_t \mu_t t\right) = \sum_{s,t} \lambda_s \mu_t \omega(s,t) st$$

and

$$\left(\sum_s \lambda_s s\right)^* = \sum_s \overline{\omega\left(s^{-1}, s\right)\lambda_s} s^{-1}.$$

We write $C_c(G,\omega)$ for $C_c(G)$ with these operations. To complete this $*$-algebra, let us first define an *ω-representation* of G to be a map V of G to the unitaries on some Hilbert space such that $V_sV_t = \omega(s,t)V_{st}$ for all $s,t \in G$. It is not hard to see that every ω-representation gives rise to a $*$-representation of $C_c(G,\omega)$ and that, conversely, every nondegenerate $*$-representation of $C_c(G,\omega)$ arises in this way.

The *(full) twisted group C^*-algebra* $C^*(G,\omega)$ is the enveloping C^*-algebra of $C_c(G,\omega)$. There is a reduced version as well, and they coincide for amenable groups. One can show that cohomologous multipliers yield isomorphic twisted group C^*-algebras.

A multiplier ω determines a subgroup G that we will make use of later. The *symmetry group* of ω is the subgroup

$$S_\omega = \{s \in G\colon\ \omega(s,t) = \omega(t,s) \text{ for all } t \in G\}.$$

The multiplier ω is called *totally skew* if S_ω is trivial. Proposition 32 of [11] shows that, for abelian G, $C^*(G,\omega)$ is simple when ω is totally skew. (The results of Green [11] that we use are stated in terms of twisted covariance algebras, but may be easily translated to results concerning twisted group algebras—see the remarks following the next example.)

Example 4.1. Let $\theta = [\theta_{ij}]$ be an $m \times m$ skew-symmetric real matrix. An *m-dimensional noncommutative torus* [21] is the universal C^*-algebra generated by m unitaries $U_1, \dots, U_m$ satisfying the generalized Weyl-commutation relations

$$U_jU_i = e^{2\pi i\theta_{ij}}U_iU_j.$$

A noncommutative torus may alternatively be regarded as a twisted group C^*-algebra of $\mathbb{Z}^m$. Indeed, let ω_θ be the multiplier given by

$$\omega_\theta(a,b) = e^{-\pi i a^{\mathrm{t}}\theta b}$$

for all $a,b \in \mathbb{Z}^m$. Let V be an ω-representation of $\mathbb{Z}^m$ and write e_i for the canonical generators of $\mathbb{Z}^m$. Then the generators satisfy

$$V_{e_j}V_{e_i} = e^{2\pi i\theta_{ij}}V_{e_i}V_{e_j}.$$

It follows that $C^*(\mathbb{Z}^m,\omega_\theta)$ is the universal C^*-algebra generated by m unitaries satisfying the same commutation relations as above. In other words, $C^*(\mathbb{Z}^m,\omega_\theta) = A_\theta$. It is well known that *every* twisted group C^*-algebra of $\mathbb{Z}^m$ is an m-dimensional noncommutative torus (see for example §2.2 of [9]).

For the next lemma we will use a result of Green [11] to pass from a primitive quotient of a twisted group algebra to a simple twisted group algebra. Since it was stated in terms of twisted covariance algebras we will briefly explain the terminology and indicate how to translate the result to the language of twisted group algebras.

A *covariant system* $(H, A, \mathcal{T})$ is a C^*-dynamical system (H, A, α) together with a homomorphism $\mathcal{T}$ of a normal subgroup N of H to the unitary group of the multiplier algebra of A, satisfying

$$\operatorname{Ad}\mathcal{T} = \alpha|_N \quad \text{and} \quad \alpha_s(\mathcal{T}(n)) = \mathcal{T}(sns^{-1})$$

for $n \in N$ and $s \in H$. Associated to such a covariant system is a *twisted covariance algebra* $C^*(H, A, \mathcal{T})$. Proposition A1 of [17] explains how certain twisted covariant algebras may be regarded as twisted group algebras. Briefly, when $A = \mathbb{C}$ (so H acts trivially), $\mathcal{T}$ takes values in $\mathbb{T}$ and one defines a multiplier ω on H/N by

$$\omega(sN, tN) = \mathcal{T}\bigl(c(s)c(t)c(st)^{-1}\bigr) \quad \text{for } sN, tN \in H/N,$$

where $c : H/N \to H$ is a Borel section. What one concludes is that $C^*(H, \mathbb{C}, \mathcal{T})$ is isomorphic to $C^*(H/N, \omega)$ (consult [17] for details).

On the other hand, one may regard twisted group algebras as twisted covariance algebras. If $C^*(G, \omega)$ is a twisted group algebra, let G_ω be the universal extension corresponding to ω: as a set G_ω is just $\mathbb{T} \times G$ and the multiplication is given by

$$(z, s)(w, t) = \bigl(zw\omega(s, t), st\bigr).$$

Let $\mathcal{T}_\omega$ be the identity map of $\mathbb{T} \subset G_\omega$. Then $C^*(G_\omega, \mathbb{C}, \mathcal{T}_\omega)$ is isomorphic to $C^*(G, \omega)$ (again, see [17]).

Proposition 4.2. *(See [11, Proposition 34(i)].) Let $(H, \mathbb{C}, \mathcal{T})$ be a covariant system with $\mathcal{T}$ an isomorphism of N onto $\mathbb{T}$ and assume that H/N is abelian (this is what Green calls a "reduced abelian system"). Let Z be the center of H. Then there is a totally skew system $(H', \mathbb{C}, \mathcal{T}')$ for which $H'/N' \cong H/Z$ and $C^*(H, \mathbb{C}, \mathcal{T})/P$ is isomorphic to $C^*(H', \mathbb{C}, \mathcal{T}')$ for every primitive ideal P of $C^*(H, \mathbb{C}, \mathcal{T})$.*

That $(H', \mathbb{C}, \mathcal{T}')$ is *totally skew* means that the normal subgroup N' of H' is exactly the center of H'; equivalently, the corresponding multiplier on H'/N' is totally skew.

Let us restate this in the form we will use below. If $C^*(G, \omega)$ is a twisted group algebra with G abelian, then $(G_\omega, \mathbb{C}, \mathcal{T}_\omega)$ is a reduced abelian system. The proposition asserts that there is a totally skew multiplier σ on $G_\omega/Z(G_\omega) \cong G/S_\omega$ such that $C^*(G/S_\omega, \sigma)$ is isomorphic to $C^*(G, \omega)/P$ for every primitive ideal P of $C^*(G, \omega)$.

We will also need a special case of a result of Poguntke that deals with general locally compact groups. We state the full result below along with the simplifications that come with restricting to discrete groups.

Proposition 4.3. *(See [20, Corollary 6].) Let G be a locally compact abelian group, and let ω be a measurable cocycle. The anti-symmetrization $(x, y) \mapsto \omega(s, t)\omega(t, s)$ of ω induces the structure of a quasi-symplectic space on G/S_ω in the terminology of* [15]. *Suppose that the invariant* $\mathrm{Inv}(G/S_\omega)$ *of this space (see below) contains $\mathbb{Z}^m$ for a certain m. Then the twisted group algebra $C^*(G, \omega)$ is isomorphic to the tensor product of $C(\widehat{S_\omega})$, an m-dimensional noncommutative torus and the algebra of compact operators on a Hilbert space $\mathcal{H}$.*

If we assume that G is finitely generated, the definition of $\mathrm{Inv}(G/S_\omega)$ [15, Definition 1.14] indicates that $\mathrm{Inv}(G/S_\omega)$ contains the free abelian group with rank equal to that of G (see also [15, Example 1.13]). The proof of Theorem 1 of [20] also shows that, when G is discrete, the space $\mathcal{H}$ is separable.

Since decomposition rank is preserved under stable isomorphism (Corollary 3.9 of [13]), we see why the decomposition rank of the twisted group algebra of a finitely generated abelian group may be estimated using the decomposition rank of noncommutative tori. This leads to our next lemma.

Lemma 4.4. *An m-dimensional noncommutative torus A_θ has decomposition rank at most $2m+1$.*

Proof. If A_θ is simple, then it is an A$\mathbb{T}$ algebra by a theorem of Phillips [19], and therefore has decomposition rank at most 1 (and in fact exactly 1 since it is not an AF algebra). Assume that A_θ is not simple.

We will reduce the nonsimple case to the simple one using a continuous field argument. Theorem 1.5 of [17] implies that the primitive ideal space of $C^*(\mathbb{Z}^m, \omega_\theta)$ is homeomorphic to $\widehat{S_{\omega_\theta}}$. By the Dauns–Hoffman theorem [2, IV.1.6.7], $C^*(\mathbb{Z}^m, \omega_\theta)$ is a $C(\widehat{S_{\omega_\theta}})$-algebra and the fibers are its primitive quotients. To get the required estimate using Lemma 3.1, it is enough to show that the primitive quotients of $C^*(\mathbb{Z}^m, \omega_\theta)$ all have decomposition rank at most 1.

We first use Proposition 4.2. We obtain a totally skew multiplier σ on $\mathbb{Z}^m/S_{\omega_\theta}$ such that $C^*(\mathbb{Z}^m, \omega_\theta)/P$ is isomorphic to $C^*(\mathbb{Z}^m/S_{\omega_\theta}, \sigma)$ for every primitive ideal P of $C^*(\mathbb{Z}^m, \omega_\theta)$. Next, applying Proposition 4.3 to the group $\mathbb{Z}^m/S_{\omega_\theta}$ and the totally skew multiplier σ, we get that $C^*(\mathbb{Z}^m/S_{\omega_\theta}, \sigma)$ is stably isomorphic to a simple noncommutative torus. But stably isomorphic C^*-algebras have the same decomposition rank [13, Corollary 3.9]. □

Let us restate a theorem mentioned in Section 2 before proving it.

Theorem 2.2. *Let G be a countable, discrete group that is a central extension*

$$1 \to N \to G \to Q \to 1$$

where N and Q are finitely generated abelian groups of ranks n and m, respectively. Then

$$\operatorname{dr} C^*(G) \leqslant (n+1)\big(2m^2+4m+1\big) - 1.$$

Proof. Theorem 1.2 of [17] implies that $C^*(G)$ is isomorphic to the C^*-algebra of a continuous field over the spectrum $\widehat{N}$ of $C^*(N)$, and moreover that every fiber has the form $C^*(Q, \omega)$ for some multiplier ω on Q. It is clear that we aim to use Lemma 3.1; to do so we only need to find an upper bound for the decomposition rank of $C^*(Q, \omega)$.

The twisted group C^*-algebra $C^*(Q, \omega)$ is stably isomorphic to $C(\widehat{S_\omega}) \otimes A_\theta$, where A_θ is a noncommutative torus of dimension at most m. This follows immediately from Proposition 4.3.

By Lemma 4.4,

$$\operatorname{dr} C^*(Q, \omega) = \operatorname{dr}\big(C(\widehat{S_\omega}) \otimes A_\theta\big) \leqslant (m+1)(2m+2) - 1,$$

again using [13, Corollary 3.9]. □

Finally, we prove Proposition 2.1. We recall the statement.

Proposition 2.1. *If $G \in \mathcal{G}$, then any generalized Bunce–Deddens algebra $C(\tilde{G}) \rtimes G$ associated with G has finite decomposition rank.*

Proof. Fix $G \in \mathcal{G}$ and a nested sequence (L_i) of finite index, normal subgroups of G with trivial intersection. Let $A = C(\tilde{G}) \rtimes G$ (where $\tilde{G}$ is the profinite completion of G with respect to (L_i)). Fix also positive integers m and n and finitely generated abelian groups N and Q such that G is a central extension as in (1) with N and Q of ranks n and m, respectively. Begin by rewriting A as an inductive limit:

$$\begin{aligned} A &= C(\tilde{G}) \rtimes G \\ &\cong \varinjlim C(G/L_i) \rtimes G \\ &\cong \varinjlim C^*(L_i) \otimes M_{r_i} \end{aligned}$$

(where $r_i = [G : L_i]$). For the first isomorphism we are using that $C(\tilde{G}) \cong \varinjlim C(G/L_i)$. For the second isomorphism one can use a theorem of Green [24, Theorem 4.30].

Remark 3.2 of [13] estimates the decomposition rank of an inductive limit as at most the limit inferior of the decomposition ranks of the limiting algebras. By Corollary 3.9 of [13], decomposition rank is invariant under tensoring with matrix algebras. Hence

$$\operatorname{dr} A \leqslant \varliminf \operatorname{dr}(C^*(L_i) \otimes M_{r_i}) = \varliminf \operatorname{dr} C^*(L_i).$$

Because N is a central subgroup of $G \in \mathcal{G}$, L_i is also a central extension of the form

$$1 \to N_i \to L_i \to Q_i \to 1$$

where N_i and Q_i are finitely generated abelian groups of ranks $n_i \leqslant n$ and $m_i \leqslant m$, respectively. The result now follows from Theorem 2.2. □

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