On locally quasi $A^*$ algebras in codimension-one over a Noetherian normal domain

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Let $R$ be Noetherian normal domain. We shall call an $R$-algebra $A$ quasi $A^*$ if $A = R[X, (aX + b)^{-1}]$ where $X \in A$ is a transcendental element over $R$, $a \in R \setminus 0$, $b \in R$ and $(a, b)R = R$. In this paper we shall describe a general structure for any faithfully flat $R$-algebra which is locally quasi $A^*$ in codimension-one over $R$. We shall also investigate minimal sufficient conditions for such an algebra to be finitely generated.

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1. Introduction

Let $R$ be an integral domain. Recall that an $R$-algebra $A$ is called $A^1$ if $A \cong R[X]$ (polynomial algebra in one variable over $R$) and is called $A^*$ if $A \cong R[X, X^{-1}]$ (Laurent polynomial algebra in one variable over $R$). Generalising this notion of $A^*$, we call an $R$-algebra $A$ to be quasi $A^*$ if $A \cong R[X, (aX + b)^{-1}]$ for some $X$ transcendental over $R$, $a \in R \setminus 0$ and $b \in R$ with $(a, b)R = R$. Note that if $a \in R^*$, then $A$ is $A^*$ over $R$. This notion of quasi $A^*$ arises naturally in the study of algebras whose generic fibres are $A^*$. To see an example, consider a discrete valuation ring $(V, \pi)$ with quotient field $K$ and a faithfully flat, finitely generated $V$-algebra $A$ such that $\pi A$ is a prime ideal of $A$ and $V/\pi V$ is algebraically closed in $A/\pi A$. Under these hypotheses, if $[A/\pi A]$ is a polynomial algebra $K[Y]$, then $A$ is a polynomial algebra $V[X]$ by [5, Theorem 2.3.1]. On the other hand, if $A[1/\pi]$ is a Laurent polynomial algebra $K[Y, Y^{-1}]$, then it can be shown (using similar methods) that $A$ is a quasi $A^*$-algebra of the form $V[X, (aX + b)^{-1}]$ for some $X$ in $A$ transcendental over $V, a \in V \setminus 0$ and $b \in V$ with $(a, b)V = V$.

Now let $R$ be a Noetherian normal domain. In [2], an integral domain $B$ containing $R$ has been called “locally $A^1$ in codimension-one” if, for every height one prime ideal $P$ of $R$, $B_P (= B \otimes_R R_P)$ is $A^1$ over $R_P$. Such an algebra $B$ has been studied extensively in [2] when $B$ is faithfully flat over $R$. In a similar fashion, we call an integral domain $A$ containing $R$ to be “locally quasi $A^*$ in codimension-one” if, for every height one prime ideal $P$ of $R$, $A_P (= A \otimes_R R_P)$ is quasi $A^*$ over $R_P$.

In this paper we investigate properties of a faithfully flat algebra $A$ over a Noetherian normal domain $R$ which is locally quasi $A^*$ in codimension-one. We first explore a general structure of $A$ and show that $A$ has an $R$-subalgebra $B$ which is faithfully flat and locally $A^1$ in codimension-one over $R$ such that $A = B[Q^{-1}]$ for some invertible ideal $Q$ of $B$ (Theorems 4.6 and 5.2). As a consequence, if $R$ is factorial then it follows (from known results about $B$) that $A$ is a direct limit of quasi $A^*$ algebras over $R$ (Corollary 4.5) and hence, if $A$ is finitely generated over $R$ then $A$ is quasi $A^*$ over $R$. It will also be seen (Proposition 5.4) that at each point $P$ of Spec $R$, $PA \subseteq$ Spec $A$, and that either $PA$ is quasi $A^*$ over $R_P$ or the fibre ring $A \otimes_R k(P') = k(P')$. As a consequence, we show that when $R$ is local, then $A$ is quasi $A^*$ under a mild hypothesis on the closed fibre. More precisely, we prove (Theorems 5.9 and 5.10):

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Theorem A. Let \((R, m)\) be a Noetherian normal local domain and \(A\) a faithfully flat \(R\)-algebra which is locally quasi \(A^*\) in codimension-one over \(R\). Then the following are equivalent:

(i) \(A\) is a quasi \(A^*\) \(R\)-algebra.
(ii) \(A\) is a finitely generated \(R\)-algebra.
(iii) \(R/m \cong A/mA\).

Moreover, if \(R\) is complete and \(A\) is Noetherian, then \(A\) is indeed quasi \(A^*\).

However if \(R\) is not complete, there exist examples of Noetherian faithfully flat \(R\)-algebras which are locally quasi \(A^*\) in codimension-one but which are not finitely generated over \(R\) (Example 6.3). Surprisingly, if we assume that \(A\) is locally \(A^*\) in codimension-one then \(A\) is actually finitely generated over \(R\) without any additional hypothesis. We prove (Theorem 4.8):

Theorem B. Let \(R\) be a Noetherian normal domain and \(A\) be a faithfully flat \(R\)-algebra which is locally \(A^*\) in codimension-one over \(R\). Then \(A = \bigoplus_{n \in \mathbb{Z}} R^n a^n\) for an invertible ideal \(I\) of \(R\). In particular, \(A\) is finitely generated over \(R\).

Theorem B was proved earlier in [1] under the additional assumption that \(A\) is finitely generated over \(R\) (cf. Remark 4.9(2)). However, even if \(R\) is complete, there exists a faithfully flat \(R\)-algebra which is locally quasi \(A^*\) in codimension-one but not finitely generated (Example 6.4).

We now give a layout of the paper. Sections 1–3 are introductory — Section 2 is on preliminaries; in Section 3, we recall results from [2] on algebras which are locally \(A^*\) in codimension-one over \(R\) and prove some results on the consequences of faithful flatness of such algebras. The main results of this paper are presented in Sections 4 and 5 — Theorem B, which requires less technical properties, will be proved in Section 4 and Theorem A in Section 5. In Section 4, we first discuss basic properties of a faithfully flat algebra \(A\) over a Noetherian normal domain \(R\) which is locally quasi \(A^*\) in codimension-one over \(R\) and establish the existence of an \(R\)-subalgebra \(B\) of \(A\) which is locally \(A^*\) in codimension-one such that \(A = B[Q^{-1}]\) for a suitable invertible ideal \(Q\) of \(B\). With the help of this presentation we prove Theorem B. In Section 5, we discuss properties of the above ring \(B\); in particular, we show that \(B\) is a faithfully flat \(R\)-algebra and deduce some results on the fibres of the map \(\text{Spec} A \to \text{Spec} R\). We also prove that \(B\) is Noetherian (respectively finitely generated over \(R\)) if and only if \(A\) is so. Finally we prove Theorem A. In Section 6, we discuss a few examples.

2. Preliminaries

We recall some standard notation to be used throughout the paper. For a ring \(R, R^*\) will denote the multiplicative group of units of \(R\). For a prime ideal \(P\) of \(R\), and an \(R\)-algebra \(A, A_P\) denotes the ring \(S^{-1}A\), where \(S = R \setminus P\) and \(k(P)\) denotes the residue field \(R_P/PR_P\). The notation \(A = R^{(1)}\) will mean that \(A\) is isomorphic, as an \(R\)-algebra, to a polynomial ring in one variable over \(R\).

For an \(R\)-module \(M\), we denote the tensor algebra of \(M\) over \(R\) by \(T_R(M)\) and the symmetric algebra by \(\text{Sym}_R(M)\). Note that if \(R\) is a domain and \(M\) is a flat \(R\)-module of rank one, then \(T_R(M) = \text{Sym}_R(M)\).

We compile below the notions mentioned in the introduction which are central to this paper.

Definition 2.1. (1) An \(R\)-algebra \(A\) is said to be \(A^*\) if there exists an element \(X\) in \(A\) which is transcendental over \(R\) such that \(A = R[X, X^{-1}]\).

(2) We shall call an \(R\)-algebra \(A\) to be “quasi \(A^*\)” if there exists an element \(X\) in \(A\) which is transcendental over \(R\) such that \(A = R[X, (aX + b)^{-1}]\), for some \(a \in R \setminus 0, b \in R\) satisfying \((a, b)R = R\). Note that \(R[X, (aX + b)^{-1}]\) is \(A^*\) over \(R\) if and only if \(a \in R^*\).

(3) We shall call an \(R\)-algebra \(A\) to be “locally quasi \(A^*\)” in codimension-one over \(R\) if \(A_P\) is quasi \(A^*\) over \(R_P\) for every height one prime ideal \(P\) in \(R\).

(4) We shall call an \(R\)-algebra \(B\) to be “locally \(A^*\)” in codimension-one if \(B_P = R_P^{[1]}\) for every height one prime ideal \(P\) in \(R\). If \(R\) is an integral domain with quotient field \(K\) and \(X\) is an element of \(B\) transcendental over \(R\) such that \(R[X] \subseteq B \subseteq K[X]\), then we say that \(X\) is a “generic variable” for \(B\).

As mentioned in the introduction, we shall discuss properties of algebras which are locally \(A^*\) in codimension-one in Section 3 and results on algebras which are locally quasi \(A^*\) in codimension-one in Sections 4 and 5.

We now mention two results on flat \(R\)-modules lying between the integral domain \(R\) and its quotient field \(K\). The first result is on the flatness of the \(R\)-subalgebra \(R[M]\) of \(K\) generated by a flat \(R\)-submodule \(M\) of \(K\).

Lemma 2.2. Let \(R\) be an integral domain with quotient field \(K\) and \(M\) a flat \(R\)-module such that \(R \subseteq M \subseteq K\). Then the \(R\)-subalgebra \(R[M]\) of \(K\) is flat over \(R\).

Proof. Since \(M \subseteq K\) and \(M\) is flat over \(R\), we can identify \(T_R(M)\) as a graded subring of the polynomial algebra \(K[W]\) with \(M \otimes_R \cdots \otimes_R M\) (\(n\) times) corresponding to \(M^nW^n (\subseteq KW^n)\). Thus

\[ T_R(M) = \{a_0 + a_1W + \cdots + a_nW^n \mid n \geq 0\ \text{and} \ a_i \in M^i\ \text{for} \ 0 \leq i \leq n\}. \]
Note that since \( R \subset M, W \in T_R(M) \); a homogeneous element of degree one. We first show that \( T_R(M) \cap (W - 1)K[W] = (W - 1)T_R(M) \). Let

\[
h = b_0 + b_1 W + \cdots + b_\ell W^\ell \in T_R(M) \cap (W - 1)K[W],
\]

with \( b_i \in M^\ell, 0 \leq i \leq \ell \). Write \( h = (W - 1)g \), where

\[
g = d_0 + d_1 W + \cdots + d_{\ell - 1} W^{\ell - 1} \in K[W].
\]

From (2.1) and (2.2), it follows that \( d_0 \in R, (d_1 - d_0) \in M, \ldots, (d_{\ell - 1} - d_{\ell - 2}) \in M^\ell \). Since \( R \subset M \subset M^2 \subset M^3 \), we have \( g \in T_R(M) \).

Let \( \phi \) be the restriction of the \( R \)-linear map \( \tilde{\phi} : K[W] \to K \) sending \( W \to 1 \). Then \( \phi(T_R(M)) = R[M] \) and hence we have the short exact sequence

\[
0 \to (W - 1)T_R(M) \to T_R(M) \xrightarrow{\phi} R[M] \to 0. \tag{2.3}
\]

Let \( I \) be an ideal of \( R \). Since \( M \) is a flat \( R \)-module, \( T_R(M) \) is a flat \( R \)-algebra. Thus \( \text{Tor}_1^R(I, T_R(M), R/I) = 0 \). Hence, tensoring (2.3) with \( R/I \), we have the exact sequence

\[
0 \to \text{Tor}_1^R(R[M], R/I) \to (W - 1)T_R(M) \otimes_R R/I \to T_R(M) \otimes_R R/I \to R[M] \otimes_R R/I \to 0.
\]

Let \( h \in T_R(M) \) be such that \((W - 1)h \in IT_R(M)\). Since \( W \) is a homogeneous element of degree one and \( IT_R(M) \) is a homogeneous ideal of the graded ring \( T_R(M) \), it follows that \( h \in IT_R(M) \). Thus the map

\[
(W - 1)T_R(M) \otimes_R R/I \to T_R(M) \otimes_R R/I
\]

is injective. Hence \( \text{Tor}_1^R(R[M], R/I) = 0 \). Thus \( R[M] \) is a flat \( R \)-algebra. \( \square \)

The next result is on the Noetherian property of flat \( R \)-subalgebras of \( R \).

**Lemma 2.3.** Let \( R \) be a Noetherian domain with quotient field \( K \), and \( D \) a flat \( R \)-algebra such that \( R \subset D \subset K \). Then \( D \) is a Noetherian ring.

**Proof.** To show that \( D \) is Noetherian it is enough to show that every prime ideal of \( D \) is finitely generated. Let \( Q \) be a prime ideal of \( D \) and \( P = Q \cap R \). Now \( D_Q \) is faithfully flat over \( R_P \) and since \( R_P \subset D_Q \subset K \), we have \( D_Q = R_P \) and hence \( D \subset R_P \). We show \( Q =PD \) for \( P \subset D \). For this it is enough to show that \( P \) is a prime ideal of \( D \). Now \( D/P \) is a flat \( R/RP \)-module and hence every element of \( R/P \) is a non-zero divisor in \( D/P \). Thus \( D/P \to R_P/PR_P \) which is a field and hence \( D/P \) is a prime ideal of \( D \). \( \square \)

Finally we recall an elementary result which will be used in the paper. (See the argument in [3, Lemma 2.8].)

**Lemma 2.4.** Let \( R \) be a Noetherian normal domain with quotient field \( K \) and let \( \Delta \) be the set of all height one prime ideals of \( R \). For a torsion free \( R \)-module \( M \), the following conditions are equivalent:

(i) \( M = \bigcap_{P \in \Delta} M_P \), where \( M \) and \( M_P = M \otimes_R R_P \) are identified with their images in \( M \otimes_R K \).

(ii) If \( a \) and \( b \) are elements of \( R \) such that \( b \) is \((R/ab)\)-regular, then \( b \) is \((M/ab)\)-regular.

In particular, if either \( M \) is \( R \)-flat or a direct limit of finitely generated reflexive \( R \)-modules, then \( M = \bigcap_{P \in \Delta} M_P \).

### 3. Locally \( A^1 \) algebras in codimension-one: some old and some new results

Throughout this section, \( R \) will denote a Noetherian normal domain with quotient field \( K \) and \( \Delta \) the set of all prime ideals in \( R \) of height one.

As in [2], we call an integral domain \( B \) containing \( R \) to be “semi-faithfully flat over \( R \)” if

(1) \( B = \bigcap_{P \in \Delta} B_P \).

(2) \( IB \subset R = I \), for every ideal \( I \) of \( R \).

In [2], properties of semi-faithfully flat algebras, which are locally \( A^1 \) in codimension-one over \( R \), were investigated. A general structure of such an \( R \)-algebra \( B \) was described ([2, Theorem 7.2]). Further, when \( B \) is faithfully flat over \( R \), conditions for \( B \) to be finitely generated were given ([2, Corollary 2.7, Theorems 2.11 and 7.12]). In this section, we shall recall some of these results and investigate some consequences (Proposition 3.10 and Lemma 3.14) when \( B \) is faithfully flat over \( R \).

Throughout this section, \( B \) will denote a semi-faithfully flat \( R \)-algebra which is locally \( A^1 \) in codimension-one. We recall from [2], some objects associated with \( B \) and a generic variable \( X \) of \( B \), (i.e., an element \( X \in B \) for which \( B \otimes_K K = K[X] \)).

For each \( P \in \Delta \), fix \( x_P \in B_P \) such that

\[
B_P = R_P[X_P].
\]

Then \( X = x_P X_P + b_P \) for some \( a_P, b_P \in R_P \).

Now set

\[
e(P) := v_P(a_P), \quad \text{where } v_P(a_P) \text{ is the valuation of } a_P \text{ in } R_P \text{ and}
\]

\[
\Delta_0(X) := \{ P \in \Delta | R_P[X] \not\subset B_P \}.
\]
We fix a generic variable $X$ and write $\Delta_0$ in place of $\Delta_0(X)$ and $e(P)$ in place of $e_X(P)$. Note that for each $P \in \Delta_0$, $a_P \notin R_P^*$ and hence $e(P) > 0$. Thus

$$\Delta_0 = \{ P \in \Delta | R_P[X] \neq B_P \} = \{ P \in \Delta | a_P \notin R_P^* \} = \{ P \in \Delta | e(P) > 0 \}.$$ 

Let $\Sigma_0$ be the set of all finite subsets of $\Delta_0$. For $\Gamma = \{P_1, \ldots, P_n\} \in \Sigma_0$, set

$$R_{\Gamma} := \bigcap_{P \in \Delta \setminus \Gamma} R_P,$$

$$B_{\Gamma} := S_{\Gamma}^{-1}B \cap R_{\Gamma}[X], \text{ where } S_{\Gamma} := R \setminus \left( \bigcup_{P \in \Gamma} P \right)$$

and

$$I_{\Gamma} := P_1^{(e(P_1))} \cap \cdots \cap P_n^{(e(P_n))}.$$ 

**Remark 3.1.** (1) $S_{\Gamma}^{-1}B = S_{\Gamma}^{-1}B_{\Gamma}$.  
(2) $I_{\Gamma}$ is a divisorial ideal of $R$ ([4, Corollary 5.5]). Hence $\text{Hom}_R(I_{\Gamma}, R) = I_{\Gamma}^{-1}$ and $\text{Hom}_R(I_{\Gamma}^{-1}, R) = I_{\Gamma}$.  
(3) For $P \in (\Delta_0 \setminus \Gamma)$, $(B_{\Gamma})_P = R_P[X] \neq B_P$ and hence if $\Gamma$ is a proper subset of $\Delta_0$ then $B_{\Gamma}$ is a proper subring of $B$.  
(4) The rings $B_{\Gamma}$, together with the inclusion maps, form a direct system $(B_{\Gamma} | \Gamma \in \Sigma_0)$ ([2, Lemma 2.1]).

The following technical result is proved in ([2, Theorem 7.2]).

**Theorem 3.2.** For each $\Gamma \in \Sigma_0$, there exists $c_{\Gamma} \in R$ such that

$$B_{\Gamma} = \bigoplus_{n \geq 0} (I_{\Gamma}^{-1})^{n-1}(X - c_{\Gamma})^n, \quad (3.1)$$

and for any $\Gamma_1, \Gamma_2 \in \Sigma_0$, with $\Gamma_1 \subseteq \Gamma_2$, we have $I_{\Gamma_1} \supseteq I_{\Gamma_2}$ and $c_{\Gamma_2} - c_{\Gamma_1} \in I_{\Gamma_1}$. Moreover,

$$B = \lim_{\Gamma \in \Sigma_0} B_{\Gamma}. \quad (3.2)$$

To a generic variable $X$ we associate the set $\Delta_0$ and the families $\{I_{\Gamma}\}_{\Gamma \in \Sigma_0}$, $\{B_{\Gamma}\}_{\Gamma \in \Sigma_0}$ and $\{c_{\Gamma}\}_{\Gamma \in \Sigma_0}$ as above. We abbreviate this as $\{\Delta_0, I_{\Gamma}, B_{\Gamma}, c_{\Gamma}\}$.

**Lemma 3.3.** $B_{\Gamma}$ is flat over $R$ if and only if $I_{\Gamma}$ is an invertible ideal of $R$. As a consequence, if $B_{\Gamma}$ is flat over $R$ then $B_{\Gamma}$ is a finitely generated $R$-algebra.

**Proof.** If $I_{\Gamma}$ is an invertible ideal of $R$ then $I_{\Gamma}^{-n}$ is an invertible ideal and hence $(I_{\Gamma}^{-1})^n = (I_{\Gamma}^{-1})^n$ is a (finitely generated) projective $R$-module and hence a flat $R$-module for every $n \geq 0$. Hence $B_{\Gamma}$ is flat over $R$. Moreover, $B_{\Gamma}$ is finitely generated over $R$ as $I_{\Gamma}^{-1}$ is a finitely generated flat $R$-module and $B_{\Gamma}$ is generated by $I_{\Gamma}^{-1}$ over $R$.

Now suppose that $B_{\Gamma}$ is flat over $R$. Then $I_{\Gamma}^{-1}$ is flat over $R$. As $R$ is Noetherian, it follows that $I_{\Gamma}^{-1}$ is a finitely generated projective $R$-module (of rank one). Hence $I_{\Gamma}$ is a finitely generated projective $R$-module (cf. **Remark 3.1(2)). Thus $I_{\Gamma}$ is invertible. □

The following result occurs in [2, Corollary 2.7] but for the sake of convenience we record a proof here.

**Proposition 3.4.** Suppose that $B$ is faithfully flat over $R$. Then $B$ is finitely generated over $R$ if and only if $\Delta_0$ is a finite (possibly empty) set.

**Proof.** Note that $\Delta_0 = \emptyset$ if and only if $R[X] = B$. We now assume that $R[X] \neq B$. By **Theorem 3.2**, 

$$B = \lim_{\Gamma \in \Sigma_0} B_{\Gamma}$$

and hence, if $B$ is finitely generated over $R$ then, there exists a finite subset $\Gamma'$ of $\Delta_0$ such that $B_{\Gamma'} = B$. Therefore, by **Remark 3.1(3),** $\Gamma' = \Delta_0$. 

Now suppose $\Delta_0$ is a finite set. Taking $\Gamma'$ to be $\Delta_0$, we see that $B_{\Gamma'} = B$. Since $B = B_{\Gamma'}$ is flat, $B$ is finitely generated over $R$ by **Lemma 3.3.** □

The following result on the $R$-algebra $B$ was stated in [3, Theorem 4.6] under the hypothesis that $B$ is faithfully flat over $R$, but the proof uses only semi-faithful flatness of $B$ (cf. [2, Remark 7.3(2)]).

**Theorem 3.5.** Let $R$ be a factorial domain. Then $B$ is a direct limit of polynomial algebras in one variable over $R$.

The next theorem, proved in [2, Theorem 3.7], gives a necessary and sufficient condition for $B$ to be finitely generated when $R$ is complete local.

**Theorem 3.6.** Let $R$ be a complete Noetherian normal local domain. Suppose that $B$ is a faithfully flat $R$-algebra. Then $B$ is Noetherian if and only if it is finitely generated over $R$. 

We now introduce some more notation to be used for the rest of this section. Choose an element $T$ of $B \otimes_R K$ such that $B \otimes_R K = K[T]$ (T need not be in $B$). Note that $B \subset B \otimes_R K = K[T]$. For $n \geq -1$, set

$$V_n := \{ g \in K[T] | \deg_{g}(g) \leq n \} \quad \text{and} \quad B_n := V_n \cap B.$$  \hspace{1cm} (3.3)

Let

$$\text{Gr}(B) = \bigoplus_{n \geq 0} B_n/B_{n-1} \subset \bigoplus_{n \geq 0} V_n/V_{n-1}.$$  \hspace{1cm} (3.4)

Note that $\text{Gr}(B)$ is independent of the choice of $T$. In fact, we observe the following:

**Remark 3.7.**

1. $B = \bigcup_{n \geq 0} B_n$.
2. The $K$–vector space $V_n$ and the $R$–module $B_n$, and hence the graded $K$–algebra $\bigoplus_{n \geq 0} V_n/V_{n-1}$ and the graded $R$–algebra $\text{Gr}(B)$ are independent of the choice of $T$.
3. Given $T$, if $W$ denotes the image of $T$ in $V_1/V_0$ then $\bigoplus_{n \geq 0} V_n/V_{n-1} = K[W]$ and hence $\text{Gr}(B) \subset K[W]$ as graded $R$–algebras.
4. For $T \in B$, we have $R[W] \subset \text{Gr}(B) \subset K[W]$ as graded $R$–algebras and hence for every $n \geq 0$, $RW^n \subset B_n/B_{n-1} \subset KW^n$. Moreover, $R[W] = \text{Gr}(B)$ if and only if $T = R[T]$.
5. $(B_n/B_{n-1})_P \cong R_P$ for every height one prime ideal $P$ of $R$.

We shall now relate faithful flatness of $B$ with that of $\text{Gr}(B)$.

**Lemma 3.8.** If $\text{Gr}(B)$ is flat over $R$ then $B$ is faithfully flat over $R$.

**Proof.** By Remark 3.7(1), it is enough to show that for each $n \geq 0$, $B_n$ is faithfully flat over $R$. Since $\text{Gr}(B)$ is $R$–flat, $B_n/B_{n-1}$ is $R$–flat for every $n \geq 0$. Moreover, we have the short exact sequence

$$0 \to B_{n-1} \to B_n \overset{\rho_n}{\to} B_n/B_{n-1} \to 0$$  \hspace{1cm} (3.5)

where $\rho_n$ is the projection map. Since $B_0 = R$, flatness of $B_1/B_0$ implies $B_1$ is faithfully flat. This in turn implies that $B_2$ is faithfully flat. Repeating this argument we see that $B_n$ is faithfully flat for every $n \geq 0$. Hence $B$ is faithfully flat over $R$. \hfill $\square$

**Remark 3.9.** We can define $(3.3)$ and $(3.4)$ for any arbitrary integral domain $R$ (not necessarily Noetherian normal) and any integral domain $B$ containing $R$ (not necessary semi-faithfully flat locally $\mathbb{A}_1$ in codimension-one) such that $B \otimes_R K = K[T]$. It is easy to see that $(1)$–$(4)$ of Remark 3.7 and Lemma 3.8 hold in this more general setup and that $(5)$ of Remark 3.7 also holds when $B$ is locally $\mathbb{A}_1$ in codimension-one.

We shall now see that the converse of Lemma 3.8 holds in our setup ($R$ is a Noetherian normal domain and $B$ is a semi-faithfully flat $R$–algebra which is locally $\mathbb{A}_1$ in codimension-one). This result was proved in [2, Corollary 3.8] under the additional hypothesis that $R$ is an analytically irreducible local domain, i.e., the completion of $R$ is an integral domain.

**Proposition 3.10.** Suppose that $B$ is faithfully flat over $R$. Then:

1. $B_n$ is flat over $R$, for every $n \geq 0$.
2. $B_n \cap B_{n-1} = B_{n-1}$, for every ideal $J$ of $R$.
3. $\text{Gr}(B)$ is faithfully flat over $R$.
4. $\text{Gr}(B) = R[L]$, where $L = B_1/R$.

**Proof.** (1) To show that $B_n$ is flat over $R$, it is enough to show that given $\sum_i a_i x_i = 0$, with $a_i \in R$ and $x_i \in B_n$, there exist $c_i \in R$ and $y_i \in B_n$ such that $\sum_i a_i c_i = 0$ for each $j$ and $x_i = \sum_j c_i y_j$ for each $i$.

Since $B$ is faithfully flat over $R$, there exist $c_i \in R$ and $z_j \in B$ such that $\sum_i a_i c_i = 0$ for each $i$ and $x_i = \sum_j c_i z_j$ for each $i$. By (3.2), we have $B = \lim_{\to \infty} B_{r'}$, and hence we can choose $r'$ such that $x_i \in B_{r'}$ for each $i$. Now by (3.1), $B_{r'}$ has a graded structure: $B_{r'} = \bigoplus_{r' \geq s} (B_{r'})^{-1} (X - c_r)'$. Note that

$$B_{r'}(n) := B_n \cap B_{r'} = \{ g \in B_{r'} | \deg_{g}(g) \leq n \} = \bigoplus_{0 \leq r \leq n} (B_{r'})^{-1} (X - c_r)'$$

Hence, $x_i \in B_n \cap B_{r'} = B_{r'}(n)$. Let $z_i = y_j + w_j$, where

$$y_j \in \bigoplus_{0 \leq r \leq n} (B_{r'})^{-1} (X - c_r)' \quad \text{and} \quad w_j \in \bigoplus_{r \geq r' + 1} (B_{r'})^{-1} (X - c_r)'$$

Now it is easy to see that the equality $x_i = \sum_j c_i z_j$ implies $x_i = \sum_j c_i y_j$, Thus $B_n$ is flat over $R$ for every $n \geq 0$.

(2) Since $B_{r'}(n) = B_n \cap B_{r'} = \bigoplus_{0 \leq r \leq n} (B_{r'})^{-1} (X - c_r)'$, for any ideal $J$ of $R$, $J B_{r'}(n) \cap B_{r'}(n - 1) = J B_{r'}(n - 1)$, Therefore, since $B$ is a direct limit of $B_{r'}$, it follows that $J B_n \cap B_{n-1} = J B_{n-1}$.
(3) Since by (1) $B_n$ is $R$-flat and by (2) for every ideal $J$ of $R$ we have $J B_n \cap B_{n-1} = J B_{n-1}$, the short exact sequence
\[ 0 \rightarrow B_{n-1} \rightarrow B_n \xrightarrow{\phi_n} B_n / B_{n-1} \rightarrow 0 \]  
(3.6)
suggests that $\text{Tor}^R_1(B_n/B_{n-1}, R/J) = 0$ for every ideal $J$ of $R$ and for every $n \geq 0$. Therefore $B_n/B_{n-1}$ is $R$-flat (of rank one) for every $n$ and hence $\text{Gr}(B) = \bigoplus_{n \geq 0} B_n/B_{n-1}$ is $R$-flat. Since $R$ is a direct summand of $\text{Gr}(B)$, it follows that $\text{Gr}(B)$ is faithfully flat over $R$.

(4) Set $L(n) := B_n/B_{n-1}$. Then $L^n \subseteq L(n)$. Since $B_n = R_n[X_n]$ for every prime ideal $P \subseteq M$, we have $\text{Gr}(B)_P = R_n[Y_P]$, where $Y_P$ denotes the image of $X_n$ in $V_n / V_0$ and $L_P = R_n(Y_P)$. Therefore, $(L^n)_P = R_n(Y^n) = L(n)_P$. By (3), $L(n)$ and $L^n$ are flat $R$-modules of rank one. Hence, by Lemma 2.4, (3.6) is $L(n)$ for $n \geq 0$. Thus $\text{Gr}(B) = R[L]$.

(5) Since $\text{Gr}(B) = R[L]$ by (4), it is now easy to see that $B = R[B_1]$. □

The following result was proved in [2, Theorem 3.10] under the additional hypothesis that $R$ is a local domain which is analytically irreducible. In view of Proposition 3.10, we now show that the hypothesis “analytically irreducible” can be dropped.

**Corollary 3.11.** Suppose that $R$ is local with maximal ideal $m$ and $B$ is faithfully flat over $R$. Then either $B$ is $\wedge^1$ over $R$ or $R/m = B/m B$.

**Proof.** By Proposition 3.10(3)–(4), $L$ is a flat $R$-module of rank one and hence by a result of Vasconcelos [6, Theorem 3.1], either $L \cong R$ or $L = mL$.

If $mL = L$ then $B_1 = R + m B_1$. Since $B = R[B_1]$ by Proposition 3.10(5), it follows that $B = R + m B$. Hence $R/m = B/m B$.

If $L \cong R$ then choose $T \subseteq B_1$ such that $L = RW$, where $W$ is the image of $T$ in $L = B_1 / R \subseteq V_1/K$. This shows that $\text{Gr}(T) \subseteq B \subseteq K[T]$ and $\text{Gr}(B) = R[W]$. Therefore $\text{Gr}(T) = B$ (cf. Remark 3.7(4)). □

**Remark 3.12.** Suppose that $B$ is a faithfully flat $R$-algebra.

(1) As a consequence of Corollary 3.11, we see that $P \subseteq B$ for every prime ideal $P$ of $R$. In fact, for each prime ideal $P$ of $R$, either $B_P$ is $\wedge^1$ over $R_P$ or the fibre ring $A \otimes_R k(P) = k(P)$ which implies that $P \subseteq B_P$ is a prime ideal of $B_P$ and hence $P$ is a prime ideal of $B$ because $P/B \subseteq B_P / P B_P$ by flatness of $B$ over $R$.

(2) If $R$ is factorial and $B$ is a Krull domain, then $B$ is factorial. Indeed, by (1), every prime element of $R$ remains a prime element of $B$. Let $S$ be the multiplicative closed set generated by all prime elements of $R$. Then $S^{-1}R = K$, and hence $S^{-1}B = K[1]$, a factorial domain. Hence $B$ is factorial by Nagata’s criterion [4, Corollary 7.3].

For an integer $n \geq 0$, recall the notation
\[ L(n) := B_n/B_{n-1}. \]  
(3.7)
Let $Y$ denote the image of generic variable $X \subseteq B$ in $V_1 / V_0$ and $M_n := \{ \lambda \in K \mid \lambda Y^n \in L(n) \}$. Then, since $KY^n \subseteq L(n) \subseteq KY^n$ (cf. Remark 3.7(4)), $R \subseteq M_n \subseteq K$. In fact
\[ M_n = \lim_{n \rightarrow 0} \left( I^n_0 \right)^{-1} = \bigcup_{n \geq 0} \left( I^n_0 \right)^{-1} \]  
(3.8)
and $L(n) = M_n Y^n$ (see [2, Remark 2.12]). Set $M := M_n$. If $B$ is faithfully flat over $R$ then by Proposition 3.10(3)–(4), it follows that $M_n$ is flat over $R$ and $M_n = M^n$ for $n \geq 1$ and hence $\text{Gr}(B) \cong \text{Sym}_R(M) = T_K(M)$.

**Corollary 3.13.** Suppose that $B$ is a faithfully flat $R$-algebra. Set $M = M_n$. Then the $R$-subalgebra $R[M]$ of $K$ is flat over $R$.

**Proof.** By Proposition 3.10(3), $\text{Gr}(B)$ is flat over $R$ and so $L = B_1 / R$ is flat over $R$ which implies that $M$ is flat over $R$ since $MY = L$. By Remark 3.7(4), $RY \subseteq L \subseteq KY$ and hence, $R \subseteq M \subseteq K$. Thus, by Lemma 2.2, $R[M]$ is flat over $R$. □

**Lemma 3.14.** Let $\Delta_1 = \{ P \in \Delta \mid R_P[X] = B_P \}$ and let $\Gamma' = \bigcap_{P \subseteq \Delta_1} R_P$. If $B$ is a faithfully flat $R$-algebra, then $\Gamma'$ has the following properties:

(1) $\Gamma'$ is flat over $R$.

(2) $\Gamma'$ is a Noetherian domain.

**Proof.** (1) By Corollary 3.13, it is enough to show that $R[M] = \Gamma'$. Note that $\Delta_1 = \Delta \setminus \Delta_0$. By (3.8), we see that $R \subseteq M$ and $R_P = M_P$ if and only if $P \subseteq \Delta_1$. Therefore $R[M]_P = R_P$ if $P \subseteq \Delta_1$ and $R[M]_P = K$ for $P \subseteq \Delta_0$. Since $R[M]$ is $R$-flat, we have $R[M] = \bigcap_{P \subseteq \Delta} R[M]_P$ by Lemma 2.4. Hence $\Gamma' = R[M]$.

(2) $\Gamma'$ is normal because it is given by intersection of normal domains. By Lemma 2.3 and (1), it follows that $\Gamma'$ is Noetherian. □
4. Locally quasi $\mathcal{A}^*$ algebras in codimension-one I: basic concepts and results; Theorem B

We first prove an elementary result on quasi $\mathcal{A}^*$ algebras over an integral domain.

**Lemma 4.1.** Let $R$ be an integral domain with quotient field $K$ and $A$ be an $R$-algebra which is quasi $\mathcal{A}^*$ over $R$. Let $T \in A \otimes_R K$ be such that $A \otimes_R K = K[T, T^{-1}]$ and $B = A \cap K[T]$. Then there exist $W \in B$, $a' \in R \setminus 0$, $b' \in R$ and $f = a'W + b'(\in B)$ such that $(a', b')R = R, B = R[W]$ and $A = R[W, f^{-1}]$. As a consequence,

$$R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] = K[T, T^{-1}]$$

and

$$A \cap KT^n = Rf^n \quad \text{and} \quad A \cap KT^{-n} = Rf^{-n} \quad \forall n \geq 0.$$

**Proof.** Since $A$ is a quasi $\mathcal{A}^*$ $R$-algebra, there exists $X \in A$, transcendental over $R$, such that $A = R[X, (aX + b)^{-1}]$ for some $a \in R \setminus 0$, $b \in R$ satisfying $(a, b)R = R$. Let $g = aX + b$. Then $K[T, T^{-1}] = A \otimes_R K = K[g, g^{-1}]$. Hence, either $K[g] = K[T]$ or $K[g] = K[T^{-1}]$.

Suppose $K[g] = K[T]$. Then $K[T] = K[g]$ and $R[X] \subseteq B \subseteq R[X, g^{-1}] = A$. Since $(a, b)R = R$, $g$ is a prime element of $R[X]$ which implies $gK[X] \cap R[X] = gR[X]$. Thus $gB \cap R[X] = gR[X]$ and since $R[X, g^{-1}] = (A) = B[g^{-1}]$, we have $B = R[X]$. Set $W := X$ and $f := g$. Then $A = R[W, f^{-1}]$ and $B = R[W]$.

Now suppose $K[g] = K[T^{-1}]$. Since $aR + bK = R$, there exist $c, d \in R$ such that $ad - bc = 1$. Set $W := (cX + d)/(aX + b)$ and $f := aW - c$. Then $f = 1/(aX + b) = 1/gX = (d - bW)/f$ and $K[W] = K[f] = K[T]$. Hence $A = (R[X, g^{-1}]) = R[W, f^{-1}]$ and arguing as before, we see that $B = R[W]$.

Now since $A = R[W, f^{-1}]$ and $f$ is linear in $W$, we have $R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] = K[T, T^{-1}]$ and hence $f = \lambda T$ for some $\lambda \in K^*$. Since $R \subseteq A \cap K \subseteq A \cap K[T] = B = R[W]$, it follows that $A \cap K = R$. If $\mu \in K$ be such that $\mu f^n \in A$, then as $f \in A^*$, we have $\mu \in (A \cap K) = R$. Therefore, $A \cap KT^n = Rf^n$ and $A \cap KT^{-n} = Rf^{-n}$ for $n \geq 0$.  

Let $R$ be an integral domain with quotient field $K$ and $A$ a faithfully flat $R$-algebra such that $A \otimes_R K = K[T, T^{-1}]$. Let $B = A \cap K[T]$. The above lemma shows that if $A$ is locally quasi $\mathcal{A}^*$ in codimension-one over $R$ then $B$ is locally $\mathcal{A}^*$ in codimension-one over $R$. One would like to know a relation between $A$ and $B$. For example, one might ask whether $A$ is (in some sense) a localisation of $B$.

Our first goal in this section is to show that if $R$ is Noetherian and normal then indeed such is the case (Theorem 4.6). We first fix some notation.

**Notation**

Throughout this section, $R$ will denote a Noetherian normal domain with quotient field $K$, $\Delta$ the set of prime ideals in $R$ of height one and $A$ a faithfully flat $R$-algebra which is locally quasi $\mathcal{A}^*$ in codimension-one.

Fix $T \in A \otimes_R K$ such that $A \otimes_R K = K[T, T^{-1}]$. For an integer $n \geq 0$, set

$$C_n := A \cap KT^n,$$

$$D_n := A \cap KT^{-n}.$$

We now prove a technical lemma on the submodules $C_n$, $D_n$.

**Lemma 4.2.** The canonical maps $C_n \otimes_R A \to C_nA$ and $D_n \otimes_R A \to D_nA$ are isomorphisms of $A$-modules and $C_nA = A = D_nA$ for each $n \geq 0$. As a consequence, $C_n$ and $D_n$ are finitely generated projective $R$-modules of rank one.

**Proof.** We show that the canonical map $C_n \otimes_R A \to C_nA$ is an isomorphism and $C_nA = A$. The results for $D_n$ will follow in a similar way.

Since $C_n \subseteq KT^n$ and $A$ is $R$-flat, we have $C_n \otimes_R A \subseteq KT^n \otimes_R A \cong K[T, T^{-1}]$, so that $C_n \otimes_R A$ is a torsion free $A$-module of rank one. Hence the map $C_n \otimes_R A$ is an isomorphism.

Since $C_n = A \cap KT^n$ and $A$ is $R$-flat, it follows that $C_n = \bigcap_{p \in \Delta} C_n p$ by Lemma 2.4. Therefore, again by Lemma 2.4, if $x, y \in R$ be such that $(aX : y) = xR$ then $(xC_n : y) = xC_n$. In particular, since $R$ is normal, if $(x, y)$ is an ideal of $R$ of height $\geq 2$ then $(x, y) = xC_n$. Since $A$ is $R$-flat and $C_n \otimes_R A \cong C_nA$, we see that $(xC_n : y) = xC_n$ for $x, y \in R$ such that $ht(x, y) \geq 2$.

For $g \in C_n, h \in D_n$, we see that $gh \in A \cap K = R$. Therefore we get an $R$-linear map $\psi : C_n \otimes_R D_n \to R$ defined by $\psi(g \otimes h) = gh$. Let $J_\psi$ be the image of $\psi$.

Since $A$ is locally quasi $\mathcal{A}^*$ in codimension-one, by Lemma 4.1, $(J_\psi)_p = R_p$ for every $p \in \Delta$. This shows that $J_\psi$ is an ideal of $R$ of height $\geq 2$. Therefore, there exist $x, y \in R$ such that $ht(x, y) \geq 2$.

Since $D_\psi \subseteq A, J_\psi A \subseteq C_nA$ and so $(x, y) \subseteq J_\psi A \subseteq C_nA$. Now since $(xC_n : y) = xC_nA$, the fact $xy \in xC_nA$ implies that $x \in xC_nA$, hence $C_nA = A$. Since $C_n \otimes_R A \cong C_nA = A$ and $A$ is faithfully flat over $R$, $C_n$ is a finitely generated projective $R$-module of rank one. Thus the lemma is proved. □

It follows from Lemma 4.2 that $J_\psi = C_nD_n$ is a locally principal ideal in $R$ of height at least two. We thus obtain the following corollary.
Corollary 4.3. Set $B := A \cap K[T]$ and let $C$ be the $R$-subalgebra $C = \bigoplus_{n \geq 0} C_n$ of $B$. Then we have:

1. $J_n = R$ for each integer $n \geq 0$.
2. The canonical map $\theta_n : C_1 \otimes_C C_1 \otimes \cdots \otimes C_1 \to C_n$ is an isomorphism of $R$-modules.
3. $C = \text{Sym}_R(C_1)$ as $R$-algebras.

Proof. (1) Since $C_n$ and $D_n$ are finitely generated projective $R$-modules of rank one, so is $C_n \otimes_R D_n$. Therefore the surjective map $\psi : C_n \otimes_R D_n \to J_n$ is an isomorphism. Thus $J_n$ is $R$-projective of rank one, i.e., an invertible ideal of $R$. Since $\text{ht}(J_n) \geq 2$, we see that $J_n = R$.

(2) For the sake of simplicity we denote $\theta_n (C_1 \otimes_R C_1 \otimes \cdots \otimes C_1)$ (n-times) by $C(n)$. Since $A$ is locally quasi $A^*$ in codimension-one, by Lemma 4.1, for every $P \in \Delta$, we have $(C_1)_P = Rf_P$ for some $f_P \in A_P$ and $(C_n)_P = Rf_P^n$ for every $n \geq 0$. Thus $C(n)_P = (C_1)_P$ for every $P \in \Delta$. This implies that $\theta_n$ is injective. Now using the fact that $C(n)$ and $C_n$ are projective $R$-modules, by Lemma 2.4, we have $C(n) = \bigcap_{P \in \Delta} C(n)_P$ and $C_n = \bigcap_{P \in \Delta} C(n)_P$ and hence $C(n) = C_n$. Thus $\theta_n$ is an isomorphism for every $n \geq 0$.

(3) Follows from (2). □

Corollary 4.4. The following statements hold:

1. $B (= A \cap K[T])$ is a semi-faithfully flat $R$-algebra which is locally $A^*$ in codimension-one.
2. $R^* \subseteq A^*$ if and only if there exists $n > 0$ such that $C_n$ is free.
3. If $C_1 = Rf$, then $Rf \subseteq B \subseteq K[f] = K[T], A = B[f^{-1}]$ and $B \cap K[T] = fB$ and hence $fB \in \text{Spec } B$.

Proof. (1) By Lemma 4.1, it is enough to show that $B$ is semi-faithfully flat over $R$. Since $A$ is faithfully flat over $R$, by Lemma 2.4, we have

$$B = A \cap K[T] = \left( \bigcap_{P \in \Delta} A_P \right) \cap K[T] = \bigcap_{P \in \Delta} (A_P \cap K[T]) = \bigcap_{P \in \Delta} B_P$$

and

$$IB \cap R \subseteq IA \cap R = I, \quad \text{for any ideal } I \text{ of } R.$$

Thus $B$ is semi-faithfully flat over $R$.

(2) Suppose that $C_n = Rh$ for some $n > 0$ and $h \in A$. Since $J_n = R$ by Corollary 4.3(1), it follows that $D_n = Rh^{-1}$. Hence $h \in A^* \setminus R^*$. Conversely, suppose that $R^* \subseteq A^*$ and let $h \in A^* \setminus R^*$. Since $A \hookrightarrow A \otimes_R K = K[T, T^{-1}], h \in KT^n$ for some $n \in \mathbb{Z}$. Replacing $h$ by $h^{-1}$ if necessary, we may assume that $h \in C_n (= A \cap KT^n)$ and $h^{-1} \in D_n$ for some integer $n > 0$. Note that $Kh = KT^n$ and $Kh^{-1} = KT^{-n}$. We then show that $C_n = Rh$. Let $g \in C_n$, then $g = \lambda h$ for some $\lambda \in K$. Hence $gh^{-1} = \lambda \in A \cap K = R$. Thus $C_n = Rh$.

(3) Since $Rf = C_1 \subseteq KT$, $K[T] = K[f]$ and $TK[T] = fK[f]$. Moreover, by (2), $f^{-1} \subseteq A$ and hence $R[f^{-1}] \subseteq A \subseteq K[f^{-1}] = K[T, T^{-1}]$. Hence, as $K[T] = K[f], R[f] \subseteq B \subseteq K[f]$. If $g \in A$, then there exists $k \geq 0$ such that $f^k g \in A \cap K[f] = (fB)$ and hence $A = B[f^{-1}]$.

Let $h \in fK[f] \setminus B$. Then $h \in A$ and, since $f \in A^*$, we have $h/f \in A \cap K[f] = B$. Thus $TK[T] \cap B (= fK[f] \cap B) = fB$. □

The following result, on factorial domain, is the quasi $A^*$ analogue of Theorem 3.5.

Corollary 4.5. Suppose that $R$ is a factorial domain. Then $A$ is a direct limit of quasi $A^*$ algebras.

Proof. By Lemma 4.2, $C_1$ is finitely generated projective $R$-module and since $R$ is factorial, we have $C_1 = Rf$ for some $f \in C_1$. Hence, by Corollary 4.4, $B$ is a semi-faithfully flat $R$-algebra which is locally $A^*$ in codimension-one such that $f$ is a generic variable of $B$ and $A = B[f^{-1}]$. By Theorem 3.5, $B$ is a direct limit of polynomial algebras in one variable over $R$ and hence, $A(= B[f^{-1}])$ is a direct limit of quasi $A^*$ algebras over $R$. □

The following theorem shows that $A$ is a localisation of $B$.

Theorem 4.6. Let $R$ be a Noetherian normal domain with quotient field $K$ and $A$ a faithfully flat $R$-algebra such that $A \otimes_R K = K[T, T^{-1}]$ for some $T$ transcendental over $K$. Suppose that $A$ is locally quasi $A^*$ in codimension-one over $R$. Let $C_1 = A \cap KT, B = A \cap K[T]$ and $Q = B \cap KT$. Then $Q = C_1B$. As a consequence, $Q$ is an invertible ideal of $B$ and $A = B[Q^{-1}]$.

Proof. It is easy to see that $C_1 = A \cap KT \subseteq A \cap K[T] = B \cap KT = Q$. Therefore $C_1B \subseteq Q$. Let $m$ be a maximal ideal of $R$ and $S = R \setminus m$. By Lemma 4.2, $C_1$ is a finitely generated projective $R$-module of rank one, hence $S^{-1}C_1$ is a free $S^{-1}R$-module of rank one and so $S^{-1}C_1 = B_m f_m$ for some $f_m \in S^{-1}C_1$. Therefore, by Corollary 4.4(3), $C_1B_m = f_m B_m = QB_m$ and $A_m = B_m [(QB_m)^{-1}] = B_m [(QB_m)^{-1}]$. Hence $C_1B = Q$ and $A = B[Q^{-1}]$. □

Corollary 4.7. The following statements hold for the ring $A$:

1. If $B$ is a Noetherian ring, then so is $A$.
2. If $B$ is a finitely generated $R$-algebra, then so is $A$.

Proof. Follows from the fact that $A = B[Q^{-1}]$ where $Q$ is an invertible ideal of $B$. □
In Section 5, we shall prove the converse of Corollary 4.7 (cf. Theorems 5.6 and 5.7).

In the special case of A being locally $A^*$ in codimension-one over R, we have Theorem B which shows that A is always finitely generated over R. In fact its proof shows that B is finitely generated over R.

**Theorem 4.8.** Let R be a Noetherian normal domain and A a faithfully flat R-algebra such that A is locally $A^*$ in codimension-one. Then $A \cong \bigoplus_{n \in \mathbb{Z}} I^n$ for an invertible ideal I of R. In particular, A is finitely generated over R.

**Proof.** Let K be the quotient field of R and $A \otimes R K = K[T, T^{-1}]$ for some $T$ transcendental over K. Let $B = \Delta \cap K[T]$, $C_\alpha = \Delta \cap K[T^\alpha]$ and $C = \bigcup_{\alpha \in \mathbb{N}_0} C_\alpha \subseteq B$. Now $A = B[C_\alpha]^{-1}$ by Theorem 4.6 and $C = \text{Sym}_K(C_1)$ by Lemma 4.3(2), with C being isomorphic to an invertible ideal of R by Lemma 4.2. Thus it suffices to show that $C = B$.

Since C is flat over R, $C = \bigcap_{P \in \Delta} C_P$ by Lemma 2.4. By Corollary 4.4(1), $B = \bigcap_{P \in \Delta} B_P$. Therefore it is enough to prove that $C_P = B_P$ for every $P \in \Delta$.

Let $P \in \Delta$. Since A is locally quasi $A^*$ in codimension-one over R, $(C_1)_P = R_\alpha f$ for some $f \in (C_1)_P$ by Lemma 4.1. Hence $C_P = R_P[f] \subseteq B_P \subseteq K[f] = K[T]$ and $fK[T] = fK[T]$. By Theorem 4.6, $B_P \cap fK[T] = Q_P = fB_P$, hence $B_P \cap C_P = fT[f] \cap C_P = fC_P$ and $R_P[f, f^{-1}] \subseteq B_P[f^{-1}] = A_P$. Therefore, to show that $C_P = B_P$, it is enough to show that $C_P[f^{-1}] = B_P[f^{-1}]$.

Since A is locally $A^*$ in codimension-one, $A_P = R_P[W, W^{-1}]$ for some W transcendental over $R_P$. Thus $R_P[f, f^{-1}] \subseteq R_P[W, W^{-1}]$. Therefore, $C_P[f^{-1}] = R_P[f, f^{-1}] = R_P[W, W^{-1}] = A_P = B_P[f^{-1}]$.

Thus the result follows. □

**Remark 4.9.** (1) Our proofs show that in all results of this section the hypothesis “R is a Noetherian normal domain” may be replaced by the weaker hypothesis “R is a Krull domain”.

(2) From Theorem 4.8, one can deduce that the hypothesis of finite generation on A in Theorem 3.4, Corollary 3.9 and Theorem 3.11 of [1] can be dropped. (There is an error in Example 3.6 of [1].)

5. **Locally quasi $A^*$ algebras in codimension-one II: Theorem A**

As in Section 4, R is a Noetherian normal domain with quotient field K, A a faithfully flat R-algebra which is locally quasi $A^*$-algebra in codimension-one, $T \in A \otimes R K$ such that $A \otimes R K = K[T, T^{-1}]$, $B = \Delta \cap K[T]$ and $Q = B \cap fK[T]$. We have seen (Theorem 4.6) that Q is an invertible ideal of B and $A = B[Q, Q^{-1}]$. Hence, if B is Noetherian (respectively finitely generated over R) then so is A. In this section, we prove a converse of this result (Theorems 5.6 and 5.7): we show that if A is Noetherian then B is Noetherian and if A is finitely generated over R then so is B. Finally we prove Theorem A.

To begin with we shall show (Theorem 5.2) that, in the above set-up, B is faithfully flat over R. We first give below (Lemma 5.1) a sufficient condition for B to be faithfully flat over R. For this we can assume that R is a local ring with maximal ideal $m$. Since, by Lemma 4.2, C is a finitely generated projective R-module and R is local, there exists $f \in C_1$ such that $C_1 = Rf$. By Corollary 4.4, B is a semi-faithfully flat R-algebra which is locally $A^1$ in codimension-one over R such that $R[f] \subseteq B \subseteq K[f]$ and $A = B[f^{-1}]$. Note that $f$ is a generic variable of B. Let $(\Delta_0, I_0, B, c)$ be the data associated to the generic variable $f$ of B.

With the hypothesis that R is local and notation as above, we prove

**Lemma 5.1.** Suppose $\Delta_0 \neq \emptyset$ and $c_\Gamma \in R^*$ for every $\Gamma \in \Delta_\Sigma$. Then B is faithfully flat over R.

**Proof.** B is R-flat if and only if $\text{Tor}_1^B(B, R/I) = 0$ for every ideal I of R. Let I be an ideal of R and $\alpha \in \text{Tor}_1^B(B, R/I)$. We show that $\alpha = 0$. Since $B[f^{-1}] = A$ is a faithfully flat R-algebra, there exists $r \geq 0$ such that $f^r \alpha = 0$ in $\text{Tor}_1^B(B, R/I)$. By Theorem 3.2, $B = \lim_{r \in \mathbb{Z}_0} B_{r^*}$ and hence

$$\text{Tor}_1^B(B, R/I) = \text{Tor}_1^B(\lim_{r \in \mathbb{Z}_0} B_{r^*}, R/I) = \lim_{r \in \mathbb{Z}_0} \text{Tor}_1^B(B_{r^*}, R/I).$$

Thus, there exists $\Gamma \in \Delta_\Sigma$ such that $\alpha \in \text{Tor}_1^B(B_{r^*}, R/I)$ and $f^r \alpha = 0$ in $\text{Tor}_1^B(B_{r^*}, R/I)$. Again by Theorem 3.2, $B_{r^*}$ is graded R-algebra:

$$B_{r^*} = \bigoplus_{n \geq 0} E_n$$

where $E_n = (I_{r^*})^{n-1} (f - c_{r^*})^n$. Therefore,

$$\text{Tor}_1^B(B_{r^*}, R/I) = \bigoplus_{n \geq 0} \text{Tor}_1^B(E_n, R/I)$$

is a graded $B_{r^*}$-module. We write $\alpha \in \text{Tor}_1^B(B_{r^*}, R/I)$ as

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_t$$

with $\alpha_i \in \text{Tor}_1^B(E_n, R/I)$, $0 \leq i \leq t$. Note that $x_{r^*} := f - c_{r^*}$ is a homogeneous element of degree one in $B_{r^*}$ and by hypothesis $c_{r^*} \in R^*$. Now $f^r \alpha = 0$ implies

$$c_{r^*} \alpha_0 = 0, \quad c_{r^*} \alpha_1 + r c_{r^*} \alpha_0 x_{r^*} = 0, \quad \ldots, \quad c_{r^*} \alpha_t = 0$$
and since \( c_T \in R^* \), we have

\[
\alpha_0 = \alpha_1 = \cdots = \alpha_r = 0
\]

showing that \( \alpha = 0 \) in \( \text{Tor}_i^R(B, \ R/I) \). Thus \( B \) is \( R \)-flat. Since, \( B \subseteq A \) and \( A \) is faithfully flat over \( R \), \( B \) is faithfully flat over \( R \). \qed

We now prove faithful flatness of \( B \) by showing that either \( B \) satisfies the hypothesis of Lemma 5.1 or \( B = R^{[1]} \).

**Theorem 5.2.** Let \( R \) be a Noetherian normal domain with quotient field \( K \) and \( A \) a faithfully flat \( R \)-algebra such that \( A \otimes_R K = K[T, T^{-1}] \) for some \( T \) transcendental over \( K \). Suppose that \( A \) is locally quasi \( \mathcal{A}^* \) in codimension-one over \( R \). Then \( B = A \cap K[T] \) is a faithfully flat \( R \)-algebra.

**Proof.** Since faithful flatness is a local property, we assume \( R \) to be a local domain. We prove faithful flatness of \( B \) by induction on the dimension of the ring \( R \).

If \( \dim R = 1 \), then there is nothing to prove since \( A \) is a faithfully flat locally quasi \( \mathcal{A}^* \) algebra in codimension-one over \( R \) and hence \( B = R^{[1]} \) by Lemma 4.1.

Now consider \( \dim R > 1 \). Let the notation be as before Lemma 5.1. If \( \Delta_0 = \emptyset \), then \( B = R[f] \) and hence a faithfully flat \( R \)-module. Now assume that \( \Delta_0 \neq \emptyset \). We show that \( c_T \in R^* \) for every \( \Gamma \in \Sigma_0 \). Then the result will follow by Lemma 5.1.

Suppose, if possible, that there exists \( \Gamma \in \Sigma_0 \) such that \( c_T \in m \) in \( m = \text{tor}^A \mathcal{A}_T \). We show that this leads to a contradiction. Since \( \dim R_T < \dim R \), by applying induction hypothesis to local rings of \( R_T \), we get that \( B_{c_T} \) is faithfully flat over \( R_{c_T} \). Also since \( B_T(= \text{tor}^A \mathcal{A}_T) \) is faithfully flat over \( R \), we have \( B_{c_T} \cap c_T \) is flat over \( R \).

Set \( I_{c} := (I_{c}B : f - c_T) \). Since \( S_{c_T}^{-1}R \) is a PID, \( I_{c}S_{c_T}^{-1}R \) is a principal ideal, say generated by \( d_T \), and hence if \( y_T = ((f - c_T)/d_T) \), then \( S_{c_T}^{-1}B = S_{c_T}^{-1}K[y_T] \) by Theorem 3.2. This shows that \( f - c_T \in I_{c}S_{c_T}^{-1}B \). Therefore we have \( I_{c}S_{c_T}^{-1}B = S_{c_T}^{-1}B \). \( \Gamma \) and there exists \( s \in S_{c_T} \) such that \( s \in \mathcal{A}_T \) so that \( I_{c}B \subseteq I_{c}B \cap R_{c_T} = R \), where \( \Delta_0, I_{c}, B_{c_T}, c_T \) is the data associated to the generic variable \( f \) of \( B \).

(2) For each \( \Gamma \in \Sigma_0, f - c_T \in I_{c}B \). In fact, from the proof of Theorem 5.2, we have \( f - c_T \in I_{c}S_{c_T}^{-1}B \cap B \), where \( I_{c}S_{c_T}^{-1}B \cap B = I_{c}B \) since \( B \) is faithfully flat over \( R \).

We now prove an analogue of Theorem 3.10 of [2] for fibre rings of algebras which are locally quasi \( \mathcal{A}^* \) in codimension-one.

**Proposition 5.4.** Let \( R \) be a Noetherian normal domain and \( A \) a faithfully flat \( R \)-algebra which is locally quasi \( \mathcal{A}^* \) in codimension-one. Then \( \mathcal{P}A \in \text{Spec} A \) for every prime ideal \( \mathcal{P} \) of \( R \). In fact, for each prime ideal \( \mathcal{P} \) of \( R \), either \( A_{\mathcal{P}} \) is quasi \( \mathcal{A}^* \) over \( R_{\mathcal{P}} \) or the fibre ring \( A \otimes_R k(\mathcal{P}) = k(\mathcal{P}) \).

**Proof.** Let \( \mathcal{P} \in \text{Spec} R \). Since \( A \) is faithfully flat over \( R \), to show that \( \mathcal{P}A \in \text{Spec} A \) it is enough to show that \( \mathcal{P}A_{\mathcal{P}} \in \text{Spec} A_{\mathcal{P}} \). Therefore, we can assume that \( R \) is a local domain with maximal ideal \( m \) and show that either \( A \) is quasi \( \mathcal{A}^* \) over \( R \) or the fibre ring \( \mathfrak{A}/mA = R/m \).

Since \( R \) is local, by Theorem 4.6, there exists \( f \in A \) such that \( R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] \) and \( A = B[f^{-1}] \), where \( B = A \cap K[f] \). By Theorem 5.2, \( B \) is a faithfully flat \( R \)-algebra which is locally \( \mathcal{A}^* \) in codimension-one over \( R \). Therefore, by Corollary 3.11, \( B \) is either \( R[T] \) for some \( T \in B \) which is transcendental over \( R \) or \( B/mB = R/m \). If \( B = R[T] \) then \( A = B[f^{-1}] = R[T, f^{-1}] \) and hence quasi \( \mathcal{A}^* \) over \( R \). In the other case, since \( A \) is faithfully flat over \( R \), \( f \notin mB \) and hence \( R/m = B/mB = A/mA \). \qed

We shall now show that if \( A \) is Noetherian then \( B \) is Noetherian and if \( A \) is finitely generated over \( R \) then so is \( B \). For this we make the following reduction.

**Remark 5.5.** To prove that \( A \) is Noetherian (resp. finitely generated over \( R \)) implies \( B \) is Noetherian (resp. finitely generated over \( R \)), it suffices to assume that \( C_1 \) is a free \( R \)-module of rank one. To see this, first note that, by Lemma 4.2, \( C_1 \) is a finitely generated projective \( R \)-module of rank one. Hence there exist \( a_1, a_2, \ldots, a_r \in R \) and \( f_1, f_2, \ldots, f_r \in C_1 \) such that \((a_1, \ldots, a_r)R = R \) and \( (C_1)_{a_i} = R_{a_i} \) is a free \( R_{a_i} \)-module for \( 1 \leq i \leq r \). Thus, if \( B_0 \) is Noetherian (resp. finitely generated over \( R \)) for each \( i \), \( 1 \leq i \leq r \), then it will follow that \( B \) is Noetherian (resp. finitely generated over \( R \)).

**Theorem 5.6.** Let \( R \) be a Noetherian normal domain with quotient field \( K \) and \( A \) a Noetherian faithfully flat locally quasi \( \mathcal{A}^* \) algebra in codimension-one over \( R \). Let \( B = A \cap K[T] \), where \( T \in A \otimes_R K = A \otimes_R K = K[T, T^{-1}] \). Then \( B \) is Noetherian.
Proof. By Remark 5.5, we may assume that $C_1(=A\cap KT)$ is a free $R$-module, say $C_1 = Rf$ for some $f \in R$. By Corollary 4.4(3), $f$ is a generic variable of $B$ and $A = B[f^{-1}]$.

Let $S = 1 + FB$. By hypothesis $B[f^{-1}](= A)$ is a Noetherian ring. So $B$ is Noetherian if and only if $S^{-1}B$ is Noetherian. By Theorem 5.2, $B$ is faithfully flat over $R$ and hence $S^{-1}B$ is a flat $R$-module. Therefore, by Lemma 2.4, $S^{-1}B = \bigcap_{P \in \Delta} (S^{-1}B)_P$, where $(S^{-1}B)_P$ is the localisation of $S^{-1}B$ by the multiplicatively closed set $R \setminus P$. Note that $(S^{-1}B)_P = S^{-1}(B_P)$.

Recall that $\Delta_0(f) = \{ P \in \Delta \mid R_P[f] \not\subseteq B_P \}$ and $\{ c_P \}_{r \in \Sigma_0}$ is the family associated with the generic variable $f$ of $B$. Then, for $P \in \Delta \setminus \Delta_0$, $B_P = R_P[f]$ and hence $(S^{-1}B)_P = S^{-1}R_P[f]$.

Let $B$ be a Noetherian normal domain and $A$ a finitely generated faithfully flat locally quasi $\mathbb{A}^*$-algebra in codimension-one over $R$. Then $B = A \otimes_R K[T]$, where $T \in A \otimes_R K$ is such that $A \otimes_R K = K[T, T^{-1}]$. Then $B$ is finitely generated over $R$. In particular, if $R$ is a local domain then $A$ is quasi $\mathbb{A}^*$ over $R$.

Proof. To show that $B$ is finitely generated over $R$, we may assume, by Remark 5.5, that $C_1 = Rf$ for some $f \in R$. By Corollary 4.4(3), $f$ is a generic variable of $B$ and $A = B[f^{-1}]$. Let $\{ \Delta_0, I_\Gamma, B_\Gamma, c_\Gamma \}$ be the data associated to the generic variable $f$ of $B$.

By Theorem 5.2, $B = \lim_{\rightarrow \in \Sigma_0} B_P$. Since $B[f^{-1}] = \lim_{\rightarrow \in \Sigma_0} B_P[f^{-1}]$ is a finitely generated $R$-algebra, there exists $I_\Gamma_0 \subseteq \Sigma_0$ such that $B[f^{-1}] = B_{I_\Gamma_0}[f^{-1}]$. We now show that $I_\Gamma_0 = \Delta_0$. Suppose that there exists $P \in (\Delta_0 \setminus I_\Gamma_0)$, then $(B_{I_\Gamma_0})_P = R_P[f] \neq B_P = R_P[X_P]$. Since $f$ is a generic variable, $f = a_P X_P + b_P$, for some $a_P, b_P \in R_P$ with $a_P \in PR_P$. This shows that $(B_{I_\Gamma_0})_P[f^{-1}] = R_P[f, f^{-1}] \neq B_P[f^{-1}]$ contradicting the fact that $B[f^{-1}] = B_{I_\Gamma_0}[f^{-1}]$. Thus $I_\Gamma_0 = \Delta_0$ and hence $B = B_{I_\Gamma_0}$ which is finitely generated over $R$ by Proposition 3.4.

If $R$ is a local domain then, since $B = B_{I_\Gamma_0} = \bigoplus_{n \geq 0} (I_{\Gamma_0})^{-1}(f - c_{\Gamma_0})^n$ is $R$-flat (cf. Theorem 5.2) and $R$ is Noetherian, $I_{\Gamma_0}$ is a finitely generated projective $R$-module. Thus $I_{\Gamma_0}$ is a free $R$-module, say $I_{\Gamma_0} = Rd_{\Gamma_0}$. Hence $B = B_{I_{\Gamma_0}} = R \left[ f - c_{\Gamma_0} \over d_{\Gamma_0} \right]$ and $A = B[f^{-1}] = R \left[ f - c_{\Gamma_0} \over d_{\Gamma_0} \right]$ is quasi $\mathbb{A}^*$ over $R$. □

Corollary 5.8. Suppose there exists $(0 \neq a \in R)$ such that $A[1/a]$ is finitely generated over $R$. Then $A$ is finitely generated over $R$.

Proof. Since $A[1/a]$ is finitely generated over $R$, by Theorem 5.7, we have that $B[1/a]$ is a finitely generated $R[1/a]$-algebra. Fix a generic variable $X \in B$ and let $\Delta_0 = \{ P \in \Delta \mid R_P[X] \not\subseteq B_P \}$. Let $\Delta_1 = \{ P \in \Delta_0 \mid a \notin P \}$. Then, since $R$ is Noetherian, $\Delta_0 \setminus \Delta_1$ is a finite set. Since $B[1/a]$ is finitely generated over $R[1/a]$, by Proposition 3.4, $\Delta_1$ is a finite set and hence $\Delta_0$ is finite. Therefore, again by Proposition 3.4, $B$ is finitely generated over $R$ and hence $A$ is finitely generated over $R$ by Corollary 4.7(2). □

We now obtain Theorem A. We state it in two parts: Theorems 5.9 and 5.10.

Theorem 5.9. Let $(R, m)$ be a Noetherian normal local domain and $A$ a faithfully flat $R$-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over $R$. Then the following are equivalent:

(i) $A$ is quasi $\mathbb{A}^*$ over $R$.
(ii) $A$ is a finitely generated $R$-algebra.
(iii) $R/m \nsubseteq A/mA$.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are trivial.
(ii) $\Rightarrow$ (i) follows from Theorem 5.7.
(iii) $\Rightarrow$ (i) follows from Proposition 5.4. □

The next result, on complete local domain, is the quasi $\mathbb{A}^*$ analogue of Theorem 3.6.

Theorem 5.10. Let $R$ be a complete local Noetherian normal domain and $A$ be a faithfully flat $R$-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over $R$. Then the following conditions are equivalent:

(i) $A$ is Noetherian.
(ii) $A$ is finitely generated over $R$.
(iii) $A$ is quasi $\mathbb{A}^*$ over $R$.

Proof. Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). For (i) $\Rightarrow$ (iii), we have by Theorem 5.6, $B$ is Noetherian. Hence, by Theorem 3.6, $B = R[W]$. Thus the result follows from Theorem 4.6. □

In Section 6, Example 6.3 shows that Theorem 5.10 does not hold if $R$ is not complete.
6. Examples

Let $R$ be a Noetherian factorial domain with quotient field $K$. Let $A$ be a faithfully flat $R$-algebra which is locally quasi $A^*$ in codimension-one, $A \otimes_R K = k[T, T^{-1}]$ and $B = A \cap K[T]$. We have seen (Theorems 4.6 and 5.2) that $B$ is a faithfully flat $R$-algebra which is locally $A^*$ in codimension-one, $B \cap TK[T] = Bf$, $A = B[f^{-1}]$ and the prime element $f$ is a generic variable of $B$.

Therefore it is natural to ask the following:

**Question 1.** Let $R$ be a Noetherian factorial domain and $B$ a faithfully flat $R$-algebra which is locally $A^*$ in codimension-one. Can we find a generic variable $f \in B$ such that $B[f^{-1}]$ is faithfully flat over $R$?

Note that if $A = B[f^{-1}]$ is faithfully flat over $R$ then $A$ is locally quasi $A^*$ in codimension-one over $R$.

We give below an example of a factorial domain $B$, which is faithfully flat locally $A^*$ in codimension-one over $k[X, Y]$ ($k$: field) such that for any generic variable $f \in B$, there exists a maximal ideal $n$ of $k[X, Y]$ such that $f \in nB$ and hence $B[f^{-1}]$ is not faithfully flat over $R$.

**Example 6.1.** Let $k$ denote the algebraic closure of $\mathbb{Q}$, $R = k[X, Y]$ and $\Delta$ the set of all height one prime ideals of $R$. $k$ being countable, can be indexed as $k = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$. Let $p_i = Y + \alpha_iX$ and $P_i = p_iR$, $i \geq 1$. Set $\Delta_0 = \{P_i\}_{i \geq 1}$.

Let $x_0$ be an indeterminate over $R$. Set $c_0 = 1$,
\[
c_n = 1 + p_1 + \cdots + p_1p_2 \cdots p_n
\]
and
\[
x_n := \frac{x_{n-1} - 1}{p_n} = \frac{x_0 - c_{n-1}}{p_1p_2 \cdots p_n}
\]
for $n \geq 1$. Now set
\[
B := R[x_0, x_1, x_2, \ldots, x_n, \ldots],
\]
a direct limit of the $R$-algebras $R[x_n]$ ($= R^{(n)}$).

Clearly, $B_p = R_p^{(n)}$ for $P \in \Delta$ and $B$ is faithfully flat over $R$ since it is a direct limit of polynomial algebras over $R$. Note that $e_{x_0}(P) = 1$ for $P \in \Delta_0$ and $e_{x_0}(P) = 0$ for $P \in \Delta \setminus \Delta_0$ (see Section 3 for definition of $e_{x_0}$). Since $x_{n+1} \notin R[x_n]$, $B$ is not finitely generated over $R$. We show that:

1. $B$ is a factorial domain.
2. For every generic variable $f$ of $B$, there exists a maximal ideal $n$ of $R$ such that $f \in nB$.
3. By Remark 3.12(2), it suffices to show that $B$ is a Krull domain. Since $B$ is faithfully flat over $R$, by Lemma 2.4, we have
\[
B = \bigcap_{P \in \Delta} B_P = \left( \bigcap_{P \in \Delta} V_P \right) \cap K[x_0],
\]
where $K$ is the quotient field of $R$ and $V_P := B_{P_R}$ is a discrete valuation ring of $K[x_0]$ for each $P \in \Delta$. Thus, since $K[x_0]$ is a Krull domain, $B$ is an intersection of discrete valuation rings. Hence, to show that $B$ is a Krull domain, it suffices to show that, for any $h \in B$, there exist only finitely many $P \in \Delta$ such that $h \in P_B$.

Now
\[
B = \bigcap_{P \in \Delta} B_P = \left( \bigcap_{P \in \Delta} B_P \right) \cap \left( \bigcap_{P \in \Delta_0} B_P \right) = \left( \bigcap_{P \in \Delta} B_P \right) \cap R[x_0],
\]
where $R' = \bigcap_{P \in \Delta \setminus \Delta_0} R_P$, $R'$ is a Noetherian normal domain by Lemma 3.14. Hence, it suffices to show that for any $h \in B$, there exists an integer $m$ such that $h \notin P_B$ for all $m \geq m$. Let $h \in B = R[x_0, x_1, \ldots]$. Multiplying $h$ by a suitable element from $R$, we may assume $h \in R[x_0]$. Let
\[
h = \phi(x_0) = a_0 + a_1x_0 + \cdots + a_rx_0^r, \quad a_i \in R.
\]
Let $d_i$ denote the degree of $a_i$ in $X$ and $Y$. Choose $m > 1$ large enough so that

(i) for $n \geq m$, the leading coefficient $a_r \notin p_nR$ and

(ii) $m > 1 + d_i \forall i \in \{0, \ldots, r\}$.

We show that $h \notin P_B$ for $n \geq m$. If $r = 0$, then clearly $h \notin P_nB$ for $n \geq m$ since $R/P_nR \leftarrow B/P_nB$ and $a_r \notin p_nR$. Suppose that $r > 0$ and $h \in P_nB$ for some $n \geq m$. Then $\phi(c_{n-1}) \in P_nR$ since $R/P_nR \leftarrow B/P_nB$ and $x_0 = c_{n-1} + p_1 \cdots p_nx_0$. Now, identifying $R/P_nR$ with $k[X]$, the image of the product $P_1 \cdots p_{n-1}$ in $R/P_nR$ is $x^m$ and hence the image of $c_{n-1}$ in $R/P_nR$ is a polynomial of degree $n - 1$. Therefore, since $m - 1 > d_i \forall i \leq r - 1$, the image of $\phi(c_{n-1})$ in $R/P_nR$ is a polynomial in $R/P_nR(\cong k[X])$ of degree at least $(n - 1)r > 1$ contradicting the fact that $\phi(c_{n-1}) \in P_nR$. Thus $h \notin P_nB$ for only finitely many $P \in \Delta$.

(2) Let $f$ be a generic variable of $B$. Then $f = a + bx_0$, for some $a, b \in R$ and $n \geq 0$ and hence for $\ell > 0$,
\[
f = a + b(1 + p_{n+1} + \cdots + p_{n+1} \cdots p_{n+\ell} + p_{n+\ell} \cdots p_{n+\ell+1}x_{n+\ell+1}).
\]
Choose \( \ell \) such that, \( \ell > \max\{\deg a, 0\} \) and \( p_j \) does not divide \( b \) for \( j = \ell + n + 1 \). It is then easy to see that the image of the element

\[
f_{\ell} := a + b(1 + p_{n+1} + \cdots + p_{n+1} \cdots p_{\ell+n})
\]

in \( R/P_{\ell+n+1} \) (identified with \( k[X] \)) is a non-constant polynomial. In fact, the \( X \)-degree of the image of \( f_{\ell} \) in \( R/P_{\ell+n+1} (= k[X]) \) is at least \( \ell (> 0) \) and hence it has a root in \( k \), say \( \beta \). Then \( f \in nB \), where \( n = (Y + \alpha_{\ell+n+1}X, X - \beta)R \).

It is therefore not possible to obtain a faithfully flat locally quasi \( A^* \) algebra in codimension-one over \( R \) by inverting some generic variable of \( B \).

We now show that if \( R \) is local or \( R \) is a retract of \( B \) (equivalently \( B \) is a graded \( R \)-algebra), then one can indeed find a generic variable \( f \) such that \( A = B[f^{-1}] \) is a faithfully flat \( R \)-algebra which is locally quasi \( A^* \) in codimension-one.

**Lemma 6.2.** Let \( R \) be a Noetherian normal domain and \( B \) a faithfully flat \( R \)-algebra which is locally \( A^1 \) in codimension-one over \( R \). Then there exists a generic variable \( f \in B \) such that \( A := B[f^{-1}] \) is faithfully flat locally quasi \( A^* \) in codimension-one over \( R \), if either

1. \( R \) is local with maximal ideal \( m \), or
2. \( R \) is a retract of \( B \).

**Proof.** It is enough to show that there exists a generic variable \( f \) such that \( B[f^{-1}] \) is faithfully flat over \( R \).

(1) Suppose that \( R \) is local with maximal ideal \( m \). Let \( X \) be a generic variable of \( B \). Then taking

\[
f = \begin{cases} 
X - 1 & \text{if } X \in mB, \\
X & \text{if } X \notin mB.
\end{cases}
\]

We see that \( B[1/f] \) is faithfully flat over \( R \).

(2) Now suppose that \( R \) is a retract of \( B \). Let \( \theta : B \rightarrow R \) be the retraction map. Let \( X \) be a generic variable for \( B \), \( c = \theta(X) \) and set \( f = X - c + 1 \). Then \( \theta(f) = 1 \) and hence \( f \notin mB \) for any maximal ideal \( m \) of \( R \). Therefore, \( A = B[f^{-1}] \) is faithfully flat over \( R \). \( \Box \)

We now show the existence of a Noetherian ring \( A \) which is locally quasi \( A^* \) algebra in codimension-one over a Noetherian normal local domain \( R \) but which is not finitely generated over \( R \).

**Example 6.3.** Example 6.2 in [2] gives us an example of a Noetherian ring \( B \) which is faithfully flat locally \( A^1 \) algebra in codimension-one over a Noetherian normal local domain \( R \) but which is not finitely generated over \( R \). By Lemma 6.2, we can get a generic variable \( f \) such that \( A = B[f^{-1}] \) is a faithfully flat locally quasi \( A^* \) algebra in codimension-one over \( R \). Now \( A \) is a Noetherian ring but \( A \) is not finitely generated over \( R \) by Theorem 5.7.

The following example shows that over a complete local domain, there exists a faithfully flat locally quasi \( A^* \) algebra in codimension-one which is not finitely generated.

**Example 6.4.** Let \( R = \mathbb{Q}[[X, Y]] \) be a Noetherian factorial complete local domain. Let \( p_n = X + nY \) and let \( W \) be an indeterminate over \( R \). Set \( W_n = W/p_n \) for \( n \geq 1 \) and \( B = R[W, W_1, \ldots, W_n, \ldots] \). Then \( B \) is a faithfully flat \( R \)-algebra which is locally \( A^1 \) in codimension-one over \( R \) but \( B \) is not finitely generated over \( R \). Hence, \( A = B[(W-1)^{-1}] \) is a faithfully flat \( R \)-algebra which is locally quasi \( A^* \) in codimension-one over \( R \) (cf. Lemma 6.2) but \( A \) is not finitely generated over \( R \) by Theorem 5.7.

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**Appendix**

We now discuss the construction of algebras which are locally quasi \( A^* \) in codimension-one. In [2], a recipe for the construction of an \( R \)-algebra \( B \) which is semi-faithfully flat and locally \( A^1 \) in codimension-one was given; in a sense, the construction characterises all such algebras. Using such a \( B \), and a finitely generated projective \( R \)-submodule \( C_1 \) of rank one such that \( C_1 \subseteq B_1 \) and \( C_1 \cap R = (0) \), one gets an \( R \)-algebra \( A = B[(C_1B)^{-1}] \) which is locally quasi \( A^* \)-algebra in codimension-one over \( R \). Note that if \( B \) is faithfully flat over \( R \), then \( A \) is flat over \( R \). However, Example 6.1 shows that it is possible to construct \( B \) as above such that \( B[(C_1B)^{-1}] \) is not faithfully flat for any choice of \( C_1 \). This leads to:

**Question 2.** Let \( B \) be a faithfully flat \( R \)-algebra which is locally \( A^1 \) in codimension-one over a Noetherian normal domain \( R \). Let \( C_1 \) be a finitely generated projective \( R \)-module of rank one such that \( C_1 \subseteq B_1 \) and \( C_1 \cap R = (0) \) and let \( A = B[(C_1B)^{-1}] \). When is \( A \) a faithfully flat locally quasi \( A^* \) algebra in codimension-one over \( R \)?

In this section we investigate Question 2.
Let the setup be as in Question 2. Let \( L = B_1/R \) where \( B_1 \) is as in (3.3). Since \( C_1 \subseteq B_1 \) and \( C_1 \cap R = (0) \), \( C_1 \) is isomorphic to its image \( \overline{C}_1 \) in \( L \). Let \( \Delta \) denote the set of all height one prime ideals of \( R \). For each \( P \in \Delta \), we define an integer \( e(P) \) by
\[
e(P) := \ell_{R_{\overline{C}_1}}(L/\overline{C}_1 \otimes R_P),
\]
the length of the \( R_P \)-module \( (L/\overline{C}_1 \otimes R_P) \). Let
\[
\Delta_0 := \{ P \in \Delta \mid e(P) > 0 \} \quad \text{and} \quad \Sigma_0 := \{ P \} \quad \text{of all finite subsets of} \ \Delta_0.
\]

For \( \Gamma \in \Sigma_0 \), let
\[
I_{\Gamma} := P_1^{(e(P_1))} \cap \cdots \cap P_n^{(e(P_n))},
\]
\[
E_{\Gamma} := I_{\Gamma}^{-1} \overline{C}_1 \subseteq L \quad \text{and} \quad D_{\Gamma} := \rho^{-1}(E_{\Gamma}), \quad \text{where} \ \rho : B_1 \to B_1/R = L \quad \text{is a canonical projection}.
\]

**Lemma A.1.** For each \( \Gamma \in \Sigma_0 \), there exists a retraction \( \theta_{\Gamma} : D_{\Gamma} \to R \) and hence \( D_{\Gamma} \cong R \oplus E_{\Gamma} \).

**Proof.** Since \( C_1 \) is a finitely generated projective \( R \)-module of rank one, \( E_{\Gamma}(= I_{\Gamma}^{-1} \overline{C}_1) \) is a finitely generated torsion-free \( R \)-module of rank one. Hence there exists a sequence \( x, y \in R \) such that \((E_{\Gamma})_x = I_{\Gamma}R_x \) and \((E_{\Gamma})_y = E_{\Gamma}R_y \) for some \( F, G \in E_{\Gamma} \).

Now \( F \in GR_y((E_{\Gamma})_y) \). Hence there exist \( a \in R \) and \( n \geq 0 \) such that
\[
F = aGy^n.
\]

Let \( f, g \in D_{\Gamma} \) be such that \( \rho(f) = F \) and \( \rho(g) = G \). Thus \( c := y^nf - ag \in \text{Ker}(\rho) = R \). Since, \( B \) is faithfully flat over \( R \) and \( c \in (y^n, a)B \cap R \), we have \( c \in (y^n, a)R \). Let \( c = y^n c_1 + ac_2 \) where \( c_1, c_2 \in R \). Then \( y^n(f - c_1) = a(g + c_2) \). Let \( f' = f - c_1 \) and \( g' = g + c_2 \), so that \( y^ng' = ag \). Then \((D_{\Gamma})_x = R_x \oplus R_yf' \) and \((D_{\Gamma})_y = R_y \oplus R_yg' \). We now show that we can give an \( R \)-linear map from \( D_{\Gamma} \to R \). Let \( h \in D_{\Gamma} \). Then \( h = \frac{r_0 + r_1 f'}{x^t} = \frac{r_0 + r_1 (ag/y^n)}{x^t} = \frac{r_0 + r_1 a}{x^t} \cdot y^n \). Now \( r_0/x^t = r_0'/y^n \in R \oplus R_y \) and hence in \( R \) since \( x, y \) is a sequence in \( R \). It is easy to see that the map \( \theta_{\Gamma} : D_{\Gamma} \to R \) defined by \( \theta_{\Gamma}(h) = r_0/x^t \) is an \( R \)-linear map. Clearly \( \theta_{\Gamma}(r) = r \) for \( r \in R \). Hence \( \theta_{\Gamma} \) is an \( R \)-retraction. \( \square \)

Note that, since \( R \subset L^{-1}, R \oplus C_1 \subset D_{\Gamma} \). Using the canonical retraction \( \theta_{C_1} : D_{\Gamma} \to R \) defined by Lemma A.1, we now define an \( R \)-linear automorphism
\[
\tilde{\theta}_{\Gamma} : R \oplus C_1 \to R \oplus C_1
\]
by \( \tilde{\theta}_{\Gamma}(r) = r \) for \( r \in R \) and \( \tilde{\theta}_{\Gamma}(c) = c - \theta_{C_1}(c) \) for \( c \in C_1 \). Set
\[
N_{\Gamma} := \tilde{\theta}_{\Gamma}(C_1)
\]
and
\[
H_{\Gamma} := \theta_{C_1}(C_1).
\]

Note that \( N_{\Gamma} \) and \( H_{\Gamma} \) are finitely generated flat \( R \)-modules such that \( N_{\Gamma} \to D_{\Gamma} \) and \( H_{\Gamma} \to R \). In fact \( D_{\Gamma} = R \oplus I_{\Gamma}^{-1}N_{\Gamma} \) and for each \( P \in \Gamma, \ (D_{\Gamma})_P = (B_1)_P \) and for \( P \notin \Gamma, \ (D_{\Gamma})_P = R_P \oplus (N_{\Gamma})_P \).

**Remark A.2.** Set \( B_{\Gamma} := \bigoplus_{n \geq 0} I_{\Gamma}^{-n}N_{\Gamma} \). Then one can show that \( B_{\Gamma} = S_{\Gamma}^{-1}B \cap R_{\Gamma}(C_1) \), where \( S_{\Gamma} := R \setminus \bigcup_{P \in \Gamma} P \) and \( B = \lim_\rightarrow B_{\Gamma} \). Moreover if \( I_{\Gamma_1} \) and \( I_{\Gamma_2} \) are finite subsets of \( \Delta_0 \) with \( I_{\Gamma_1} \subseteq I_{\Gamma_2} \), then \( \theta_{\Gamma_1 - \Gamma_2}(C_1) = (\theta_{\Gamma_1}(\Gamma_1 \setminus \Gamma_2)) \cap I_{\Gamma_1} \).

This is an analogue of Theorem 3.2 giving the structure of \( B \) in terms of a finitely generated projective \( R \)-module \( C_1 \) such that \( C_1 \subseteq B_1 \) and \( C_1 \cap R = (0) \) in place of a generic variable.

**Proposition A.3.** With notation as above, the following are equivalent:

(i) \( A = B[(C_1B)^{-1}] \) is a faithfully flat \( R \)-algebra which is locally quasi \( A^* \) in codimension-one.

(ii) \( I_{\Gamma} + H_{\Gamma} = R \) for each \( \Gamma \in \Sigma_0 \).

As a consequence, if \( A = B[(C_1B)^{-1}] \) is a faithfully flat \( R \)-algebra then \( C_1/I_{\Gamma}C_1 \) is a free \( R/I_{\Gamma} \)-module for each \( \Gamma \in \Sigma_0 \).

**Proof.** (i) \( \Rightarrow \) (ii) follows from Remark 5.3(1) by reducing to the local case.

(ii) \( \Rightarrow \) (i): Let \( m \) be a maximal ideal of \( R \). Since \( C_1 \) is a finitely generated projective \( R \)-module of rank one, \( (C_1)_m = R_m f \) for some \( f \in C_1 \) and for each \( \Gamma \in \Sigma_0, H_{\Gamma}R_m = R_m c_{\Gamma} \) for some \( c_{\Gamma} \in R \). If \( I_{\Gamma}R_m = R_m \) for each \( \Gamma \in \Sigma_0 \), then \( B_m = R_m [f] \) and hence \( A_m = B_m [f^{-1}] = R_m [f, f^{-1}] \) is faithfully flat over \( R_m \). Suppose that there exists a \( \Gamma \in \Sigma_0 \) such that \( I_{\Gamma}R_m \subseteq R_m \), then \( c_{\Gamma} \in R_m \) by hypothesis. Since \( B_m \) is faithfully flat over \( R_m \), we have \( f - c_{\Gamma} \in I_{\Gamma}S_{\Gamma}^{-1}B_m \cap B_m = I_{\Gamma}B_m \) and hence \( f \notin mB_m \) as \( c_{\Gamma} \in R_m \). Thus \( A_m = B_m [f^{-1}] \) is faithfully flat over \( R_m \) and hence, since faithful flatness is a local property, \( A = B[(C_1B)^{-1}] \) is faithfully flat over \( R \). \( \square \)
References