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On locally quasi $\mathbb{A}^*$ algebras in codimension-one over a Noetherian normal domain

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ABSTRACT

Let R be Noetherian normal domain. We shall call an R-algebra A quasi $\mathbb{A}^*$ if $A = R[X, (aX + b)^{-1}]$ where $X \in A$ is a transcendental element over R, $a \in R \setminus 0$, $b \in R$ and $(a, b)R = R$. In this paper we shall describe a general structure for any faithfully flat R-algebra A which is locally quasi $\mathbb{A}^*$ in codimension-one over R. We shall also investigate minimal sufficient conditions for such an algebra to be finitely generated.

1. Introduction

Let R be an integral domain. Recall that an R-algebra A is called $\mathbb{A}^1$ if $A \cong R[X]$ (polynomial algebra in one variable over R) and is called $\mathbb{A}^*$ if $A \cong R[X, X^{-1}]$ (Laurent polynomial algebra in one variable over R). Generalising this notion of $\mathbb{A}^*$, we call an R-algebra A to be quasi $\mathbb{A}^*$ if $A \cong R[X, (aX + b)^{-1}]$ for some X transcendental over R, $a \in R \setminus 0$ and $b \in R$ with $(a, b)R = R$. Note that if $a \in R^*$, then A is $\mathbb{A}^*$ over R. This notion of quasi $\mathbb{A}^*$ arises naturally in the study of algebras whose generic fibres are $\mathbb{A}^*$. To see an example, consider a discrete valuation ring (V, π) with quotient field K and a faithfully flat, finitely generated V-algebra A such that πA is a prime ideal of A and $V/\pi V$ is algebraically closed in $A/\pi A$. Under these hypotheses, if $A[1/\pi]$ is a polynomial algebra $K[Y]$, then A is a polynomial algebra $V[X]$ by [5, Theorem 2.3.1]; on the other hand, if $A[1/\pi]$ is a Laurent polynomial algebra $K[Y, Y^{-1}]$, then it can be shown (using similar methods) that A is a quasi $\mathbb{A}^*$-algebra of the form $V[X, (aX + b)^{-1}]$ for some X in A transcendental over V, $a \in V \setminus 0$ and $b \in V$ with $(a, b)V = V$.

Now let R be a Noetherian normal domain. In [2], an integral domain B containing R has been called "locally $\mathbb{A}^1$ in codimension-one" if, for every height one prime ideal P of R, $B_P\ (= B \otimes_R R_P)$ is $\mathbb{A}^1$ over R_P. Such an algebra B has been studied extensively in [2] when B is faithfully flat over R. In a similar fashion, we call an integral domain A containing R to be "locally quasi $\mathbb{A}^*$ in codimension-one" if, for every height one prime ideal P of R, $A_P\ (= A \otimes_R R_P)$ is quasi $\mathbb{A}^*$ over R_P.

In this paper we investigate properties of a faithfully flat algebra A over a Noetherian normal domain R which is locally quasi $\mathbb{A}^*$ in codimension-one. We first explore a general structure of A and show that A has an R-subalgebra B which is faithfully flat and locally $\mathbb{A}^1$ in codimension-one over R such that $A = B[Q^{-1}]$ for some invertible ideal Q of B (Theorems 4.6 and 5.2). As a consequence, if R is factorial then it follows (from known results about B) that A is a direct limit of quasi $\mathbb{A}^*$ algebras over R (Corollary 4.5) and hence, if A is finitely generated over R then A is quasi $\mathbb{A}^*$ over R. It will also be seen (Proposition 5.4) that at each point $\mathcal{P}$ of Spec R, $\mathcal{P}A \in$ Spec A, and that either $A_{\mathcal{P}}$ is quasi $\mathbb{A}^*$ over $R_{\mathcal{P}}$ or the fibre ring $A \otimes_R k(\mathcal{P}) = k(\mathcal{P})$. As a consequence, we show that when R is local, then A is quasi $\mathbb{A}^*$ under a mild hypothesis on the closed fibre. More precisely, we prove (Theorems 5.9 and 5.10):

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Theorem A. *Let (R, m) be a Noetherian normal local domain and A a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over R. Then the following are equivalent:*

(i) *A is a quasi $\mathbb{A}^*$ R-algebra.*
(ii) *A is a finitely generated R-algebra.*
(iii) *$R/m \subsetneqq A/mA$.*

Moreover, if R is complete and A is Noetherian, then A is indeed quasi $\mathbb{A}^*$.

However if R is not complete, there exist examples of Noetherian faithfully flat R-algebras which are locally quasi $\mathbb{A}^*$ in codimension-one but which are not finitely generated over R (Example 6.3). Surprisingly, if we assume that A is locally $\mathbb{A}^*$ in codimension-one then A is actually finitely generated over R without any additional hypothesis. We prove (Theorem 4.8):

Theorem B. *Let R be a Noetherian normal domain and A be a faithfully flat R-algebra which is locally $\mathbb{A}^*$ in codimension-one over R. Then $A = \bigoplus_{n\in\mathbb{Z}} I^n u^n$ for an invertible ideal I of R. In particular, A is finitely generated over R.*

Theorem B was proved earlier in [1] under the additional assumption that A is finitely generated over R (cf. Remark 4.9(2)).

However, even if R is complete, there exists a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one but not finitely generated (Example 6.4).

We now give a layout of the paper. Sections 1–3 are introductory — Section 2 is on preliminaries; in Section 3, we recall results from [2] on algebras which are locally $\mathbb{A}^1$ in codimension-one over R and prove some results on the consequences of faithful flatness of such algebras. The main results of this paper are presented in Sections 4 and 5 — Theorem B, which requires less technical properties, will be proved in Section 4 and Theorem A in Section 5. In Section 4, we first discuss basic properties of a faithfully flat algebra A over a Noetherian normal domain R which is locally quasi $\mathbb{A}^*$ in codimension-one over R and establish the existence of an R-subalgebra B of A which is locally $\mathbb{A}^1$ in codimension-one such that $A = B[Q^{-1}]$ for a suitable invertible ideal Q of B. With the help of this presentation we prove Theorem B. In Section 5, we discuss properties of the above ring B; in particular, we show that B is a faithfully flat R-algebra and deduce some results on the fibres of the map $\operatorname{Spec} A \to \operatorname{Spec} R$. We also prove that B is Noetherian (respectively finitely generated over R) if and only if A is so. Finally we prove Theorem A. In Section 6, we discuss a few examples.

2. Preliminaries

We recall some standard notation to be used throughout the paper. For a ring R, R^* will denote the multiplicative group of units of R. For a prime ideal P of R, and an R-algebra A, A_P denotes the ring $S^{-1}A$, where $S = R \setminus P$ and $k(P)$ denotes the residue field R_P/PR_P. The notation $A = R^{[1]}$ will mean that A is isomorphic, as an R-algebra, to a polynomial ring in one variable over R.

For an R-module M, we denote the tensor algebra of M over R by $T_R(M)$ and the symmetric algebra by $Sym_R(M)$. Note that if R is a domain and M is a flat R-module of rank one, then $T_R(M) = Sym_R(M)$.

We compile below the notions mentioned in the introduction which are central to this paper.

Definition 2.1. (1) An R-algebra A is said to be $\mathbb{A}^*$ if there exists an element X in A which is transcendental over R such that $A = R[X, X^{-1}]$.

(2) We shall call an R-algebra A to be "quasi $\mathbb{A}^*$" if there exists an element X in A which is transcendental over R such that

$$A = R[X, (aX + b)^{-1}],$$

for some $a \in R \setminus 0$, $b \in R$ satisfying $(a, b)R = R$. Note that $R[X, (aX + b)^{-1}]$ is $\mathbb{A}^*$ over R if and only if $a \in R^*$.

(3) We shall call an R-algebra A to be "locally quasi $\mathbb{A}^*$ in codimension-one" over R if A_P is quasi $\mathbb{A}^*$ over R_P for every height one prime ideal P in R.

(4) We shall call an R-algebra B to be "locally $\mathbb{A}^1$ in codimension-one" if $B_P = {R_P}^{[1]}$ for every height one prime ideal P in R. If R is an integral domain with quotient field K and X is an element of B transcendental over R such that $R[X] \subseteq B \subseteq K[X]$, then we say that X is a "generic variable" for B.

As mentioned in the introduction, we shall discuss properties of algebras which are locally $\mathbb{A}^1$ in codimension-one in Section 3 and results on algebras which are locally quasi $\mathbb{A}^*$ in codimension-one in Sections 4 and 5.

We now mention two results on flat R-modules lying between the integral domain R and its quotient field K. The first result is on the flatness of the R-subalgebra $R[M]$ of K generated by a flat R-submodule M of K.

Lemma 2.2. *Let R be an integral domain with quotient field K and M a flat R-module such that $R \subseteq M \subseteq K$. Then the R-subalgebra $R[M]$ of K is flat over R.*

Proof. Since $M \subseteq K$ and M is flat over R, we can identify $T_R(M)$ as a graded subring of the polynomial algebra $K[W]$ with $M \otimes_R \cdots \otimes_R M$ (n times) corresponding to $M^n W^n$ ($\subseteq KW^n$). Thus

$$T_R(M) = \{a_0 + a_1 W + \cdots + a_n W^n \mid n \geq 0 \text{ and } a_i \in M^i \text{ for } 0 \leq i \leq n\}.$$

Note that since $R \subset M$, $W \in T_R(M)$; a homogeneous element of degree one. We first show that $T_R(M) \cap (W-1)K[W] = (W-1)T_R(M)$. Let

$$h = b_0 + b_1 W + \cdots + b_\ell W^\ell \in T_R(M) \cap (W-1)K[W], \tag{2.1}$$

with $b_i \in M^i$, $0 \le i \le \ell$. Write $h = (W-1)g$, where

$$g = d_0 + d_1 W + \cdots + d_{\ell-1} W^{\ell-1} \in K[W]. \tag{2.2}$$

From (2.1) and (2.2), it follows that $d_0 \in R$, $(d_1 - d_0) \in M, \ldots, (d_{\ell-1} - d_{\ell-2}) \in M^{\ell-1}$. Since $R \subseteq M \subseteq M^2 \subseteq M^3 \cdots$, we have $g \in T_R(M)$.

Let ϕ be the restriction of the R-linear map $\tilde{\phi} : K[W] \to K$ sending $W \to 1$. Then $\phi(T_R(M)) = R[M]$ and hence we have the short exact sequence

$$0 \to (W-1)T_R(M) \to T_R(M) \xrightarrow{\phi} R[M] \to 0. \tag{2.3}$$

Let I be an ideal of R. Since M is a flat R-module, $T_R(M)$ is a flat R-algebra. Thus $\mathrm{Tor}_1^R(T_R(M), R/I) = 0$. Hence, tensoring (2.3) with R/I, we have the exact sequence

$$0 \to \mathrm{Tor}_1^R(R[M], R/I) \to (W-1)T_R(M) \otimes_R R/I \to T_R(M) \otimes_R R/I \to R[M] \otimes_R R/I \to 0.$$

Let $h \in T_R(M)$ be such that $(W-1)h \in IT_R(M)$. Since W is a homogeneous element of degree one and $IT_R(M)$ is a homogeneous ideal of the graded ring $T_R(M)$, it follows that $h \in IT_R(M)$. Thus the map

$$(W-1)T_R(M) \otimes_R R/I \to T_R(M) \otimes_R R/I$$

is injective. Hence $\mathrm{Tor}_1^R(R[M], R/I) = 0$. Thus $R[M]$ is a flat R-algebra. □

The next result is on the Noetherian property of flat R-subalgebras of the quotient field of R.

Lemma 2.3. *Let R be a Noetherian domain with quotient field K, and D a flat R-algebra such that $R \subseteq D \subseteq K$. Then D is a Noetherian ring.*

Proof. To show that D is Noetherian it is enough to show that every prime ideal of D is finitely generated. Let Q be a prime ideal of D and $P = Q \cap R$. Now D_Q is faithfully flat over R_P and since $R_P \subseteq D_Q \subseteq K$, we have $D_Q = R_P$ and hence $D \subseteq R_P$. We show $Q = PD$. For this it is enough to show that PD is a prime ideal of D. Now D/PD is flat R/PR-module and hence every element of $R \setminus P$ is a non-zero divisor in D/PD. Thus $D/PD \hookrightarrow R_P/PR_P$ which is a field and hence PD is a prime ideal of D. □

Finally, we recall an elementary result which will be used in the paper. (See the argument in [3, Lemma 2.8].)

Lemma 2.4. *Let R be a Noetherian normal domain with quotient field K and let Δ be the set of all height one prime ideals of R. For a torsion free R-module M, the following conditions are equivalent:*

(i) *$M = \bigcap_{P \in \Delta} M_P$, where M and $M_P = M \otimes_R R_P$ are identified with their images in $M \otimes_R K$.*
(ii) *If a and b are elements of R such that b is (R/aR)-regular, then b is (M/aM)-regular.*

In particular, if either M is R-flat or a direct limit of finitely generated reflexive R-modules, then $M = \bigcap_{P \in \Delta} M_P$.

3. Locally $\mathbb{A}^1$ algebras in codimension-one: some old and some new results

Throughout this section, R will denote a **Noetherian normal domain** *with quotient field K and Δ the set of all prime ideals in R of height one.*

As in [2], we call an integral domain B containing R to be "semi-faithfully flat over R" if

(1) $B = \bigcap_{P \in \Delta} B_P$.
(2) $IB \cap R = I$, for every ideal I of R.

In [2], properties of semi-faithfully flat algebras, which are locally $\mathbb{A}^1$ in codimension-one over R, were investigated. A general structure of such an R-algebra B was described ([2, Theorem 7.2]). Further, when B is faithfully flat over R, conditions for B to be finitely generated were given ([2, Corollary 2.7, Theorems 2.11 and 7.12]). In this section, we shall recall some of these results and investigate some consequences (Proposition 3.10 and Lemma 3.14) when B is faithfully flat over R.

Throughout this section, B will denote a **semi-faithfully flat** *R-algebra which is* **locally $\mathbb{A}^1$ in codimension-one**. We recall from [2], some objects associated with B and a generic variable X of B, (i.e., an element $X \in B$ for which $B \otimes_R K = K[X]$).

For each $P \in \Delta$, fix $X_P \in B_P$ such that

$$B_P = R_P[X_P]. \quad \text{Then } X = a_P X_P + b_P \quad \text{for some } a_P, b_P \in R_P.$$

Now set

$$e_X(P) := v_P(a_P), \quad \text{where } v_P(a_P) \text{ is the valuation of } a_P \text{ in } R_P \text{ and}$$
$$\Delta_0(X) := \{P \in \Delta \mid R_P[X] \subsetneqq B_P\}.$$

We fix a generic variable X and write Δ_0 in place of $\Delta_0(X)$ and $e(P)$ in place of $e_X(P)$. Note that for each $P \in \Delta_0$, $a_P \notin {R_P}^*$ and hence $e(P) > 0$. Thus

$$\Delta_0 = \{P \in \Delta \mid R_P[X] \subsetneq B_P\} = \{P \in \Delta \mid a_P \notin {R_P}^*\} = \{P \in \Delta \mid e(P) > 0\}.$$

Let Σ_0 be the set of all finite subsets of Δ_0. For $\Gamma = \{P_1, \dots, P_n\} \in \Sigma_0$, set

$$R_\Gamma := \bigcap_{P \in \Delta \setminus \Gamma} R_P,$$

$$B_\Gamma := {S_\Gamma}^{-1} B \cap R_\Gamma[X], \quad \text{where } S_\Gamma := R \setminus \left(\bigcup_{P \in \Gamma} P \right) \quad \text{and}$$

$$I_\Gamma := {P_1}^{(e(P_1))} \cap \cdots \cap {P_n}^{(e(P_n))}.$$

Remark 3.1. (1) ${S_\Gamma}^{-1}B = {S_\Gamma}^{-1}B_\Gamma$.
(2) I_Γ is a divisorial ideal of R ([4, Corollary 5.5]). Hence $\mathrm{Hom}_R(I_\Gamma, R) = {I_\Gamma}^{-1}$ and $\mathrm{Hom}_R({I_\Gamma}^{-1}, R) = I_\Gamma$.
(3) For $P \in (\Delta_0 \setminus \Gamma)$, $(B_\Gamma)_P = R_P[X] \neq B_P$ and hence if Γ is a proper subset of Δ_0 then B_Γ is a proper subring of B.
(4) The rings B_Γ, together with the inclusion maps, form a direct system $\{B_\Gamma \mid \Gamma \in \Sigma_0\}$ ([2, Lemma 2.1]).

The following technical result is proved in ([2, Theorem 7.2]).

Theorem 3.2. *For each $\Gamma \in \Sigma_0$, there exists $c_\Gamma \in R$ such that*

$$B_\Gamma = \bigoplus_{n \geq 0} ({I_\Gamma}^n)^{-1} \, (X - c_\Gamma)^n, \tag{3.1}$$

and for any $\Gamma_1, \Gamma_2 \in \Sigma_0$, with $\Gamma_1 \subseteq \Gamma_2$, we have $I_{\Gamma_1} \supseteq I_{\Gamma_2}$ and $c_{\Gamma_2} - c_{\Gamma_1} \in I_{\Gamma_1}$. Moreover,

$$B = \varinjlim_{\Gamma \in \Sigma_0} B_\Gamma. \tag{3.2}$$

To a generic variable X we associate the set Δ_0 and the families $\{I_\Gamma\}_{\Gamma \in \Sigma_0}$, $\{B_\Gamma\}_{\Gamma \in \Sigma_0}$ and $\{c_\Gamma\}_{\Gamma \in \Sigma_0}$ as above. We abbreviate this as $\{\Delta_0, I_\Gamma, B_\Gamma, c_\Gamma\}$.

Lemma 3.3. *B_Γ is flat over R if and only if I_Γ is an invertible ideal of R. As a consequence, if B_Γ is flat over R then B_Γ is a finitely generated R-algebra.*

Proof. If I_Γ is an invertible ideal of R then ${I_\Gamma}^n$ is an invertible ideal and hence $({I_\Gamma}^n)^{-1} = ({I_\Gamma}^{-1})^n$ is a (finitely generated) projective R-module and hence a flat R-module for every $n \geq 0$. Hence B_Γ is flat over R. Moreover, B_Γ is finitely generated over R as ${I_\Gamma}^{-1}$ is a finitely generated flat R-module and B_Γ is generated by ${I_\Gamma}^{-1}$ over R.

Now suppose that B_Γ is flat over R. Then ${I_\Gamma}^{-1}$ is flat over R. As R is Noetherian, it follows that ${I_\Gamma}^{-1}$ is a finitely generated projective R-module (of rank one). Hence I_Γ is a finitely generated projective R-module (cf. Remark 3.1(2)). Thus I_Γ is invertible. □

The following result occurs in [2, Corollary 2.7] but for the sake of convenience we record a proof here.

Proposition 3.4. *Suppose that B is faithfully flat over R. Then B is finitely generated over R if and only if Δ_0 is a finite (possibly empty) set.*

Proof. Note that $\Delta_0 = \emptyset$ if and only if $R[X] = B$. We now assume that $R[X] \neq B$. By Theorem 3.2,

$$B = \varinjlim_{\Gamma \in \Sigma_0} B_\Gamma$$

and hence, if B is finitely generated over R then, there exists a finite subset Γ' of Δ_0 such that $B_{\Gamma'} = B$. Therefore, by Remark 3.1(3), $\Gamma' = \Delta_0$.

Now suppose Δ_0 is a finite set. Taking Γ to be Δ_0, we see that $B_\Gamma = B$. Since $B = B_\Gamma$ is flat, B is finitely generated over R by Lemma 3.3. □

The following result on the R-algebra B was stated in [3, Theorem 4.6] under the hypothesis that B is faithfully flat over R, but the proof uses only semi-faithful flatness of B (cf. [2, Remark 7.3(2)]).

Theorem 3.5. *Let R be a factorial domain. Then B is a direct limit of polynomial algebras in one variable over R.*

The next theorem, proved in [2, Theorem 3.7], gives a necessary and sufficient condition for B to be finitely generated when R is complete local.

Theorem 3.6. *Let R be a complete Noetherian normal local domain. Suppose that B is a faithfully flat R-algebra. Then B is Noetherian if and only if it is finitely generated over R.*

We now introduce some more notation to be used for the rest of this section. Choose an element T of $B \otimes_R K$ such that $B \otimes_R K = K[T]$ (T need not be in B). Note that $B \subset B \otimes_R K = K[T]$. For $n \geq -1$, set

$$V_n := \{g \in K[T] \mid \deg_T(g) \leq n\} \quad \text{and} \quad B_n := V_n \cap B. \tag{3.3}$$

Let

$$\mathrm{Gr}(B) = \bigoplus_{n\geq 0} B_n/B_{n-1} \subset \bigoplus_{n\geq 0} V_n/V_{n-1}. \tag{3.4}$$

Note that $\mathrm{Gr}(B)$ is independent of the choice of T. In fact, we observe the following:

Remark 3.7. (1) $B = \bigcup_{n\geq 0} B_n$.

(2) The K-vector space V_n and the R-module B_n, and hence the graded K-algebra $\bigoplus_{n\geq 0} V_n/V_{n-1}$ and the graded R-algebra $\mathrm{Gr}(B)$ are independent of the choice of T.

(3) Given T, if W denotes the image of T in V_1/V_0 then $\bigoplus_{n\geq 0} V_n/V_{n-1} = K[W]$ and hence $\mathrm{Gr}(B) \subset K[W]$ as graded R-algebras.

(4) For $T \in B$, we have $R[W] \subset \mathrm{Gr}(B) \subset K[W]$ as graded R-algebras and hence for every $n \geq 0$, $RW^n \subset B_n/B_{n-1} \subset KW^n$. Moreover, $R[W] = \mathrm{Gr}(B)$ if and only if $B = R[T]$.

(5) $(B_n/B_{n-1})_P \cong R_P$ for every height one prime ideal P of R.

We shall now relate faithful flatness of B with that of $\mathrm{Gr}(B)$.

Lemma 3.8. *If $\mathrm{Gr}(B)$ is flat over R then B is faithfully flat over R.*

Proof. By Remark 3.7(1), it is enough to show that for each $n \geq 0$, B_n is faithfully flat over R. Since $\mathrm{Gr}(B)$ is R-flat, B_n/B_{n-1} is R-flat for every $n \geq 0$. Moreover, we have the short exact sequence

$$0 \to B_{n-1} \to B_n \xrightarrow{\rho_n} B_n/B_{n-1} \to 0 \tag{3.5}$$

where ρ_n is the projection map. Since $B_0 = R$, flatness of B_1/B_0 implies B_1 is faithfully flat. This in turn implies that B_2 is faithfully flat. Repeating this argument we see that B_n is faithfully flat for every $n \geq 0$. Hence B is faithfully flat over R. □

Remark 3.9. We can define (3.3) and (3.4) for any arbitrary integral domain R (not necessarily Noetherian normal) and any integral domain B containing R (not necessary semi-faithfully flat locally $\mathbb{A}^1$ in codimension-one) such that $B \otimes_R K = K[T]$. It is easy to see that (1)–(4) of Remark 3.7 and Lemma 3.8 hold in this more general setup and that (5) of Remark 3.7 also holds when B is locally $\mathbb{A}^1$ in codimension-one.

We shall now see that the converse of Lemma 3.8 holds in our setup (R is a Noetherian normal domain and B is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one). This result was proved in [2, Corollary 3.8] under the additional hypothesis that R is an analytically irreducible local domain, i.e., the completion of R is an integral domain.

Proposition 3.10. *Suppose that B is faithfully flat over R. Then:*

(1) B_n is flat over R, for every $n \geq 0$.

(2) $JB_n \cap B_{n-1} = JB_{n-1}$, for every ideal J of R.

(3) $\mathrm{Gr}(B)$ is faithfully flat over R.

(4) $\mathrm{Gr}(B) = R[L]$, where $L = B_1/R$.

(5) $B = R[B_1]$.

Proof. (1) To show that B_n is flat over R, it is enough to show that given $\sum_i a_i x_i = 0$, with $a_i \in R$ and $x_i \in B_n$, there exist $c_{ij} \in R$ and $y_j \in B_n$ such that $\sum_i a_i c_{ij} = 0$ for each j and $x_i = \sum_j c_{ij} y_j$ for each i.

Since B is faithfully flat over R, there exist $c_{ij} \in R$ and $z_j \in B$ such that $\sum_i a_i c_{ij} = 0$ for each j and $x_i = \sum_j c_{ij} z_j$ for each i. By (3.2), we have $B = \varinjlim_{\Gamma\in\Sigma_0} B_\Gamma$, and hence we can choose Γ such that $x_i,\ z_j \in B_\Gamma$ for each i, j. Now by (3.1), B_Γ has a graded structure: $B_\Gamma = \bigoplus_{r\geq 0} ({I_\Gamma}^r)^{-1} (X - c_\Gamma)^r$. Note that

$$B_\Gamma(n) := B_n \cap B_\Gamma = \{g \in B_\Gamma \mid \deg_X(g) \leq n\} = \bigoplus_{0\leq r\leq n} ({I_\Gamma}^r)^{-1} (X - c_\Gamma)^r.$$

Hence, $x_i \in B_n \cap B_\Gamma = B_\Gamma(n)$. Let $z_j = y_j + w_j$, where

$$y_j \in \bigoplus_{0\leq r\leq n} ({I_\Gamma}^r)^{-1} (X - c_\Gamma)^r \quad \text{and} \quad w_j \in \bigoplus_{t\geq n+1} ({I_\Gamma}^t)^{-1} (X - c_\Gamma)^t.$$

Now it is easy to see that the equality $x_i = \sum_j c_{ij} z_j$ implies $x_i = \sum_j c_{ij} y_j$. Thus B_n is flat over R for every $n \geq 0$.

(2) Since $B_\Gamma(n) = B_n \cap B_\Gamma = \bigoplus_{0\leq r\leq n} ({I_\Gamma}^r)^{-1} (X - c_\Gamma)^r$, for any ideal J of R, $JB_\Gamma(n) \cap B_\Gamma(n-1) = JB_\Gamma(n-1)$. Therefore, since B is a direct limit of B_Γ, it follows that $JB_n \cap B_{n-1} = JB_{n-1}$.

(3) Since by (1) B_n is R-flat and by (2) for every ideal J of R we have $JB_n \cap B_{n-1} = JB_{n-1}$, the short exact sequence

$$0 \to B_{n-1} \to B_n \xrightarrow{\rho_n} B_n/B_{n-1} \to 0 \tag{3.6}$$

shows that $\mathrm{Tor}_1^R(B_n/B_{n-1}, R/J) = 0$ for every ideal J of R and for every $n \geq 0$. Therefore B_n/B_{n-1} is R-flat (of rank one) for every n and hence $\mathrm{Gr}(B) = \bigoplus_{n\geq 0} B_n/B_{n-1}$ is R-flat. Since R is a direct summand of $\mathrm{Gr}(B)$, it follows that $\mathrm{Gr}(B)$ is faithfully flat over R.

(4) Set $L(n) := B_n/B_{n-1}$. Then $L^n \subseteq L(n)$. Since $B_P = R_P[X_P]$ for every prime ideal $P \in \Delta$, we have $\mathrm{Gr}(B)_P = R_P[Y_P]$, where Y_P denotes the image of X_P in V_1/V_0 and $L_P = R_P Y_P$. Therefore, $(L^n)_P = R_P {Y_P}^n = L(n)_P$. By (3), $L(n)$ and L^n are flat R-modules of rank one. Hence, by Lemma 2.4, $L^n = L(n)$ for $n \geq 0$. Thus $\mathrm{Gr}(B) = R[L]$.

(5) Since $\mathrm{Gr}(B) = R[L]$ by (4), it is now easy to see that $B = R[B_1]$. □

The following result was proved in [2, Theorem 3.10] under the additional hypothesis that R is a local domain which is analytically irreducible. In view of Proposition 3.10, we now show that the hypothesis "analytically irreducible" can be dropped.

Corollary 3.11. *Suppose that R is local with maximal ideal $\mathpzc{m}$ and B is faithfully flat over R. Then either B is $\mathbb{A}^1$ over R or $R/\mathpzc{m} = B/\mathpzc{m}B$.*

Proof. By Proposition 3.10(3)–(4), L is a flat R-module of rank one and hence by a result of Vasconcelos [6, Theorem 3.1], either $L \cong R$ or $L = \mathpzc{m}L$.

If $\mathpzc{m}L = L$ then $B_1 = R + \mathpzc{m}B_1$. Since $B = R[B_1]$ by Proposition 3.10(5), it follows that $B = R + \mathpzc{m}B$. Hence $R/\mathpzc{m} = B/\mathpzc{m}B$.

If $L \cong R$ then choose $T \in B_1$ such that $L = RW$, where W is the image of T in $L = B_1/R \subseteq V_1/K$. This shows that $R[T] \subseteq B \subseteq K[T]$ and $\mathrm{Gr}(B) = R[W]$. Therefore $R[T] = B$ (cf. Remark 3.7(4)). □

Remark 3.12. Suppose that B is a faithfully flat R-algebra.

(1) As a consequence of Corollary 3.11, we see that $\mathcal{P}B \in \mathrm{Spec}\ B$ for every prime ideal $\mathcal{P}$ of R. In fact, for each prime ideal $\mathcal{P}$ of R, either $B_{\mathcal{P}}$ is $\mathbb{A}^1$ over $R_{\mathcal{P}}$ or the fibre ring $A \otimes_R k(\mathcal{P}) = k(\mathcal{P})$ which implies that $\mathcal{P}B_{\mathcal{P}}$ is a prime ideal of $B_{\mathcal{P}}$ and hence $\mathcal{P}B$ is a prime ideal of B because $B/\mathcal{P}B \hookrightarrow B_{\mathcal{P}}/\mathcal{P}B_{\mathcal{P}}$ by flatness of B over R.

(2) If R is factorial and B is a Krull domain, then B is factorial. Indeed, by (1), every prime element of R remains a prime element of B. Let S be the multiplicative closed set generated by all prime elements of R. Then $S^{-1}R = K$, and hence $S^{-1}B = K^{[1]}$, a factorial domain. Hence B is factorial by Nagata's criterion [4, Corollary 7.3].

For an integer $n \geq 0$, recall the notation

$$L(n) := B_n/B_{n-1}. \tag{3.7}$$

Let Y denote the image of generic variable $X \in B$ in V_1/V_0 and $M_n := \{\lambda \in K \mid \lambda Y^n \in L(n)\}$. Then, since $RY^n \subseteq L(n) \subseteq KY^n$ (cf. Remark 3.7(4)), $R \subseteq M_n \subseteq K$. In fact

$$M_n = \varinjlim_{\Gamma\in\Sigma_0} ({I_\Gamma}^n)^{-1} \left(= \bigcup_{\Gamma\in\Sigma_0} ({I_\Gamma}^n)^{-1} \right) \tag{3.8}$$

and $L(n) = M_n Y^n$ (see [2, Remark 2.12]). Set $M := M_1$. If B is faithfully flat over R then by Proposition 3.10(3)–(4), it follows that M_n is flat over R and $M_n = M^n$ for $n \geq 1$ and hence $\mathrm{Gr}(B) \cong Sym_R(M) = T_R(M)$.

Corollary 3.13. *Suppose that B is a faithfully flat R-algebra. Set $M = M_1$. Then the R-subalgebra $R[M]$ of K is flat over R.*

Proof. By Proposition 3.10(3), $\mathrm{Gr}(B)$ is flat over R and so $L = B_1/R$ is flat over R which implies that M is flat over R since $MY = L$. By Remark 3.7(4), $RY \subseteq L \subseteq KY$ and hence, $R \subseteq M \subseteq K$. Thus, by Lemma 2.2, $R[M]$ is flat over R. □

Lemma 3.14. *Let $\Delta_1 = \{P \in \Delta \mid R_P[X] = B_P\}$ and let $R' = \bigcap_{P\in\Delta_1} R_P$. If B is a faithfully flat R-algebra, then R' has the following properties:*

(1) *R' is flat over R.*
(2) *R' is a Noetherian normal domain.*

Proof. (1) By Corollary 3.13, it is enough to show that $R[M] = R'$. Note that $\Delta_1 = \Delta \setminus \Delta_0$. By (3.8), we see that $R \subseteq M$ and $R_P = M_P$ if and only if $P \in \Delta_1$. Therefore $R[M]_P = R_P$ if $P \in \Delta_1$ and $R[M]_P = K$ for $P \in \Delta_0$. Since $R[M]$ is R-flat, we have $R[M] = \bigcap_{P\in\Delta} R[M]_P$ by Lemma 2.4. Hence $R' = R[M]$.

(2) R' is normal because it is given by intersection of normal domains. By Lemma 2.3 and (1), it follows that R' is Noetherian. □

4. Locally quasi $\mathbb{A}^*$ algebras in codimension-one I: basic concepts and results; Theorem B

We first prove an elementary result on quasi $\mathbb{A}^*$ algebras over an integral domain.

Lemma 4.1. *Let R be an integral domain with quotient field K and A be an R-algebra which is quasi $\mathbb{A}^*$ over R. Let $T \in A \otimes_R K$ be such that $A \otimes_R K = K[T, T^{-1}]$ and $B = A \cap K[T]$. Then there exist $W \in B$, $a' \in R \setminus 0$, $b' \in R$ and $f = a'W + b'(\in B)$ such that $(a', b')R = R$, $B = R[W]$ and $A = R[W, f^{-1}]$. As a consequence,*

$$R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] = K[T, T^{-1}]$$

and

$$A \cap KT^n = Rf^n \quad \text{and} \quad A \cap KT^{-n} = Rf^{-n} \quad \forall\, n \geq 0.$$

Proof. Since A is a quasi $\mathbb{A}^*$ R-algebra, there exists $X \in A$, transcendental over R, such that $A = R[X, (aX + b)^{-1}]$ for some $a \in R \setminus 0$, $b \in R$ satisfying $(a, b)R = R$. Let $g = aX + b$. Then $K[T, T^{-1}] = A \otimes_R K = K[g, g^{-1}]$. Hence, either $K[g] = K[T]$ or $K[g] = K[T^{-1}]$.

Suppose $K[g] = K[T]$. Then $K[T](= K[g]) = K[X]$ and $R[X] \subseteq B \subseteq R[X, g^{-1}] = A$. Since $(a, b)R = R$, g is a prime element of $R[X]$ which implies $gK[X] \cap R[X] = gR[X]$. Thus $gB \cap R[X] = gR[X]$ and since $R[X, g^{-1}](= A) = B[g^{-1}]$, we have $B = R[X]$. Set $W := X$ and $f := g$. Then $A = R[W, f^{-1}]$ and $B = R[W]$.

Now suppose $K[g] = K[T^{-1}]$. Since $aR+bR = R$, there exist $c, d \in R$ such that $ad-bc = 1$. Set $W := (cX+d)/(aX+b)$ and $f := aW - c$. Then $f = 1/(aX + b) = 1/g$, $X = (d - bW)/f$ and $K[W] = K[f] = K[T]$. Hence $A(= R[X, g^{-1}]) = R[W, f^{-1}]$ and arguing as before, we see that $B = R[W]$.

Now since $A = R[W, f^{-1}]$ and f is linear in W, we have $R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] = K[T, T^{-1}]$ and hence $f = \lambda T$ for some $\lambda \in K^*$. Since $R \subset A \cap K \subset A \cap K[T] = B = R[W]$, it follows that $A \cap K = R$. If $\mu \in K$ be such that $\mu f^n \in A$, then as $f \in A^*$, we have $\mu \in (A \cap K =)R$. Therefore, $A \cap KT^n = Rf^n$ and $A \cap KT^{-n} = Rf^{-n}$ for $n \geq 0$. □

Let R be an integral domain with quotient field K and A a faithfully flat R-algebra such that $A \otimes_R K = K[T, T^{-1}]$. Let $B = A \cap K[T]$. The above lemma shows that if A is locally quasi $\mathbb{A}^*$ in codimension-one over R then B is locally $\mathbb{A}^1$ in codimension-one over R. One would like to know a relation between A and B. For example, one might ask whether A is (in some sense) a localisation of B.

Our first goal in this section is to show that if R is Noetherian and normal then indeed such is the case (Theorem 4.6). We first fix some notation.

Notation

Throughout this section, R will denote a **Noetherian normal domain** *with quotient field K, Δ the set of prime ideals in R of height one and A a* **faithfully flat** *R-algebra which is* **locally quasi $\mathbb{A}^*$ in codimension-one**.

Fix $T \in A \otimes_R K$ such that $A \otimes_R K = K[T, T^{-1}]$. For an integer $n \geq 0$, set

$$C_n := A \cap KT^n.$$
$$D_n := A \cap KT^{-n}.$$

We now prove a technical lemma on the submodules C_n, D_n.

Lemma 4.2. *The canonical maps $C_n \otimes_R A \to C_nA$ and $D_n \otimes_R A \to D_nA$ are isomorphisms of A-modules and $C_nA = A = D_nA$ for each $n \geq 0$. As a consequence, C_n and D_n are finitely generated projective R-modules of rank one.*

Proof. We show that the canonical map $C_n \otimes_R A \to C_nA$ is an isomorphism and $C_nA = A$. The results for D_n will follow in a similar way.

Since $C_n \hookrightarrow KT^n$ and A is R-flat, we have $C_n \otimes_R A \hookrightarrow KT^n \otimes_R A \cong K[T, T^{-1}]$, so that $C_n \otimes_R A$ is a torsion free A-module of rank one. Hence the map $C_n \otimes_R A \to C_nA$ is an isomorphism.

Since $C_n = A \cap KT^n$ and A is R-flat, it follows that $C_n = \bigcap_{P \in \Delta}(C_n)_P$ by Lemma 2.4. Therefore, again by Lemma 2.4, if $x, y \in R$ be such that $(xR : y) = xR$ then $(xC_n : y) = xC_n$. In particular, since R is normal, if (x, y) is an ideal of R of height ≥ 2 then $(xC_n : y) = xC_n$. Since A is R-flat and $C_n \otimes_R A \cong C_nA$, we see that $(xC_nA : y) = xC_nA$ for $x, y \in R$ such that $\text{ht}(x, y) \geq 2$.

For $g \in C_n$, $h \in D_n$, we see that $gh \in A \cap K = R$. Therefore we get an R-linear map $\psi : C_n \otimes_R D_n \to R$ defined by $\psi(g \otimes h) = gh$. Let J_n be the image of ψ.

Since A is locally quasi $\mathbb{A}^*$ in codimension-one, by Lemma 4.1, $(J_n)_P = R_P$ for every $P \in \Delta$. This shows that J_n is an ideal of R of height ≥ 2. Therefore, there exist $x, y \in J_n$ such that $\text{ht}(x, y) \geq 2$.

Since $D_nA \subseteq A$, $J_nA \subseteq C_nA$ and so $(x, y) \subseteq J_nA \subseteq C_nA$. Now since $(xC_nA : y) = xC_nA$, the fact $xy \in xC_nA$ implies that $x \in xC_nA$, hence $C_nA = A$. Since $C_n \otimes_R A \cong C_nA = A$ and A is faithfully flat over R, C_n is a finitely generated projective R-module of rank one. Thus the lemma is proved. □

It follows from Lemma 4.2 that $J_n = C_nD_n$ is a locally principal ideal in R of height at least two. We thus obtain the following corollary.

Corollary 4.3. *Set $B := A \cap K[T]$ and let C be the R-subalgebra $C = \bigoplus_{n \geq 0} C_n$ of B. Then we have:*
(1) *$J_n = R$ for each integer $n \geq 0$.*
(2) *The canonical map $\theta_n : C_1 \otimes C_1 \otimes \cdots \otimes C_1 \to C_n$ is an isomorphism of R-modules.*
(3) *$C = Sym_R(C_1)$ as R-algebras.*

Proof. (1) Since C_n and D_n are finitely generated projective R-modules of rank one, so is $C_n \otimes_R D_n$. Therefore the surjective map $\psi : C_n \otimes_R D_n \to J_n$ is an isomorphism. Thus J_n is R-projective of rank one, i.e., an invertible ideal of R. Since $\mathrm{ht}(J_n) \geq 2$, we see that $J_n = R$.

(2) For the sake of simplicity we denote $\theta_n(C_1 \otimes C_1 \otimes \cdots \otimes C_1)$ (n-times) by $C(n)$. Since A is locally quasi $\mathbb{A}^*$ in codimension-one, by Lemma 4.1, for every $P \in \Delta$, we have $(C_1)_P = R_P f_P$ for some $f_P \in A_P$ and $(C_n)_P = R_P {f_P}^n$ for every $n \geq 0$. Thus $C(n)_P = (C_n)_P$ for every $P \in \Delta$. This implies that θ_n is injective. Now using the fact that $C(n)$ and C_n are projective R-modules, by Lemma 2.4, we have $C(n) = \bigcap_{P \in \Delta} C(n)_P$ and $C_n = \bigcap_{P \in \Delta} (C_n)_P$ and hence $C(n) = C_n$. Thus θ_n is an isomorphism for every $n \geq 0$.

(3) Follows from (2). □

Corollary 4.4. *The following statements hold:*
(1) *$B (= A \cap K[T])$ is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one.*
(2) *$R^* \subsetneqq A^*$ if and only if there exists $n > 0$ such that C_n is free.*
(3) *If $C_1 = Rf$, then $R[f] \subseteq B \subseteq K[f] = K[T]$, $A = B[f^{-1}]$ and $B \cap TK[T] = fB$ and hence $fB \in \mathrm{Spec}\, B$.*

Proof. (1) By Lemma 4.1, it is enough to show that B is semi-faithfully flat over R. Since A is faithfully flat over R, by Lemma 2.4, we have

$$B = A \cap K[T] = \left(\bigcap_{P \in \Delta} A_P \right) \cap K[T] = \bigcap_{P \in \Delta} (A_P \cap K[T]) = \bigcap_{P \in \Delta} B_P$$

and

$$IB \cap R \subseteq IA \cap R = I, \quad \text{for any ideal } I \text{ of } R.$$

Thus B is semi-faithfully flat over R.

(2) Suppose that $C_n = Rh$ for some $n > 0$ and $h \in A$. Since $J_n = R$ by Corollary 4.3(1), it follows that $D_n = Rh^{-1}$. Hence $h \in A^* \setminus R^*$. Conversely, suppose that $R^* \subsetneqq A^*$ and let $h \in A^* \setminus R^*$. Since $A \hookrightarrow A \otimes_R K = K[T, T^{-1}]$, $h \in KT^n$ for some $n \in \mathbb{Z}$. Replacing h by h^{-1} if necessary, we may assume that $h \in C_n (= A \cap KT^n)$ and $h^{-1} \in D_n$ for some integer $n > 0$. Note that $Kh = KT^n$ and $Kh^{-1} = KT^{-n}$. We now show that $C_n = Rh$. Let $g \in C_n$, then $g = \lambda h$ for some $\lambda \in K$. Hence $gh^{-1} = \lambda \in A \cap K = R$. Thus $C_n = Rh$.

(3) Since $Rf = C_1 \subseteq KT$, $K[T] = K[f]$ and $TK[T] = fK[f]$. Moreover, by (2), $f^{-1} \in A$ and hence $R[f, f^{-1}] \subseteq A \subseteq K[f, f^{-1}] = K[T, T^{-1}]$. Hence, as $K[T] = K[f]$, $R[f] \subseteq B \subseteq K[f]$. If $g \in A$, then there exists $k \geq 0$ such that $f^k g \in A \cap K[f] (= B)$ and hence $A = B[f^{-1}]$.

Let $h \in fK[f] \cap B$. Then $h \in A$ and, since $f \in A^*$, we have $h/f \in A \cap K[f] = B$. Thus $TK[T] \cap B (= fK[f] \cap B) = fB$. □

The following result, on factorial domain, is the quasi $\mathbb{A}^*$ analogue of Theorem 3.5.

Corollary 4.5. *Suppose that R is a factorial domain. Then A is a direct limit of quasi $\mathbb{A}^*$ algebras.*

Proof. By Lemma 4.2, C_1 is finitely generated projective R-module and since R is factorial, we have $C_1 = Rf$ for some $f \in C_1$. Hence, by Corollary 4.4, B is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one such that f is a generic variable of B and $A = B[f^{-1}]$. By Theorem 3.5, B is a direct limit of polynomial algebras in one variable over R and hence, $A (= B[f^{-1}])$ is a direct limit of quasi $\mathbb{A}^*$ algebras over R. □

The following theorem shows that A is a *localisation* of B.

Theorem 4.6. *Let R be a Noetherian normal domain with quotient field K and A a faithfully flat R-algebra such that $A \otimes_R K = K[T, T^{-1}]$ for some T transcendental over K. Suppose that A is locally quasi $\mathbb{A}^*$ in codimension-one over R. Let $C_1 = A \cap KT$, $B = A \cap K[T]$ and $Q = B \cap TK[T]$. Then $Q = C_1 B$. As a consequence, Q is an invertible ideal of B and $A = B[Q^{-1}]$.*

Proof. It is easy to see that $C_1 = A \cap KT \subset A \cap TK[T] = B \cap TK[T] = Q$. Therefore $C_1 B \subseteq Q$. Let m be a maximal ideal of R and $S = R \setminus m$. By Lemma 4.2, C_1 is a finitely generated projective R-module of rank one, hence $S^{-1}C_1$ is a free $S^{-1}R$-module of rank one and so $S^{-1}C_1 = R_m f_m$ for some $f_m \in S^{-1}C_1$. Therefore, by Corollary 4.4(3), $C_1 B_m = f_m B_m = QB_m$ and $A_m = B_m[{f_m}^{-1}] = B_m[(QB_m)^{-1}]$. Hence $C_1 B = Q$ and $A = B[Q^{-1}]$. □

Corollary 4.7. *The following statements hold for the ring A:*
(1) *If B is a Noetherian ring, then so is A.*
(2) *If B is a finitely generated R-algebra, then so is A.*

Proof. Follows from the fact that $A = B[Q^{-1}]$ where Q is an invertible ideal of B. □

In Section 5, we shall prove the converse of Corollary 4.7 (cf. Theorems 5.6 and 5.7).

In the special case of A being locally $\mathbb{A}^*$ in codimension-one over R, we have Theorem B which shows that A is always finitely generated over R. In fact its proof shows that B is finitely generated over R.

Theorem 4.8. *Let R be a Noetherian normal domain and A a faithfully flat R-algebra such that A is locally $\mathbb{A}^*$ in codimension-one. Then $A \cong \bigoplus_{n\in\mathbb{Z}} I^n$ for an invertible ideal I of R. In particular, A is finitely generated over R.*

Proof. Let K be the quotient field of R and $A \otimes_R K = K[T, T^{-1}]$ for some T transcendental over K. Let $B = A \cap K[T]$, $C_n = A \cap KT^n$ and $C = \bigoplus_{n\geq 0} C_n \subseteq B$. Now $A = B[(C_1B)^{-1}]$ by Theorem 4.6 and $C = Sym_R(C_1)$ by Lemma 4.3(2), with C_1 being isomorphic to an invertible ideal of R by Lemma 4.2. Thus it suffices to show that $C = B$.

Since C is flat over R, $C = \bigcap_{P\in\Delta} C_P$ by Lemma 2.4. By Corollary 4.4(1), $B = \bigcap_{P\in\Delta} B_P$. Therefore it is enough to prove that $C_P = B_P$ for every $P \in \Delta$.

Let $P \in \Delta$. Since A is locally quasi $\mathbb{A}^*$ in codimension-one over R, $(C_1)_P = R_Pf$ for some $f \in (C_1)_P$ by Lemma 4.1. Hence $C_P = R_P[f] \subseteq B_P \subseteq K[f] = K[T]$ and $fK[T] = TK[T]$. By Theorem 4.6, $B_P \cap TK[T] = Q_P = fB_P$, hence $fB_P \cap C_P (= TK[T] \cap C_P) = fC_P$ and $R_P[f, f^{-1}] \subseteq B_P[f^{-1}] = A_P$. Therefore, to show that $C_P = B_P$, it is enough to show that $C_P[f^{-1}] = B_P[f^{-1}]$.

Since A is locally $\mathbb{A}^*$ in codimension-one, $A_P = R_P[W, W^{-1}]$ for some W transcendental over R_P. Thus $R_P[f, f^{-1}] \subseteq R_P[W, W^{-1}] \subseteq K[f, f^{-1}]$. Therefore, $C_P[f^{-1}] = R_P[f, f^{-1}] = R_P[W, W^{-1}] = A_P = B_P[f^{-1}]$.

Thus the result follows. □

Remark 4.9. (1) Our proofs show that in all results of this section the hypothesis "R is a Noetherian normal domain" may be replaced by the weaker hypothesis "R is a Krull domain".

(2) From Theorem 4.8, one can deduce that the hypothesis of finite generation on A in Theorem 3.4, Corollary 3.9 and Theorem 3.11 of [1] can be dropped. (There is an error in Example 3.6 of [1].)

5. Locally quasi $\mathbb{A}^*$ algebras in codimension-one II: Theorem A

As in Section 4, *R is a* **Noetherian normal domain** *with quotient field K, A a* **faithfully flat** *R-algebra which is* **locally quasi $\mathbb{A}^*$ algebra in codimension-one**, *$T \in A \otimes_R K$ is such that $A \otimes_R K = K[T, T^{-1}]$, $B = A \cap K[T]$ and $Q = B \cap TK[T]$.* We have seen (Theorem 4.6) that Q is an invertible ideal of B and $A = B[Q^{-1}]$. Hence, if B is Noetherian (respectively finitely generated over R) then so is A. In this section, we prove a converse of this result (Theorems 5.6 and 5.7): we show that if A is Noetherian then B is Noetherian and if A is finitely generated over R then so is B. Finally we prove Theorem A.

To begin with we shall show (Theorem 5.2) that, in the above set-up, B is faithfully flat over R. We first give below (Lemma 5.1) a sufficient condition for B to be faithfully flat over R. For this we can assume that R is a local ring with maximal ideal m. Since, by Lemma 4.2, C_1 is a finitely generated projective R-module and R is local, there exists $f \in C_1$ such that $C_1 = Rf$. By Corollary 4.4, B is a semi-faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one over R such that $R[f] \subseteq B \subseteq K[f]$ and $A = B[f^{-1}]$. Note that f is a generic variable of B. Let $\{\Delta_0, I_\Gamma, B_\Gamma, c_\Gamma\}$ be the data associated to the generic variable f of B.

With the hypothesis that R is local and notation as above, we prove

Lemma 5.1. *Suppose $\Delta_0 \neq \emptyset$ and $c_\Gamma \in R^*$ for every $\Gamma \in \Sigma_0$. Then B is faithfully flat over R.*

Proof. B is R-flat if and only if $\mathrm{Tor}_1^R(B, R/I) = 0$ for every ideal I of R. Let I be an ideal of R and $\alpha \in \mathrm{Tor}_1^R(B, R/I)$. We show that $\alpha = 0$. Since $B[f^{-1}](= A)$ is a faithfully flat R-algebra, there exists $r \geq 0$ such that $f^r\alpha = 0$ in $\mathrm{Tor}_1^R(B, R/I)$. By Theorem 3.2, $B = \varinjlim_{\Gamma\in\Sigma_0} B_\Gamma$ and hence

$$\mathrm{Tor}_1^R(B, R/I) = \mathrm{Tor}_1^R(\varinjlim_{\Gamma\in\Sigma_0} B_\Gamma, R/I) = \varinjlim_{\Gamma\in\Sigma_0} \mathrm{Tor}_1^R(B_\Gamma, R/I).$$

Thus, there exists $\Gamma \in \Sigma_0$ such that $\alpha \in \mathrm{Tor}_1^R(B_\Gamma, R/I)$ and $f^r\alpha = 0$ in $\mathrm{Tor}_1^R(B_\Gamma, R/I)$. Again by Theorem 3.2, B_Γ is graded R-algebra:

$$B_\Gamma = \bigoplus_{n\geq 0} E_n \tag{5.1}$$

where $E_n = ({I_\Gamma}^n)^{-1} (f - c_\Gamma)^n$. Therefore,

$$\mathrm{Tor}_1^R(B_\Gamma, R/I) = \bigoplus_{n\geq 0} \mathrm{Tor}_1^R(E_n, R/I)$$

is a graded B_Γ-module. We write $\alpha \in \mathrm{Tor}_1^R(B_\Gamma, R/I)$ as

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_t \quad \text{with } \alpha_i \in \mathrm{Tor}_1^R(E_i, R/I), 0 \leq i \leq t.$$

Note that $x_\Gamma := f - c_\Gamma$ is a homogeneous element of degree one in B_Γ and by hypothesis $c_\Gamma \in R^*$. Now $f^r\alpha = 0$ implies

$${c_\Gamma}^r\alpha_0 = 0, \qquad {c_\Gamma}^r\alpha_1 + r{c_\Gamma}^{r-1}\alpha_0 x_\Gamma = 0, \qquad \ldots, \qquad \alpha_t {x_\Gamma}^r = 0$$

and since $c_\Gamma \in R^*$, we have

$$\alpha_0 = \alpha_1 = \cdots = \alpha_t = 0$$

showing that $\alpha = 0$ in $\mathrm{Tor}_1^R(B_\Gamma, R/I)$. Thus B is R-flat. Since, $B \subseteq A$ and A is faithfully flat over R, B is faithfully flat over R. □

We now prove faithful flatness of B by showing that either B satisfies the hypothesis of Lemma 5.1 or $B = R^{[1]}$.

Theorem 5.2. *Let R be a Noetherian normal domain with quotient field K and A a faithfully flat R-algebra such that $A \otimes_R K = K[T, T^{-1}]$ for some T transcendental over K. Suppose that A is locally quasi $\mathbb{A}^*$ in codimension-one over R. Then $B = A \cap K[T]$ is a faithfully flat R-algebra.*

Proof. Since faithful flatness is a local property, we assume R to be a local domain. We prove faithful flatness of B by induction on the dimension of the ring R.

If $\dim R = 1$, then there is nothing to prove since A is a faithfully flat locally quasi $\mathbb{A}^*$ algebra in codimension-one over R and hence $B = R^{[1]}$ by Lemma 4.1.

Now consider $\dim R > 1$. Let the notation be as before Lemma 5.1. If $\Delta_0 = \emptyset$, then $B = R[f]$ and hence a faithfully flat R-module. Now assume that $\Delta_0 \neq \emptyset$. We show that $c_\Gamma \in R^*$ for every $\Gamma \in \Sigma_0$. Then the result will follow by Lemma 5.1.

Suppose, if possible, that there exists $\Gamma \in \Sigma_0$ such that $c_\Gamma \in m$. We show that this leads to a contradiction. Since $\dim R_{c_\Gamma} < \dim R$, by applying induction hypothesis to local rings of R_{c_Γ}, we get that B_{c_Γ} is faithfully flat over R_{c_Γ}. Also since $B_f (= A)$ is faithfully flat over R, we have B_{f-c_Γ} is flat over R.

Set $\mathcal{I}_\Gamma := (I_\Gamma B : f - c_\Gamma)$. Since $S_\Gamma^{-1}R$ is a PID, $I_\Gamma S_\Gamma^{-1}R$ is a principal ideal, say generated by d_Γ, and hence if $y_\Gamma = (f - c_\Gamma)/d_\Gamma$, then $S_\Gamma^{-1}B = S_\Gamma^{-1}R[y_\Gamma]$ by Theorem 3.2. This shows that $f - c_\Gamma \in I_\Gamma S_\Gamma^{-1}B$. Therefore we have $\mathcal{I}_\Gamma S_\Gamma^{-1}B = S_\Gamma^{-1}B$ and hence there exists $s \in S_\Gamma$ such that $s \in \mathcal{I}_\Gamma$ so that $I_\Gamma \subsetneq \mathcal{I}_\Gamma \cap R$. Further $I_\Gamma B_{f-c_\Gamma} (= \mathcal{I}_\Gamma B_{f-c_\Gamma}) \neq B_{f-c_\Gamma}$. If it were so, then there would exist $\ell \in \mathbb{N}$ such that $(f - c_\Gamma)^\ell \in I_\Gamma B \subseteq mB$. Since $c_\Gamma \in m$, this would imply that $f^\ell \in mB$. But $f^\ell \notin mB$ as $B[f^{-1}]$ is a faithfully flat R-algebra. Thus $B_{f-c_\Gamma}/I_\Gamma B_{f-c_\Gamma}$ is a non-zero ring and the image of s in $B_{f-c_\Gamma}/I_\Gamma B_{f-c_\Gamma}$ is zero. But this contradicts the fact that $B_{f-c_\Gamma}/I_\Gamma B_{f-c_\Gamma}$ is flat over R/I_Γ and the image of s in R/I_Γ is not a zero-divisor. Hence $c_\Gamma \in R^*$ for every $\Gamma \in \Sigma_0$. □

Remark 5.3. (1) The proof of Theorem 5.2 shows that if R is local then either $\Delta_0 = \emptyset$ and $B = R^{[1]}$ or for each $\Gamma \in \Sigma_0$, $c_\Gamma \in R^*$. Hence for R which need not be local, if A is a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over R such that $A = B[f^{-1}]$, where $B = A \cap K[f]$, then for each $\Gamma \in \Sigma_0$, $I_\Gamma + Rc_\Gamma = R$, where $\{\Delta_0, I_\Gamma, B_\Gamma, c_\Gamma\}$ is the data associated to the generic variable f of B.

(2) For each $\Gamma \in \Sigma_0$, $f - c_\Gamma \in I_\Gamma B$. In fact, from the proof of Theorem 5.2, we have $f - c_\Gamma \in I_\Gamma {S_\Gamma}^{-1}B \cap B$, where $I_\Gamma {S_\Gamma}^{-1}B \cap B = I_\Gamma B$ since B is faithfully flat over R.

We now prove an analogue of Theorem 3.10 of [2] for fibre rings of algebras which are locally quasi $\mathbb{A}^*$ in codimension-one.

Proposition 5.4. *Let R be a Noetherian normal domain and A a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one. Then $\mathcal{P}A \in \mathrm{Spec}\ A$ for every prime ideal $\mathcal{P}$ of R. In fact, for each prime ideal $\mathcal{P}$ of R, either $A_\mathcal{P}$ is quasi $\mathbb{A}^*$ over $R_\mathcal{P}$ or the fibre ring $A \otimes_R k(\mathcal{P}) = k(\mathcal{P})$.*

Proof. Let $\mathcal{P} \in \mathrm{Spec}\ R$. Since A is faithfully flat over R, to show that $\mathcal{P}A \in \mathrm{Spec}\ A$ it is enough to show that $\mathcal{P}A_\mathcal{P} \in \mathrm{Spec}\ A_\mathcal{P}$. Therefore, we can assume that R is a local domain with maximal ideal m and show that either A is quasi $\mathbb{A}^*$ over R or the fibre ring $A/mA = R/m$.

Since R is local, by Theorem 4.6, there exists $f \in A$ such that $R[f, f^{-1}] \subset A \subset K[f, f^{-1}]$ and $A = B[f^{-1}]$, where $B = A \cap K[f]$. By Theorem 5.2, B is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one over R. Therefore, by Corollary 3.11, B is either $R[T]$ for some $T \in B$ which is transcendental over R or $B/mB = R/m$. If $B = R[T]$ then $A = B[f^{-1}] = R[T, f^{-1}]$ and hence quasi $\mathbb{A}^*$ over R. In the other case, since A is faithfully flat over R, $f \notin mB$ and hence $R/m = B/mB = A/mA$. □

We shall now show that if A is Noetherian then B is Noetherian and if A is finitely generated over R then so is B. For this we make the following reduction.

Remark 5.5. To prove that A is Noetherian (resp. finitely generated over R) implies B is Noetherian (resp. finitely generated over R), it suffices to assume that C_1 is a free R-module of rank one. To see this, first note that, by Lemma 4.2, C_1 is a finitely generated projective R-module of rank one. Hence there exist $a_1, a_2, \ldots, a_r \in R$ and $f_1, f_2 \ldots, f_r \in C_1$ such that $(a_1, \ldots, a_r)R = R$ and $(C_1)_{a_i} = R_{a_i} f_i$ is a free R_{a_i}-module for $1 \le i \le r$. Thus, if B_{a_i} is Noetherian (resp. finitely generated over R) for each i, $1 \le i \le r$, then it will follow that B is Noetherian (resp. finitely generated over R).

Theorem 5.6. *Let R be a Noetherian normal domain with quotient field K and A a Noetherian faithfully flat locally quasi $\mathbb{A}^*$-algebra in codimension-one over R. Let $B = A \cap K[T]$, where $T \in A \otimes_R K$ is such that $A \otimes_R K = K[T, T^{-1}]$. Then B is Noetherian.*

Proof. By Remark 5.5, we may assume that $C_1(= A \cap KT)$ is a free R-module, say $C_1 = Rf$ for some $f \in R$. By Corollary 4.4(3), f is a generic variable of B and $A = B[f^{-1}]$.

Let $S = 1 + fB$. By hypothesis $B[f^{-1}](= A)$ is a Noetherian ring. So B is Noetherian if and only if $S^{-1}B$ is Noetherian. By Theorem 5.2, B is faithfully flat over R and hence $S^{-1}B$ is a flat R-module. Therefore, by Lemma 2.4, $S^{-1}B = \bigcap_{P \in \Delta} (S^{-1}B)_P$, where $(S^{-1}B)_P$ is the localisation of $S^{-1}B$ by the multiplicatively closed set $R \setminus P$. Note that $(S^{-1}B)_P = S^{-1}(B_P)$.

Recall that $\Delta_0(f) = \{P \in \Delta \mid R_P[f] \subsetneqq B_P\}$ and $\{c_\Gamma\}_{\Gamma \in \Sigma_0}$ is the family associated with the generic variable f of B. Then, for $P \in \Delta \setminus \Delta_0$, $B_P = R_P[f]$ and hence $(S^{-1}B)_P = S^{-1}R_P[f]$.

Note that, by Remark 5.3, for $P \in \Delta_0$, $(c_{\{P\}}, P) = R$ and $f - c_{\{P\}} \in PB$ and hence $fB + PB = B$. Therefore $S \cap PB \neq \emptyset$. Moreover, R_P is a local PID with maximal ideal PR_P and $B_P = R_P[W]$ for some $W \in B_P$ which is transcendental over R_P. Putting these facts together we see that $S^{-1}B_P = S^{-1}K[W] = S^{-1}K[f]$.

Thus $S^{-1}B = S^{-1}R'[f]$, where $R' = \bigcap_{P \in \Delta \setminus \Delta_0} R_P$. Now by Lemma 3.14(2), R' is Noetherian and hence B is Noetherian. □

Theorem 5.7. *Let R be a Noetherian normal domain and A a finitely generated faithfully flat locally quasi $\mathbb{A}^*$-algebra in codimension-one over R. Let $B = A \cap K[T]$, where $T \in A \otimes_R K$ is such that $A \otimes_R K = K[T, T^{-1}]$. Then B is finitely generated over R. In particular, if R is a local domain then A is quasi $\mathbb{A}^*$ over R.*

Proof. To show that B is finitely generated over R, we may assume, by Remark 5.5, that $C_1 = Rf$ for some $f \in R$. By Corollary 4.4(3), f is a generic variable of B and $A = B[f^{-1}]$. Let $\{\Delta_0, I_\Gamma, B_\Gamma, c_\Gamma\}$ be the data associated to the generic variable f of B.

By Theorem 3.2, $B = \varinjlim_{\Gamma \in \Sigma_0} B_\Gamma$. Since $B[f^{-1}] = \varinjlim_{\Gamma \in \Sigma_0} B_\Gamma[f^{-1}]$ is a finitely generated R-algebra, there exists $\Gamma_0 \in \Sigma_0$ such that $B[f^{-1}] = B_{\Gamma_0}[f^{-1}]$. We now show that $\Gamma_0 = \Delta_0$. Suppose that there exists $P \in (\Delta_0 \setminus \Gamma_0)$. Then $(B_{\Gamma_0})_P = R_P[f] \neq B_P = R_P[X_P]$. Since f is a generic variable, $f = a_p X_P + b_P$, for some $a_P, b_P \in R_P$ with $a_P \in PR_P$. This shows that $(B_{\Gamma_0})_P[f^{-1}] = R_P[f, f^{-1}] \neq B_P[f^{-1}]$ contradicting the fact that $B[f^{-1}] = B_{\Gamma_0}[f^{-1}]$. Thus $\Gamma_0 = \Delta_0$ and hence $B = B_{\Gamma_0}$ which is finitely generated over R by Proposition 3.4.

If R is a local domain then, since $B = B_{\Gamma_0} = \bigoplus_{n \geq 0} ({I_{\Gamma_0}}^n)^{-1} (f - c_{\Gamma_0})^n$ is R-flat (cf. Theorem 5.2) and R is Noetherian, I_{Γ_0} is a finitely generated projective R-module. Thus I_{Γ_0} is a free R-module, say $I_{\Gamma_0} = Rd_{\Gamma_0}$. Hence $B(= B_{\Gamma_0}) = R\left[\frac{f - c_{\Gamma_0}}{d_{\Gamma_0}}\right]$ and $A = B[f^{-1}] = R\left[\frac{f - c_{\Gamma_0}}{d_{\Gamma_0}}, f^{-1}\right]$ is quasi $\mathbb{A}^*$ over R. □

Corollary 5.8. *Suppose there exists $(0 \neq) a \in R$ such that $A[1/a]$ is finitely generated over R. Then A is finitely generated over R.*

Proof. Since $A[1/a]$ is finitely generated over R, by Theorem 5.7, we have that $B[1/a]$ is a finitely generated $R[1/a]$-algebra. Fix a generic variable $X \in B$ and let $\Delta_0 = \{P \in \Delta \mid R_P[X] \subsetneqq B_P\}$. Let $\Delta_1 = \{P \in \Delta_0 \mid a \notin P\}$. Then, since R is Noetherian, $\Delta_0 \setminus \Delta_1$ is a finite set. Since $B[1/a]$ is finitely generated over $R[1/a]$, by Proposition 3.4, Δ_1 is a finite set and hence Δ_0 is finite. Therefore, again by Proposition 3.4, B is finitely generated over R and hence A is finitely generated over R by Corollary 4.7(2). □

We now obtain Theorem A. We state it in two parts: Theorems 5.9 and 5.10.

Theorem 5.9. *Let (R, m) be a Noetherian normal local domain and A a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over R. Then the following are equivalent:*

(i) *A is quasi $\mathbb{A}^*$ over R.*
(ii) *A is a finitely generated R-algebra.*
(iii) *$R/m \subsetneqq A/mA$.*

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are trivial.
(ii) ⇒ (i) follows from Theorem 5.7.
(iii) ⇒ (i) follows from Proposition 5.4. □

The next result, on complete local domain, is the quasi $\mathbb{A}^*$ analogue of Theorem 3.6.

Theorem 5.10. *Let R be a complete local Noetherian normal domain and A be a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over R. Then the following conditions are equivalent:*

(i) *A is Noetherian.*
(ii) *A is finitely generated over R.*
(iii) *A is quasi $\mathbb{A}^*$ over R.*

Proof. Clearly (iii) ⇒ (ii) ⇒ (i). For (i) ⇒ (iii), we have by Theorem 5.6, B is Noetherian. Hence, by Theorem 3.6, $B = R[W]$. Thus the result follows from Theorem 4.6. □

In Section 6, Example 6.3 shows that Theorem 5.10 does not hold if R is not complete.

6. Examples

Let R be a Noetherian *factorial* domain with quotient field K. Let A be a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one, $A \otimes_R K = K[T, T^{-1}]$ and $B = A \cap K[T]$. We have seen (Theorems 4.6 and 5.2) that B is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one, $B \cap TK[T] = Bf$, $A = B[f^{-1}]$ and the prime element f is a generic variable of B.

Therefore it is natural to ask the following:

Question 1. Let R be a Noetherian factorial domain and B a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one. Can we find a generic variable $f \in B$ such that $B[f^{-1}]$ is *faithfully* flat over R?

Note that if $A = B[f^{-1}]$ is faithfully flat over R then A is locally quasi $\mathbb{A}^*$ in codimension-one over R.

We give below an example of a factorial domain B, which is faithfully flat locally $\mathbb{A}^1$ in codimension-one over $k[X, Y]$ (k: field) such that for any generic variable $f \in B$, there exists a maximal ideal n of $k[X, Y]$ such that $f \in nB$ and hence $B[f^{-1}]$ is **not faithfully** flat over R.

Example 6.1. Let k denote the algebraic closure of $\mathbb{Q}$, $R = k[X, Y]$ and Δ the set of all height one prime ideals of R. k, being countable, can be indexed as $k = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$. Let $p_i = Y + \alpha_i X$ and $P_i = p_i R$, $i \geq 1$. Set $\Delta_0 = \{P_i\}_{i\geq 1}$.

Let x_0 be an indeterminate over R. Set $c_0 = 1$,

$$c_n = 1 + p_1 + \cdots + p_1 p_2 \cdots p_n$$

and

$$x_n := \frac{x_{n-1} - 1}{p_n} = \frac{x_0 - c_{n-1}}{p_1 p_2 \cdots p_n}$$

for $n \geq 1$. Now set

$$B := R[x_0, x_1, x_2, \ldots, x_n, \ldots],$$

a direct limit of the R-algebras $R[x_n] (= R^{[1]})$.

Clearly, $B_P = {R_P}^{[1]}\ \forall P \in \Delta$ and B is faithfully flat over R since it is a direct limit of polynomial algebras over R. Note that $e_{x_0}(P) = 1$ for $P \in \Delta_0$ and $e_{x_0}(P) = 0$ for $P \in \Delta \setminus \Delta_0$ (see Section 3 for definition of e_{x_0}). Since $x_{n+1} \notin R[x_n]$, B is not finitely generated over R. We show that:

(1) B is a factorial domain.

(2) For every generic variable f of B, there exists a maximal ideal n of R such that $f \in nB$.

(1) By Remark 3.12(2), it suffices to show that B is a Krull domain. Since B is faithfully flat over R, by Lemma 2.4, we have

$$B = \bigcap_{P\in\Delta} B_P = \left(\bigcap_{P\in\Delta} V_P\right) \cap K[x_0],$$

where K is the quotient field of R and $V_P := B_{PB}$ is a discrete valuation ring of $K(x_0)$ for each $P \in \Delta$. Thus, since $K[x_0]$ is a Krull domain, B is an intersection of discrete valuation rings. Hence, to show that B is a Krull domain, it suffices to show that, for any $h \in B$, there exist only finitely many $P \in \Delta$ such that $h \in PB$. Now

$$B = \bigcap_{P\in\Delta} B_P = \left(\bigcap_{P\in\Delta_0} B_P\right) \cap \left(\bigcap_{P\in\Delta\setminus\Delta_0} B_P\right) = \left(\bigcap_{P\in\Delta_0} B_P\right) \cap R'[x_0],$$

where $R' = \bigcap_{P\in\Delta\setminus\Delta_0} R_P$. R' is a Noetherian normal domain by Lemma 3.14. Hence, it suffices to show that for any $h \in B$, there exists an integer m such that $h \notin P_n B\ \forall\, n \geq m$. Let $h \in B = R[x_0, x_1, \ldots]$. Multiplying h by a suitable element from R, we may assume $h \in R[x_0]$. Let

$$h = \phi(x_0) = a_0 + a_1 x_0 + \cdots + a_r {x_0}^r, \quad a_i \in R.$$

Let d_i denote the degree of a_i in X and Y. Choose $m > 1$ large enough so that

(i) for $n \geq m$, the leading coefficient $a_r \notin p_n R$, and

(ii) $m > 1 + d_i\ \forall\, i, 0 \leq i \leq r - 1$.

We show that $h \notin P_n B$ for $n \geq m$. If $r = 0$, then clearly $h \notin P_n B$ for $n \geq m$ since $R/P_nR \hookrightarrow B/P_nB$ and $a_r \notin p_n R$. Suppose that $r > 0$ and $h \in P_n B$ for some $n \geq m$. Then $\phi(c_{n-1}) \in P_n R$ since $R/P_nR \hookrightarrow B/P_nB$ and $x_0 = c_{n-1} + p_1 \cdots p_n x_n$. Now, identifying R/P_nR with $k[X]$, the image of the product $p_1 \cdots p_{n-1}$ in R/P_nR is λX^{n-1}, where $\lambda = (\alpha_1 - \alpha_n) \cdots (\alpha_{n-1} - \alpha_n) \in k^*$ and hence the image of c_{n-1} in R/P_nR is a polynomial of degree $n - 1$. Therefore, since $m - 1 > d_i\ \forall\ i \leq r - 1$, the image of $\phi(c_{n-1})$ in R/P_nR is a polynomial in $R/P_nR (= k[X])$ of degree at least $(n - 1)r > 1$ contradicting the fact that $\phi(c_{n-1}) \in P_n R$. Thus $h \in PB$ for only finitely many $P \in \Delta$.

(2) Let f be a generic variable of B. Then $f = a + bx_n$ for some $a, b \in R$ and $n \geq 0$ and hence for $\ell > 0$,

$$f = a + b(1 + p_{n+1} + \cdots + p_{n+1} \ldots p_{n+\ell} + p_{n+1} \ldots p_{n+\ell+1} x_{n+\ell+1}).$$

Choose ℓ such that, $\ell > \max\{\deg a, 0\}$ and p_j does not divide b for $j = \ell + n + 1$. It is then easy to see that the image of the element

$$f_\ell := a + b(1 + p_{n+1} + \cdots + p_{n+1} \ldots p_{\ell+n})$$

in $R/P_{\ell+n+1}$ (identified with $k[X]$) is a non-constant polynomial. In fact, the X-degree of the image of f_ℓ in $R/P_{\ell+n+1}$ $(= k[X])$ is at least $\ell(> 0)$ and hence it has a root in k, say β. Then $f \in \mathfrak{n}B$, where $\mathfrak{n} = (Y + \alpha_{\ell+n+1}X, X - \beta)R$.

It is therefore not possible to obtain a faithfully flat locally quasi $\mathbb{A}^*$ algebra in codimension-one over R by inverting some generic variable of B.

We now show that if R is local or R is a retract of B (equivalently B is a graded R-algebra), then one can indeed find a generic variable f such that $A = B[f^{-1}]$ is a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one.

Lemma 6.2. *Let R be a Noetherian normal domain and B a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one over R. Then there exists a generic variable $f \in B$ such that $A := B[f^{-1}]$ is faithfully flat locally quasi $\mathbb{A}^*$ in codimension-one over R, if either*

(1) *R is local with maximal ideal $\mathfrak{m}$, or*

(2) *R is a retract of B.*

Proof. It is enough to show that there exists a generic variable f such that $B[f^{-1}]$ is faithfully flat over R.

(1) Suppose that R is local with maximal ideal $\mathfrak{m}$. Let X be a generic variable of B. Then taking

$$f = \begin{cases} X - 1 & \text{if } X \in \mathfrak{m}B. \\ X & \text{if } X \notin \mathfrak{m}B. \end{cases}$$

We see that $B[1/f]$ is faithfully flat over R.

(2) Now suppose that R is a retract of B. Let $\theta : B \to R$ be the retraction map. Let X be a generic variable for B, $c = \theta(X)$ and set $f = X - c + 1$. Then $\theta(f) = 1$ and hence $f \notin \mathfrak{m}B$ for any maximal ideal $\mathfrak{m}$ of R. Therefore, $A = B[f^{-1}]$ is faithfully flat over R. □

We now show the existence of a *Noetherian* ring A which is locally quasi $\mathbb{A}^*$ algebra in codimension-one over a Noetherian normal local domain R but which is not finitely generated over R.

Example 6.3. Example 6.2 in [2] gives us an example of a Noetherian ring B which is faithfully flat locally $\mathbb{A}^1$ algebra in codimension-one over a Noetherian normal local domain R but which is not finitely generated over R. By Lemma 6.2, we can get a generic variable f such that $A = B[f^{-1}]$ is a faithfully flat locally quasi $\mathbb{A}^*$ algebra in codimension-one over R. Now A is a Noetherian ring but A is not finitely generated over R by Theorem 5.7.

The following example shows that over a complete local domain, there exists a faithfully flat locally quasi $\mathbb{A}^*$ algebra in codimension-one which is not finitely generated.

Example 6.4. Let $R = \mathbb{Q}[[X, Y]]$ be a Noetherian factorial complete local domain. Let $p_n = X + nY$ and let W be an indeterminate over R. Set $W_n = W/p_n$ for $n \geq 1$ and $B = R[W, W_1, \ldots, W_n, \ldots]$. Then B is a faithfully flat R-algebra which is locally $\mathbb{A}^1$ in codimension-one over R but B is not finitely generated over R. Hence, $A = B[(W - 1)^{-1}]$ is a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one over R (cf. Lemma 6.2) but A is not finitely generated over R by Theorem 5.7.

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Appendix

We now discuss the construction of algebras which are locally quasi $\mathbb{A}^*$ in codimension-one. In [2], a recipe for the construction of an R-algebra B which is semi-faithfully flat and locally $\mathbb{A}^1$ in codimension-one was given; in a sense, the construction characterises all such algebras. Using such a B, and a finitely generated projective R-submodule C_1 of rank one such that $C_1 \subseteq B_1$ and $C_1 \cap R = (0)$, one gets an R-algebra $A = B[(C_1B)^{-1}]$ which is locally quasi $\mathbb{A}^*$-algebra in codimension-one over R. Note that if B is faithfully flat over R, then A is flat over R. However, Example 6.1 shows that it is possible to construct B as above such that $B[(C_1B)^{-1}]$ is not *faithfully* flat for any choice of C_1. This leads to:

Question 2. Let B be a faithfully flat algebra which is locally $\mathbb{A}^1$ in codimension-one over a Noetherian normal domain R. Let C_1 be a finitely generated projective R-module of rank one such that $C_1 \subseteq B_1$ and $C_1 \cap R = (0)$ and let $A = B[(C_1B)^{-1}]$. When is A a *faithfully* flat locally quasi $\mathbb{A}^*$ algebra in codimension-one over R?

In this section we investigate Question 2.

Let the setup be as in Question 2. Set $L = B_1/R$ where B_1 is as in (3.3). Since $C_1 \subseteq B_1$ and $C_1 \cap R = (0)$, C_1 is isomorphic to its image $\overline{C_1}$ in L. Let Δ denote the set of all height one prime ideals of R. For each $P \in \Delta$, we define an integer $e(P)$ by

$$e(P) := \ell_{R_P}(L/\overline{C_1} \otimes_R R_P),$$

the length of the R_P-module $(L/\overline{C_1} \otimes_R R_P)$. Let

$$\Delta_0 := \{P \in \Delta \mid e(P) > 0\} \quad \text{and}$$
$$\Sigma_0 := \text{the set of all finite subsets of } \Delta_0.$$

For $\Gamma \in \Sigma_0$, let

$$I_\Gamma := {P_1}^{(e(P_1))} \cap \cdots \cap {P_n}^{(e(P_n))},$$
$$E_\Gamma := I_\Gamma^{-1}\overline{C_1} \subset L \quad \text{and}$$
$$D_\Gamma := \rho^{-1}(E_\Gamma), \quad \text{where } \rho : B_1 \to B_1/R = L \text{ is a canonical projection.}$$

Lemma A.1. *For each $\Gamma \in \Sigma_0$, there exists a retraction $\theta_\Gamma : D_\Gamma \to R$ and hence $D_\Gamma \cong R \oplus E_\Gamma$.*

Proof. Since C_1 is a finitely generated projective R-module of rank one, $E_\Gamma (= I_\Gamma^{-1}\overline{C_1})$ is a finitely generated torsion-free R-module of rank one. Hence there exists a sequence $x, y \in R$ such that $(E_\Gamma)_x = FR_x$ and $(E_\Gamma)_y = GR_y$ for some $F, G \in E_\Gamma$. Now $F \in GR_y (= (E_\Gamma)_y)$. Hence there exist $a \in R$ and $n \geq 0$ such that

$$F = aG/y^n.$$

Let $f, g \in D_\Gamma$ be such that $\rho(f) = F$ and $\rho(g) = G$. Thus $c := y^n f - ag \in Ker(\rho) = R$. Since, B is faithfully flat over R and $c \in (y^n, a)B \cap R$, we have $c \in (y^n, a)R$. Let $c = y^n c_1 + ac_2$ where $c_1, c_2 \in R$. Then $y^n(f - c_1) = a(g + c_2)$. Let $f' = f - c_1$ and $g' = g + c_2$, so that $y^n f' = ag'$. Then $(D_\Gamma)_x = R_x \oplus R_x f'$ and $(D_\Gamma)_y = R_y \oplus R_y g'$. We now show that we can give an R-linear map from $D_\Gamma \to R$. Let $h \in D_\Gamma$. Then $h = \frac{r_0 + r_1 f'}{x^\ell} = \frac{r_0' + r_1' g'}{y^m}$ for some $r_0, r_1, r_0', r_1' \in R$ and $\ell, m \geq 0$. Now

$$\frac{r_0 + r_1 f'}{x^\ell} = \frac{r_0 + r_1(ag'/y^n)}{x^\ell} = \frac{r_0}{x^\ell} + \frac{ar_1}{x^\ell y^n} g' = \frac{r_0' + r_1' g'}{y^m}.$$

Now $r_0/x^\ell = r_0'/y^m \in R_x \cap R_y$ and hence in R since x, y is a sequence in R. It is easy to see that the map $\theta_\Gamma : D_\Gamma \to R$ defined by $\theta_\Gamma(h) = r_0/x^\ell$ is an R-linear map. Clearly $\theta_\Gamma(r) = r$ for $r \in R$. Hence θ_Γ is an R-retraction. □

Note that, since $R \subset {I_\Gamma}^{-1}$, $R \oplus C_1 \subset D_\Gamma$. Using the canonical retraction $\theta_\Gamma : D_\Gamma \to R$ defined by Lemma A.1, we now define an R-linear automorphism

$$\bar{\theta}_\Gamma : R \oplus C_1 \to R \oplus C_1 \tag{A.1}$$

by $\bar{\theta}_\Gamma(r) = r$ for $r \in R$ and $\bar{\theta}_\Gamma(c) = c - \theta_\Gamma(c)$ for $c \in C_1$. Set

$$N_\Gamma := \bar{\theta}_\Gamma(C_1)$$

and

$$H_\Gamma := \theta_\Gamma(C_1).$$

Note that N_Γ and H_Γ are finitely generated flat R-modules such that $N_\Gamma \hookrightarrow D_\Gamma$ and $H_\Gamma \hookrightarrow R$. In fact $D_\Gamma = R \oplus {I_\Gamma}^{-1} N_\Gamma$ and for each $P \in \Gamma$, $(D_\Gamma)_P = (B_1)_P$ and for $P \notin \Gamma$, $(D_\Gamma)_P = R_P \oplus (N_\Gamma)_P$.

Remark A.2. Set $B_\Gamma := \bigoplus_{n \geq 0} I_\Gamma^{-n} {N_\Gamma}^n$. Then one can show that $B_\Gamma = {S_\Gamma}^{-1} B \cap R_\Gamma[C_1]$, where $S_\Gamma := R \setminus (\bigcup_{P \in \Gamma} P)$ and $B = \varinjlim_{\Gamma \in \Sigma} B_\Gamma$. Moreover if Γ_1 and Γ_2 are finite subsets of Δ_0 with $\Gamma_1 \subseteq \Gamma_2$, then $(\bar{\theta}_{\Gamma_1} - \bar{\theta}_{\Gamma_2})(C_1) = (\theta_{\Gamma_1} - \theta_{\Gamma_2})(C_1) \subseteq I_{\Gamma_1}$. This is an analogue of Theorem 3.2 giving the structure of B in terms of a finitely generated projective R-module C_1 such that $C_1 \subset B_1$ and $C_1 \cap R = (0)$ in place of a generic variable.

Proposition A.3. *With notation as above, the following are equivalent:*
(i) *$A = B[(C_1 B)^{-1}]$ is a faithfully flat R-algebra which is locally quasi $\mathbb{A}^*$ in codimension-one.*
(ii) *$I_\Gamma + H_\Gamma = R$ for each $\Gamma \in \Sigma_0$.*
As a consequence, if $A = B[(C_1 B)^{-1}]$ is a faithfully flat R-algebra then $C_1/I_\Gamma C_1$ is a free R/I_Γ-module for each $\Gamma \in \Sigma_0$.

Proof. (i) ⇒ (ii) follows from Remark 5.3(1) by reducing to the local case.

(ii) ⇒ (i): Let m be a maximal ideal of R. Since C_1 is a finitely generated projective R-module of rank one, $(C_1)_m = R_m f$ for some $f \in C_1$ and for each $\Gamma \in \Sigma_0$, $H_\Gamma R_m = R_m c_\Gamma$ for some $c_\Gamma \in R$. If $I_\Gamma R_m = R_m$ for each $\Gamma \in \Sigma_0$, then $B_m = R_m[f]$ and hence $A_m = B_m[f^{-1}] = R_m[f, f^{-1}]$ is faithfully flat over R_m. Suppose that there exists a $\Gamma \in \Sigma_0$ such that $I_\Gamma R_m \subsetneq R_m$, then $c_\Gamma \in {R_m}^*$ by hypothesis. Since B_m is faithfully flat over R_m, we have $f - c_\Gamma \in I_\Gamma {S_\Gamma}^{-1} B_m \cap B_m = I_\Gamma B_m$ and hence $f \notin mB_m$ as $c_\Gamma \in {R_m}^*$. Thus $A_m = B_m[f^{-1}]$ is faithfully flat over R_m and hence, since faithful flatness is a local property, $A (= B[(C_1 B)^{-1}])$ is faithfully flat over R. □

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