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Cosimplicial versus DG-rings: a version of the Dold–Kan correspondence

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Abstract

The (dual) Dold–Kan correspondence says that there is an equivalence of categories $K: \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\Delta}$ between nonnegatively graded cochain complexes and cosimplicial abelian groups, which is inverse to the normalization functor. We show that the restriction of K to DG-rings can be equipped with an associative product and that the resulting functor $DGR^* \to \mathrm{Rings}^{\Delta}$, although not itself an equivalence, does induce one at the level of homotopy categories. In other words both DGR^* and Rings^{Δ} are Quillen closed model categories and the total left derived functor of K is an equivalence:

$$\mathbb{L}K: \mathrm{Ho}\ DGR^* \xrightarrow{\sim} \mathrm{Ho}\ \mathrm{Rings}^{\Delta}.$$

The dual of this result for chain DG and simplicial rings was obtained independently by Schwede and Shipley, Algebraic and Geometric Topology 3 (2003) 287, through different methods. Our proof is based on a functor $Q: DGR^* \to \mathrm{Rings}^{\Delta}$, naturally homotopy equivalent to K, and which preserves the closed model structure. It also has other interesting applications. For example, we use Q to prove a noncommutative version of the Hochschild–Kostant–Rosenberg and Loday–Quillen theorems. Our version applies to the cyclic module $[n] \mapsto \coprod_R^n S$ that arises from a homomorphism $R \to S$ of not necessarily commutative rings, using the coproduct $\coprod_R$ of associative R-algebras. As another application of the properties of Q, we obtain a simple, braid-free description of a product on the tensor power $S^{\otimes_R^n}$ originally defined by Nuss K-theory 12 (1997) 23, using braids.

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1. Introduction

The (dual) Dold–Kan correspondence is an equivalence between the category $\mathrm{Ch}^{\geqslant 0}$ of nonnegatively graded cochain complexes of abelian groups and the category $\mathfrak{Ab}^{\Delta}$ of cosimplicial abelian groups. This equivalence is defined by a pair of inverse functors

$$N : \mathfrak{Ab}^{\Delta} \leftrightarrows \mathrm{Ch}^{\geqslant 0} : K. \tag{1}$$

Here N is the normalized or Moore complex (see (24) below). The functor K is described in [17], 8.4.4; if $A = (A, d) \in \mathrm{Ch}^{\geqslant 0}$ and $n \geqslant 0$, then

$$K^n A = \bigoplus_{i=0}^{n} \binom{n}{i} A^i \cong \bigoplus_{i=0}^{n} A^i \otimes \Lambda^i \mathbb{Z}^n. \tag{2}$$

If in addition A happens to be a DG-ring, then $K^n A$ can be equipped with a product, namely that coming from the tensor product of rings $A \otimes \Lambda\mathbb{Z}^n$:

$$(a \otimes x)(b \otimes y) = ab \otimes x \wedge y. \tag{3}$$

This product actually makes $[n] \mapsto K^n A$ into a cosimplicial ring (see 5.3). Thus K can be viewed as a functor from DG- to cosimplicial rings:

$$K : DGR^* \to \mathrm{Rings}^{\Delta}, \quad A \mapsto KA. \tag{4}$$

Note that for all n, $K^n A$ is a nilpotent extension of A^0. As there are cosimplicial rings which are not codimensionwise nilpotent extensions of constant cosimplicial rings, $A \mapsto KA$ is not a category equivalence. However we prove (Theorem 9.8) that it induces one upon inverting weak equivalences. Precisely, K carries quasi-isomorphisms to maps inducing an isomorphism at the cohomotopy level, and therefore induces a functor $\mathbb{L}K$ between the localizations $\mathrm{Ho}DGR^*$ and $\mathrm{HoRings}^{\Delta}$ obtained by formally inverting such maps, and we prove that $\mathbb{L}K$ is an equivalence:

$$\mathbb{L}K : \mathrm{Ho}DGR^* \xrightarrow{\sim} \mathrm{HoRings}^{\Delta}. \tag{5}$$

The dual of this result, that is, the equivalence between the homotopy categories of chain DG and simplicial rings, was obtained independently by Schwede and Shipley through different methods (see [15] and also Remark 9.4 below).

To prove (5) we use Quillen's formalism of closed model categories [14]. We consider in each of DGR^* and Rings^{Δ} a closed model structure, in which weak equivalences are as above, fibrations are surjective maps and cofibrations are appropriately defined to fit Quillen's axioms. There is a technical problem in that the functor K does not preserve cofibrations. To get around this, we replace K by a certain functor Q. As is the case of the Dold–Kan functor, Q too is defined for all cochain complexes A, even if they may not be DG-rings. If $A \in \mathrm{Ch}^{\geqslant 0}$ then

$$Q^n A = \bigoplus_{i=0}^{\infty} A^i \otimes T^i(\mathbb{Z}^n). \tag{6}$$

We show that any set map $\alpha : [n] \to [m]$ induces a group homomorphism $Q^n A \to Q^m A$, so that $[n] \mapsto Q^n A$ is not only a functor on Δ but on the larger category $\mathfrak{Fin}$ with the same objects, where a homomorphism $[n] \to [m]$ is just any set map. The projection $T\mathbb{Z}^n \to \Lambda\mathbb{Z}^n$ induces a homomorphism

$$\hat{p} : QA \xrightarrow{\sim} KA. \tag{7}$$

We show $\hat{p}$ induces an isomorphism of cohomotopy groups. If moreover A is a DG-ring, Q^nA has an obvious product coming from $A \otimes T\mathbb{Z}^n$; however this product is not well-behaved with respect to the $\mathfrak{Fin}$ nor the cosimplicial structure. In order to get a $\mathfrak{Fin}$-ring we perturb the product by a Hochschild 2-cocycle $f : A^* \otimes T^*V \to A^{*+1} \otimes T^{*+1}V$. We obtain a product $\circ$ of the form

$$(a \otimes x) \circ (b \otimes y) = ab \otimes xy + f(a \otimes x, b \otimes y). \tag{8}$$

For a definition of f see (48) below. It turns out that the map $\hat{p}$ is a ring homomorphism (see 5.3). This implies that the derived functors of K and of the functor $\tilde{Q}$ obtained from Q by restriction of its $\mathfrak{Fin}$-structure to a cosimplicial one, are isomorphic (see 9.3):

$$\mathbb{L}\tilde{Q} \cong \mathbb{L}K. \tag{9}$$

We show further that $\mathbb{L}\tilde{Q}$ is an equivalence. We deduce this from the stronger result (Theorem 9.6) that $\tilde{Q}$ is the left adjoint of a Quillen equivalence (as defined in Hovey's book [7, 1.3.12]).

Next we review other results obtained in this paper. As mentioned above, for $A \in \mathrm{Ch}^{\geqslant 0}$, QA is not only a cosimplicial group but a $\mathfrak{Fin}$-group. In particular the cyclic permutation $t_n := (0 \cdots n) : [n] \to [n]$ acts on Q^nA, and we may view QA as a cyclic module in the sense of [17, 9.6.1]. Consider the associated normalized mixed complex (NQA, μ, B). We show that there is a weak equivalence of mixed complexes

$$(A, 0, d) \xrightarrow{\sim} (NQA, \mu, B). \tag{10}$$

In particular these two mixed complexes have the same Hochschild homology:

$$A^* \cong H_*(NQA, \mu). \tag{11}$$

If A happens to be a DG-ring then the shuffle product induces a graded ring structure on $H_*(NQA, \mu)$; we show in 6.1 that (11) is a ring isomorphism for the product of A and the shuffle product of $H_*(NQA, \mu)$.

A specially interesting case is that of the DG-ring of noncommutative differential forms $\Omega_R S$ relative to a ring homomorphism $R \to S$ (as defined in [3]). We show in 7.6 that $Q\Omega_R S$ is the coproduct $\mathfrak{Fin}$-ring:

$$Q\Omega_R S \cong \coprod\nolimits_R S : [n] \mapsto \coprod_{i=0}^{n}{}_R\, S. \tag{12}$$

In particular, by (11), there is an isomorphism of graded rings

$$\Omega_R S \xrightarrow{\cong} H_*\left(N \coprod\nolimits_R S, \mu\right). \tag{13}$$

The particular case of (13) when R is commutative and $R \to S$ is central and flat was proved in 1994 by Guccione et al. [6]. More generally, by (10) we have a mixed complex equivalence

$$(\Omega_R S, 0, d) \xrightarrow{\sim} \left(N \coprod_R S, \mu, B\right). \tag{14}$$

We view (13) and (14) as noncommutative versions of the Hochschild–Kostant–Rosenberg and Loday–Quillen theorems [17, 9.4.13, 9.8.7].

As another application, we give a simple formulation for a product structure defined by Nuss [13] on each term of the Amitsur complex associated to a homomorphism $R \to S$ of not necessarily commutative rings R and S

$$\bigotimes_R S : [n] \mapsto \bigotimes_{i=0}^{n} {}_R S. \tag{15}$$

Nuss constructs his product using tools from the theory of quantum groups. We show here (see Section 8) that the canonical Dold–Kan isomorphism maps product (3) to that defined by Nuss. Thus

$$K\Omega_R S = KN\left(\bigotimes_R S\right) \cong \bigotimes_R S \tag{16}$$

is an isomorphism of cosimplicial rings.

The remainder of this paper is organized as follows. Basic notations are fixed in Section 2. In Section 3 the functor Q is defined. The homotopy equivalence of the cosimplicial groups KA and QA as well as that of the mixed complexes (10) is proved in Section 4. In Section 5 we show that the functor $Q: \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ is strong monoidal (5.2). We use this to introduce, for $A \in DGR^*$, the product (8) on QA (5.3). The graded ring isomorphism (11) is proved in Section 6. Isomorphism (12) and its corollaries (13) and (14) are proved in Section 7. The reformulation of Nuss' product is the subject of Section 8. In Section 5 we prove that $\tilde{Q}$ is the left adjoint of a Quillen equivalence (Theorem 9.6) and deduce from this that $\mathbb{L}K$ is a category equivalence (Corollary 9.8).

2. Cochain complexes and cosimplicial abelian groups

We write Δ for the simplicial category, and $\mathfrak{Fin}$ for the category with the same objects as Δ, but where the homomorphisms $[n] \to [m]$ are just the set maps. The inclusion

$$\hom_\Delta([n],[m]) \subset \mathrm{Map}([n],[m]) = \hom_{\mathfrak{Fin}}([n],[m])$$

gives a faithful embedding $\Delta \subset \mathfrak{Fin}$. If I and $\mathfrak{C}$ are categories, we shall write $\mathfrak{C}^I$ to denote the category of functors $I \to \mathfrak{C}$, to which we refer as I-objects of $\mathfrak{C}$. If $C : I \to \mathfrak{C}$ is an I-object, we write C^i for $C(i)$. We use the same letter for a map $\alpha : [n] \to [m] \in I$ as for its image under C. The canonical embedding $\Delta \subset \mathfrak{Fin}$ mentioned above makes $[n] \mapsto [n]$ into a cosimplicial object of $\mathfrak{Fin}$. We write $\partial_i : [n] \to [n+1]$, $i = 0, \dots, n+1$ and $\mu_j : [n] \to [n-1]$, $j = 0, \dots, n-1$, for the coface and codegeneracy maps. We also consider the map $\mu_n:[n]\to[n-1]$ defined by

$$\mu_n(i)=\begin{cases} i & \text{if } i<n,\\ 0 & \text{if } i=n.\end{cases} \tag{17}$$

One checks that $d_i := \mu_i:[n]\to[n-1]$, $i=0,\dots,n$ and $s_j=\partial_{j+1}:[n]\to[n+1]$, $j=0,\dots,n$ satisfy the simplicial identities, with the d_i as faces and the s_i as degeneracies. Thus there is a functor $\Delta^{op}\to\mathfrak{Fin}$, $[n]\mapsto[n]$. Moreover the cyclic permutation $t_n=(0\cdots n):[n]\to[n]$ extends this simplicial structure to a cyclic one (see [17, 9.6.3]). Composing with these functors and with the inclusion $\Delta\subset\mathfrak{Fin}$ mentioned above we have a canonical way of regarding any $\mathfrak{Fin}$-object in a category $\mathfrak{C}$ as either a cosimplicial, a simplicial, or a cyclic object.

If $\mathfrak{C}$ is a category with finite coproducts, and $A\in\mathfrak{C}$, we write $\coprod A$ for the functor

$$\coprod A:\mathfrak{Fin}\to\mathfrak{C},\qquad [n]\mapsto\coprod_{i=0}^{n}A. \tag{18}$$

Here $\coprod$ may be replaced by whatever sign denotes the coproduct of $\mathfrak{C}$; for example if $\mathfrak{C}$ is abelian, we write $\oplus A$ for $\coprod A$.

If $A=\oplus_{n=0}^{\infty}A_n$ and $B=\oplus_{n=0}^{\infty}B_n$ are graded abelian groups, we write

$$A\boxtimes B:=\bigoplus_{n=0}^{\infty}A_n\otimes B_n. \tag{19}$$

If A,B are graded I-abelian groups, we put $A\boxtimes B$ for the graded I-abelian group $i\mapsto A^i\boxtimes B^i$.

3. The functor Q

We are going to define a functor $Q:Ch^{\geqslant 0}\to\mathfrak{Ab}^{\mathfrak{Fin}}$; first we need some auxiliary constructions. Write $V:=\ker(\oplus\mathbb{Z}\to\mathbb{Z})$ for the kernel of the canonical map to the constant $\mathfrak{Fin}$-abelian group, and $\{e_i:0\leqslant i\leqslant n\}$ for the canonical basis of $\oplus_{i=0}^{n}\mathbb{Z}$. Put $v_i=e_i-e_0$, $0\leqslant i\leqslant n$. Note $v_0=0$ and $\{v_1,\dots,v_n\}$ is a basis of V^n. The action of a map $\alpha:[n]\to[m]\in\mathfrak{Fin}$ on V is given by

$$\alpha v_i=v_{\alpha(i)}-v_{\alpha(0)}\quad(0\leqslant i\leqslant n). \tag{20}$$

Applying to V the tensor algebra functor T in each codimension yields a graded $\mathfrak{Fin}$-ring TV. If $A=(A,d)\in\mathrm{Ch}^{\geqslant 0}$, we put

$$Q^nA:=A\boxtimes TV^n. \tag{21}$$

If $\alpha:[n]\to[m]\in\mathfrak{Fin}$, we set

$$\alpha(a\otimes x)=a\otimes\alpha x+da\otimes v_{\alpha(0)}\alpha x. \tag{22}$$

If $\beta:[m]\to[p]\in\mathfrak{Fin}$, then

$$\begin{aligned}\beta(\alpha(a\otimes x))&=a\otimes\beta\alpha x+da\otimes v_{\beta(0)}\beta\alpha x+da\otimes\beta(v_{\alpha(0)})\beta\alpha x,\\ &=(\beta\alpha)(a\otimes x).\end{aligned}$$

Thus QA is a $\mathfrak{Fin}$-abelian group, and $Q:\ \mathrm{Ch}^{\geqslant 0}\to\mathfrak{Ab}^{\mathfrak{Fin}}$ a functor. We have a filtration on QA by $\mathfrak{Fin}$-subgroups, given by

$$\mathscr{F}_nQA=\bigoplus_{i=n}^{\infty}A^i\otimes T^iV. \tag{23}$$

The associated graded $\mathfrak{Fin}$-abelian group is $G_{\mathscr{F}}QA=A\boxtimes TV$.

4. Comparison between Q and the Dold–Kan functor K

The Dold–Kan correspondence is a pair of inverse functors (see [17, 8.4]):

$$K:\ \mathrm{Ch}^{\geqslant 0}\leftrightarrows\mathfrak{Ab}^{\Delta}:N.$$

If $C\in\mathfrak{Ab}^{\Delta}$ then NC can be equivalently described as the normalized complex or as the Moore complex

$$N^nC=C^n\Big/\sum_{i=1}^{n}\partial_iC^{n-1}\cong\bigcap_{i=0}^{n-1}\ker(\mu_i:C^n\to C^{n-1}). \tag{24}$$

In either version the coboundary map $N^nC\to N^{n+1}C$ is induced by

$$\partial=\sum_{i=0}^{n}(-1)^i\partial_i. \tag{25}$$

In the first version this is the same map as that induced by ∂_0. A description of the inverse functor K (in the simplicial case) is given in [17, 8.4.4], and another in [9, 1.5]. Here is yet another. Let ΛV be the exterior algebra, $p:TV\to\Lambda V$ the canonical projection. One checks that $\ker(1\otimes p)\subset QA$ is a $\mathfrak{Fin}$-subgroup. Thus

$$K^*A:=A\boxtimes\Lambda V^* \tag{26}$$

inherits a $\mathfrak{Fin}$-structure. Moreover,

$$\hat{p}:=1\otimes p:QA\twoheadrightarrow KA \tag{27}$$

is a natural surjection of $\mathfrak{Fin}$-abelian groups. To see that the resulting cosimplicial abelian group KA is indeed the same as (i.e. is naturally isomorphic to) that of [17], it suffices to show that $NKA=A$. Put

$$V_j^n=\bigoplus_{i\neq j}\mathbb{Z}v_i\subset\bigoplus_{i=1}^{n}\mathbb{Z}v_i=V^n.$$

We have

$$\begin{aligned}NK^nA&=A\boxtimes\Lambda V^n\Big/\sum_{i=1}^{n}A\boxtimes\partial_i(\Lambda V^n)\\&=A\boxtimes\left(\Lambda V^n\Big/\sum_{i=1}^{n}\Lambda(V_i^n)\right)\\&=A^n\otimes v_1\wedge\cdots\wedge v_n\cong A^n.\end{aligned}$$

Furthermore it is clear that the coboundary map induced by ∂_0 is $d: A^* \to A^{*+1}$. Thus our KA is the same cosimplicial abelian group as that of [17]. But since in our construction KA has a $\mathfrak{Fin}$-structure, we may also regard it as a simplicial or cyclic abelian group. From our definition of faces and degeneracies, it is clear that the normalized complex of KA considered as a simplicial group has the abelian group $N^n KA = A^n$ in each dimension. One checks that the alternating sum μ of the faces induces the trivial boundary. Thus the normalized chain complex of the simplicial group KA is $(A, 0)$. Consider the Connes operator $B: NQ^*A \to NQ^{*+1}A$,

$$B = \partial_0 \circ \sum_{i=0}^{n} (-1)^{ni} t_n^i. \tag{28}$$

We show in 4.2 below that $\hat{p}B = D\hat{p}$, where $D := (n+1)d$ on A^n. Hence we have a map of mixed complexes

$$\hat{p}: (NQA, \mu, B) \to (A, 0, D). \tag{29}$$

We shall see in 4.2 below that (29) is a *rational equivalence of mixed complexes*. We recall that a map of mixed complexes is an equivalence if it induces an isomorphism at the level of Hochschild homology; this automatically implies it also induces an isomorphism at the level of cyclic, periodic cyclic and negative cyclic homologies. In 4.2 we also consider the map

$$l: A \to NQA, \quad l(a) = a \otimes \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v_{\sigma 1} \cdots v_{\sigma n}. \tag{30}$$

We show in Theorem 4.2 below that l is an integral equivalence

$$l: (A, 0, d) \xrightarrow{\sim} (NQA, \mu, B).$$

Remark 4.1. Note that if A is a complex of $\mathbb{Q}$-vectorspaces, then $\hat{p}$ can be rescaled as $(1/n!)\hat{p}$ on NQ^nA to give a mixed complex map $(NQA, \mu, B) \to (A, 0, d)$ which is left inverse to l.

Theorem 4.2. *Let A be a cochain complex of abelian groups, $\hat{p}: QA \twoheadrightarrow KA$ the map of $\mathfrak{Fin}$-abelian groups defined in (27) above. Then*:

(i) *There are a natural cochain map $j: (A, d) \to (NQA, \partial)$ such that $\hat{p}j = 1_A$ and a natural cochain homotopy $h: N^*QA \to N^{*-1}QA$ such that $[h, \partial] = 1 - j\hat{p}$.*
(ii) *Map (29) is a rational equivalence of mixed complexes. On the other hand map (30) is a natural integral equivalence $l: (A, 0, d) \to (NQA, \mu, B)$.*

Proof. First we compute NQA. A similar argument as that given in Section 4 to compute NKA, shows that

$$N^n QA = A \boxtimes \left(TV^n \middle/ \sum_{i=1}^{n} TV_i^n \right). \tag{31}$$

On the other hand, we have a canonical identification between the rth tensor power of $V^n = \mathbb{Z}^n$ and the free abelian group on the set of all maps $\{1,\dots,r\} \to \{1,\dots,n\}$:

$$T^r V^n \cong \mathbb{Z}[\mathrm{Map}(\{1,\dots,r\},\{1,\dots,n\})]. \tag{32}$$

Using (32), $T^r V^n / \sum_{j=1}^n T^r V_j^n$ becomes the free module on all surjective maps $\{1,\dots,r\} \to \{1,\dots,n\}$; we get

$$N^n QA = A \boxtimes \mathbb{Z}\left[\mathrm{Sur}_{*,n}\right] = \bigoplus_{r=n}^{\infty} A^r \otimes \mathbb{Z}\left[\mathrm{Sur}_{r,n}\right]. \tag{33}$$

Here $\mathrm{Sur}_{p,q}$ is the set of all surjections $\{1,\dots,p\} \to \{1,\dots,q\}$. Note that in particular $\mathrm{Sur}_{n,n} = S_n$, the symmetric group on n letters. To prove (i), regard NQA as a cochain complex. We may view NQA as the direct sum total complex of a second quadrant double complex

$$C^{p,q} = \begin{cases} A & \text{if } p = q = 0, \\ A^q \otimes \mathbb{Z}[\mathrm{Sur}_{q,q+p}] & \text{if } (p,q) \neq (0,0). \end{cases}$$

Here $1 \otimes \partial_0$ and $d \otimes v_1 \partial_0$ are, respectively, the horizontal and the vertical coboundary operators. Filtration (23) is the row filtration. If we regard $A = NKA$ as a double cochain complex concentrated in the zero column, then $\hat{p}$ becomes a map of double complexes. By definition, $\hat{p} = 1 \otimes p$; at the nth row, p is a map:

$$p : \mathbb{Z}\left[\mathrm{Sur}_{n,*}\right] \twoheadrightarrow \mathbb{Z}[n]. \tag{34}$$

The only nonzero component of p is $p(\sigma) = \mathrm{sign}(\sigma)$. We claim (34) is a cochain homotopy equivalence. To prove this note first that because both $\mathbb{Z}[\mathrm{Sur}_{n,*}]$ and $\mathbb{Z}[n]$ are complexes of free abelian groups, to show p is a homotopy equivalence it suffices to check it is a quasi-isomorphism. Next note that

$$\begin{aligned} H^*\left(\mathbb{Z}\left[\mathrm{Sur}_{n,*}\right]\right) &= H^*(NT^n V) = \pi^*(T^n V) \\ &= T^n \pi^*(V) \\ &= T^n H^*(NV) \\ &= T^n H^*(\mathbb{Z}[1]) = \mathbb{Z}[n]. \end{aligned} \tag{35}$$

Thus, to prove p is a cochain equivalence it suffices to show that

$$\ker(p : \mathbb{Z}[S_n] \to \mathbb{Z}) = \partial_0\left(\mathbb{Z}\left[\mathrm{Sur}_{n,n-1}\right]\right). \tag{36}$$

The inclusion $\supset$ of (36) holds because p is a cochain map. To prove the other inclusion, proceed as follows. First note the identification

$$\mathbb{Z}[S_n] \cong \bigoplus_{\sigma \in S_n} \mathbb{Z} v_{\sigma 1} \cdots v_{\sigma n}.$$

Next, observe that the kernel of p is generated by elements of the form

$$\cdots v_1 \cdots v_i \cdots + \cdots v_i \cdots v_1 \cdots \equiv -\partial_0(\cdots v_{i-1} \cdots v_{i-1} \cdots) \quad (i > 1).$$

Here congruence is taken modulo $\sum_{j\geqslant 1}\partial_j TV$. Thus p is a surjective homotopy equivalence, as claimed. Therefore, we may choose a cochain map $j':\mathbb{Z}[n]\to\mathbb{Z}[\mathrm{Sur}_{n,*}]$ such that $pj'=1$ and a cochain homotopy $h':\mathbb{Z}[\mathrm{Sur}_{n,*}]\to\mathbb{Z}[\mathrm{Sur}_{n,*-1}]$ such that $[h',\partial_0]=1-pj'$. One checks that the following maps satisfy the requirements of part (i) of the theorem:

$$j := 1\otimes j' + (1\otimes h')((1\otimes j')d - d\otimes v_1\partial_0 j'),$$

$$h := (1\otimes h' - (1\otimes h')(d\otimes v_1\partial_0)(1\otimes h'))(1\otimes j'p-1).$$

Next we prove part (ii). Observe the face maps of NQA are of the form $1\otimes\mu_i$, where μ_i is the face map in TV. Hence we have a direct sum decomposition of chain complexes

$$(NQA,\mu)=\bigoplus_{n=0}^{\infty}A^n\otimes\left(\mathbb{Z}\left[\mathrm{Sur}_{n,*}\right],\mu\right). \tag{37}$$

The homology version of the argument used in (35) shows that

$$H_*\left(\mathbb{Z}\left[\mathrm{Sur}_{n,*}\right]\right)=\mathbb{Z}[n].$$

In particular $L_n := \ker(\mu:\mathbb{Z}[S_n]\to\mathbb{Z}[\mathrm{Sur}_{n,n-1}])$ is free of rank one. By definition, to prove $\hat{p}$ is a rational mixed complex equivalence, we must prove that $\hat{p}\mu=0$, which is straightforward, that $\hat{p}B=D\hat{p}$, which we leave for later, and finally that $\hat{p}=1\otimes p:(NQA,\mu)\to(A,0)$ is a rational chain equivalence, which in turn reduces to proving $p(L_n)\neq 0$ for $n\geqslant 1$. Consider the element

$$\varepsilon_n := \sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\sigma\in\mathbb{Z}[S_n]. \tag{38}$$

We have $p(\varepsilon_n)=n!$; one checks further that $\varepsilon_n\in L_n$. It follows that $\hat{p}:(NQA,\mu)\to(A,0)$ is a rational equivalence, as we had to prove. Moreover, as every coefficient of ε_n is invertible, and L_n has rank one, we have $L_n=\mathbb{Z}\varepsilon_n$. It follows that the map $l':\mathbb{Z}[n]\to(\mathbb{Z}[\mathrm{Sur}_{n,*}],\mu)$ which sends $1\in\mathbb{Z}$ to ε_n is a quasi-isomorphism, whence a homotopy equivalence. To finish the proof, we must show that $ld=Bl$ and $\hat{p}B=D\hat{p}$. Both of these follow once one has proven formula (39) below, which in turn is derived from identities (40), which are proved by induction. The inclusion $\{1\}\subseteq\{1,\ldots,n+1\}$ together with the map $\{1,\ldots,n\}\to\{1,\ldots,n+1\}$, $i\mapsto i+1$, define a bijection $\{1\}\coprod\{1,\ldots,n\}\to\{1,\ldots,n+1\}$. We identify $\{1\}\coprod\{1,\ldots,n\}=\{1,\ldots,n+1\}$ using this bijection. If $\sigma\in S_n$, we denote by $1\coprod\sigma$ the coproduct map.

$$B(a\otimes\sigma)=da\otimes\sum_{i=0}^{n}(-1)^{in}(1,\ldots,n+1)^i\left(1\coprod\sigma\right),$$

$$t_n^i(v_j)=\begin{cases} v_{i+j}-v_i, & \text{if } i\leqslant n-j,\\ v_{p-1}-v_i, & \text{if } i=n-j+p \quad j\geqslant p\geqslant 1,\end{cases}$$

$$t_n^i(a\otimes x)=a\otimes t_n^i x+da\otimes v_i t_n^i x, \tag{39}$$

$$B(a\otimes x)=da\otimes\sum_{i=0}^{n}(-1)^{in}v_{i+1}\partial_0 t^i x. \qquad \square \tag{40}$$

Notation 4.3. Let $B = (B, d) \in \mathrm{Ch}^{\geqslant 0}$. Put $PB^n = B^n \oplus B^{n-1} \oplus B^n$. Equip PB with the coboundary operator $\partial : PB^* \to PB^{*+1}$ given by the matrix

$$\partial = \begin{bmatrix} d & 0 & 0 \\ 1 & -d & -1 \\ 0 & 0 & d \end{bmatrix}.$$

We note PB comes equipped with a natural map $\varepsilon = (\varepsilon_0, \varepsilon_1) : PB \to B \oplus B$, and that two maps $f_0, f_1 : A \to B$ are cochain homotopic if and only if there exists a cochain homomorphism $H : A \to PB$ such that $\varepsilon H = (f_0, f_1)$. □

The next corollary says that, for $A, B \in \mathrm{Ch}^{\geqslant 0}$, every cosimplicial map $f : QA \to QB$ has a canonically associated cochain map $\bar{f}$, such that $NQ\bar{f}$ and Nf are naturally homotopic. Moreover, if $f = Qg$, then $\bar{f} = g$.

Corollary 4.4. *Let $A, B \in \mathrm{Ch}^{\geqslant 0}$. Consider the functors*

$$(\mathrm{Ch}^{\geqslant 0})^{\mathrm{op}} \times \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}$$

$$(A, B) \mapsto \hom_{\mathfrak{Ab}^\Delta}(QA, QB)$$

$$(A, B) \mapsto \hom_{\mathrm{Ch}^{\geqslant 0}}(NQA, PNQB).$$

There are two natural transformations

$$^- : \hom_{\mathfrak{Ab}^\Delta}(QA, QB) \to \hom_{\mathrm{Ch}^{\geqslant 0}}(A, B),$$

$$H : \hom_{\mathfrak{Ab}^\Delta}(QA, QB) \to \hom_{\mathrm{Ch}^{\geqslant 0}}(NQA, PNQB).$$

These are such that $\overline{Qg} = g$ and that the following diagram commutes:

$$\begin{tikzcd} & \hom_{\mathrm{Ch}\geq 0}(NQA, PNQB) \arrow[d, "\epsilon"] \\ \hom_{\mathfrak{Ab}^\Delta}(QA, QB) \arrow[ur, "H"] \arrow[r] & \hom_{\mathrm{Ch}\geq 0}(NQA, NQB \oplus NQB) \\ f \arrow[r, mapsto] & (Nf, NQ\bar{f}) \end{tikzcd} \tag{41}$$

Proof. Let $f \in \hom_{\mathfrak{Ab}^\Delta}(QA, QB)$ and $j, \hat{p}$ and h be as in the theorem. Define $\bar{f} := \hat{p}N(f)j$. Because $\hat{p}j = 1$, $\overline{Qg} = g$. Using the naturality of j and $\hat{p}$, one checks further that $f \mapsto \bar{f}$ is natural. Let $\delta = N(f) - NQ(\bar{f})$ and put

$$\kappa = \kappa_f := h\delta + \delta h - [h\delta, \partial]h.$$

One checks that $[\kappa, \partial] = \delta$, whence $H_f := (Nf, \kappa, NQ\bar{f})$ is a homomorphism $NQA \to PNQB$ with $\varepsilon H_f = (Nf, NQ\bar{f})$. The naturality of $H : f \mapsto H_f$ follows from that of h. □

Simplicial powers and cosimplicial homotopies 4.5. Let $A \in \mathfrak{Ab}^\Delta$, $X \in \mathrm{Sets}^{\Delta^{op}}$. Put

$$(A^X)^n := \prod_{x \in X_n} A^n. \tag{42}$$

If $\alpha \in \hom_{\Delta}([n],[m])$ and $a \in (A^X)^n$, define $\alpha(a)_x = \alpha(a_{\alpha x})$ $(x \in X_m)$. The dual $\mathbb{Z}[X]^\vee$: $[n] \mapsto \hom_{\mathbb{Z}}(\mathbb{Z}[X_n],\mathbb{Z})$ of the simplicial free abelian group $\mathbb{Z}[X]$ is a cosimplicial group. Consider the cosimplicial tensor product $A \otimes \mathbb{Z}[X]^\vee : [n] \mapsto A^n \otimes \mathbb{Z}[X_n]^\vee$. There is a natural homomorphism

$$\eta : A \otimes \mathbb{Z}[X]^\vee \to A^X \quad \eta(a \otimes \phi)_x = a\phi(x). \tag{43}$$

In case each X_n is finite, η is an isomorphism. Dualizing the statement in [12]—next after 8.9—we get that the composite of the normalized shuffle map $NA \otimes N\mathbb{Z}[X]^\vee \to N(A \otimes \mathbb{Z}[X]^\vee)$ with Alexander–Whitney map $N(A \otimes \mathbb{Z}[X]^\vee) \to NA \otimes N\mathbb{Z}[X]^\vee$ is the identity. Thus $NA \otimes N\mathbb{Z}[X]^\vee$ is a deformation retract of $N(A \otimes \mathbb{Z}[X]^\vee)$. In particular $PNA = NA \otimes N\mathbb{Z}[\Delta[1]]^\vee$ is a deformation retract of $N(A^{\Delta[1]})$. Recall two cosimplicial maps $f_0, f_1 : A \to B$ are called *homotopic* if $(f_0, f_1) : A \to B \times B = B^{\Delta[0] \coprod \Delta[0]}$ can be lifted to a map $H : A \to B^{\Delta[1]}$. From what we have just seen it is clear that f_0, f_1 are homotopic in this sense if and only if Nf_0, Nf_1 are cochain homotopic. (The dual of this assertion is proved in [5].) Let $\mathfrak{C}$ be either of $\mathrm{Ch}^{\geqslant 0}$, $\mathfrak{Ab}^{\Delta}$. We write $[\mathfrak{C}]$ for the category with the same objects as $\mathfrak{C}$, but where the homomorphisms are the homotopy classes of maps in $\mathfrak{C}$.

Proposition 4.6. *The functor Q induces an equivalence of categories* $[\mathrm{Ch}^{\geqslant 0}] \to [\mathfrak{Ab}^{\Delta}]$.

Proof. If $A \in \mathfrak{Ab}^{\Delta}$, then $A = KNA$. By Theorem 4.2, NA is homotopy equivalent to NQA. Thus A is homotopy equivalent to $KNQA = QA$. It remains to show that the following map is a bijection

$$[Q] : \hom_{[\mathrm{Ch}^{\geqslant 0}]}(A,B) \to \hom_{[\mathfrak{Ab}^{\Delta}]}(QA,QB).$$

It is clear from the previous corollary that the composite of Q with

$$[N] : \hom_{[\mathfrak{Ab}^{\Delta}]}(QA,QB) \to \hom_{[\mathrm{Ch}^{\geqslant 0}]}(NQA,NQB) \tag{44}$$

is a bijection. But (44) is bijective by 4.5. □

Definition 4.7. Give $\mathrm{Ch}^{\geqslant 0}$ the closed model category structure in which a map is a *fibration* if it is surjective codimensionwise, a *weak equivalence* if it is a quasi-isomorphism, and a cofibration if it has Quillen's left lifting property (*LLP*, see [14]) with respect to those fibrations which are also weak equivalences (*trivial fibrations*). All this structure carries over to $\mathfrak{Ab}^{\Delta}$ using the category equivalence $N : \mathfrak{Ab}^{\Delta} \to \mathrm{Ch}^{\geqslant 0}$. In the lemma below *RLP* stands for right lifting property in the sense of [14].

Notation 4.8. In the next lemma and further below, we use the following notation. If $n \geqslant 0$, we write $\mathbb{Z}\langle n, n+1\rangle$ for the mapping cone of the identity map $\mathbb{Z}[n] \to \mathbb{Z}[n]$.

Lemma 4.9. *Let $f : E \to B$ be a homomorphism of cosimplicial abelian groups. We have*:

(i) *f is a fibration if and only if for all $n \geqslant 1$ f has the RLP with respect to* $0 \to Q\mathbb{Z}\langle n-1, n\rangle$.

(ii) *f is a trivial fibration if and only if for all $n \geqslant 1$ f has the RLP with respect to the natural inclusion $Q\mathbb{Z}[n] \hookrightarrow Q\mathbb{Z}\langle n-1,n\rangle$.*

Proof. Let $f : C \to D$ be a cochain map. By the theorem, Kf is a retract of Qf. Thus every map having the *RLP* with respect to Qf also has it with respect to Kf. The lemma follows from this applied to the cochain maps $0 \to \mathbb{Z}\langle n-1,n\rangle$ and $\mathbb{Z}[n] \hookrightarrow \mathbb{Z}\langle n-1,n\rangle$. □

5. Monoidal structure

Consider the map $\theta : TV^* \to TV^*$,

$$\theta(v_i) = v_i^2, \quad \theta(xy) = \theta(x)y + (-1)^{|x|}x\theta(y). \tag{45}$$

The second identity says that θ is a homogeneous derivation of degree $+1$. Note it follows from (45) that $\theta^2 = 0$.

Lemma 5.1. *For every $\alpha \in Map([n],[m])$ and $x \in TV^n$, $[\alpha,\theta](x) = [v_{\alpha(0)}, \alpha(x)]$.*

Proof. Both sides of the identity we have to prove are derivations. Thus it suffices to show they agree on the generators v_i, and this is straightforward. □

Theorem 5.2. *Let $A,B \in \mathrm{Ch}^{\geqslant 0}$ and θ as defined in (45) above. Consider the tensor product of $\mathfrak{Fin}$-abelian groups $QA \otimes QB : [n] \to Q^nA \otimes Q^nB$. The map $\upsilon : QA \otimes QB \to Q(A \otimes B)$ given by the following formula is an isomorphism in $\mathfrak{Ab}^{\mathfrak{Fin}}$, and makes $Q : \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ a strong monoidal functor*

$$\upsilon((a \otimes x) \otimes (b \otimes y)) = a \otimes b \otimes xy + (-1)^{|a|} a \otimes db \otimes \theta(x)y.$$

Proof. It is clear that the following map is an isomorphism of abelian groups:

$$g : (a \otimes x) \otimes (b \otimes y) \mapsto (a \otimes x) \cdot (b \otimes y) := a \otimes b \otimes xy.$$

Because $h := \upsilon - g$ is homogeneous of degree $+1$ and $h^2 = 0$, υ is a group isomorphism. That υ is a homomorphism in $\mathfrak{Ab}^{\mathfrak{Fin}}$ follows straightforwardly using Lemma 5.1. In order to see that Q is strong monoidal, we must check that the two diagrams involving the unit object of $\mathfrak{Ab}^{\mathfrak{Fin}}$ commute, which is immediate, and also the following associativity condition for $\alpha \in QA$, $\beta \in QB$ and $\gamma \in QC$

$$\upsilon(\upsilon(\alpha \otimes \beta) \otimes \gamma) = \upsilon(\alpha \otimes \upsilon(\beta \otimes \gamma)). \tag{46}$$

Writing this in terms of g and h, and because g is associative, we obtain

$$\begin{aligned} h(h(\alpha \otimes \beta) \otimes \gamma) - h(\alpha \otimes h(\beta \otimes \gamma)) &= \alpha \cdot h(\beta \otimes \gamma) - h(\alpha \cdot \beta \otimes \gamma) \\ &\quad + h(\alpha \otimes \beta \cdot \gamma) - h(\alpha \otimes \beta) \cdot \gamma. \end{aligned} \tag{47}$$

For $\alpha = a \otimes x$, $\beta = b \otimes y$ and $\gamma = c \otimes z$, the left-hand side of (47) is

$$(-1)^{|y|+1} adbdc \otimes \theta(\theta(x)y)z + (-1)^{|x|+|y|+1} adbdc \otimes \theta(x)\theta(y)z = 0.$$

This is zero because θ is a square-zero derivation. Thus (47) says that h is a Hochschild 2-cocycle, which follows from the fact that both d and θ are derivations. □

Product structure 5.3. Let $A \in DGR^*$, $m : A \otimes A \to A$ the multiplication map. Consider the composite

$$\circ : QA \otimes QA \xrightarrow{\upsilon} Q(A \otimes A) \xrightarrow{Qm} QA.$$

We have

$$(\omega \otimes x) \circ (\eta \otimes y) := \omega\eta \otimes xy + (-1)^{|x|}\omega d\eta \otimes \theta(x)y. \tag{48}$$

By construction, $(QA, \circ)$ is a $\mathfrak{Fin}$-ring. Note that each term $\mathscr{F}_n QA$ of the filtration (23) is a $\mathfrak{Fin}$-ideal. The associated graded $\mathfrak{Fin}$-ring is $A \boxtimes TV$ equipped with the product inherited from $A \boxtimes TV \subset A \otimes TV$. Thus we may view QA as a deformation of $A \boxtimes TV$. One checks that the kernel of the map $\hat{p} : QA \twoheadrightarrow KA$ of (27) is an ideal for $\circ$. Hence KA inherits a $\mathfrak{Fin}$-ring structure; using the definition of θ we get that the induced product on $KA = A \boxtimes \Lambda V$ is just that coming from $A \otimes \Lambda V$:

$$(a \otimes x)(b \otimes y) = ab \otimes xy. \tag{49}$$

6. Comparison with the shuffle product

Let R be a simplicial ring. Consider the direct sum of its homotopy groups

$$\pi R := \bigoplus_{n=0}^{\infty} \pi_n R. \tag{50}$$

Recall that the shuffle product $\bigstar$ makes πR into a graded ring. If moreover R is a $\mathfrak{Fin}$-ring, then the Connes operator $B : \pi_* R \to \pi_{*+1} R$ is a derivation, so that $\pi R = (\pi R, \bigstar, B)$ becomes in fact a DG-ring. This follows from the version of [10, 4.3.3]. for cyclic modules, the same which is used without further proof in [10, 4.3.7-8]. Hence we have a functor

$$\mathrm{Rings}^{\mathfrak{Fin}} \to DGR^*, \quad R \mapsto \pi R. \tag{51}$$

Proposition 6.1. *Let $A \in DGR^*$. Consider the natural isomorphism of graded abelian groups induced by the map l of* 4.2(ii).

$$l : A \xrightarrow{\sim} \pi QA. \tag{52}$$

Map (52) is an isomorphism of DG-rings. In particular functor (51) is a left inverse of Q.

Proof. By 4.2, l induces a cochain isomorphism $(A, d) \cong (\pi QA, B)$. It remains to show that the induced map is a ring homomorphism. Recall the formula for the shuffle product $\bigstar$ involves degeneracies and shuffles. Keeping in mind that the degeneracies in QA are of the form $s_i = 1 \otimes \partial_{i+1}$ with ∂_j the coface of TV, we get the following identity for $a \in A^n$, $b \in A^m$:

$$\begin{aligned} l(a) \star l(b) &= (a \otimes \varepsilon_n) \star (b \otimes \varepsilon_m), \\ &\equiv ab \otimes \varepsilon_n \star \varepsilon_m \operatorname{mod} N\mathscr{F}_{n+m+1}Q^{n+m}A, \\ &= ab \otimes \varepsilon_{n+m} = l(ab). \end{aligned}$$

This finishes the proof, since $\pi_{n+m}N\mathscr{F}_{n+m+1}QA = 0$ by the proof of 4.2. □

7. Noncommutative Hochschild–Kostant–Rosenberg and Loday–Quillen theorems

Recall from [17] that for every algebra S over a commutative ring R which is central in S there is defined a cyclic R-module $C_*(S/R)$. Recall also that the normalization of $C_*(S/R)$ is the mixed complex of noncommutative differential forms [4] $NC_*(S/R) = \Omega_R S$. The Hochschild–Kostant–Rosenberg theorem ([17], Ex. 9.4.2) says that if R and S are commutative, R noetherian, and $R \to S$ an essentially of finite type, smooth homomorphism, then the canonical map from commutative differential forms to Hochschild homology induced by the shuffle product is an isomorphism:

$$\Omega^*_{S/R} = \Lambda^* HH_1(S/R) \xrightarrow{\sim} HH_*(S/R). \tag{53}$$

If $R \supset \mathbb{Q}$ the inverse of (53) is induced by the homomorphism

$$\Omega_R S \to \Omega_{S/R} \quad a_0 da_1 \cdots da_n \mapsto \frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n. \tag{54}$$

Here the boundary operators are the Hochschild boundary b on $\Omega_R S$ and the trivial boundary on $\Omega_{S/R}$. Moreover, as (54) maps B to d, it is in fact a mixed complex equivalence

$$(\Omega_R S, b, B) \xrightarrow{\sim} (\Omega_{S/R}, 0, d).$$

We will prove an analogue of this which holds for not necessarily commutative R and S. Note that if R and S are commutative then $C_*(S/R)$ is just the coproduct $\mathfrak{Fin}$-algebra $\otimes_R S$ considered as a cyclic module. The analogue concerns the coproduct $\mathfrak{Fin}$-ring $\coprod_R S$ which arises from a ring homomorphism $R \to S$ of not necessarily commutative rings. We show in 7.7 below that there is an equivalence of mixed complexes $(\Omega_R S, 0, d) \to (N \coprod_R S, \mu, B)$, valid without restrictions on the characteristic. We deduce this from 4.2 and from 7.6 below, where we show that $\coprod_R S = Q\Omega_R S$. In particular the isomorphism $\Omega^*_R S \cong HH_*(N \coprod_R S, \mu, B) = \pi_* \coprod_R S$ is (52), which is a ring homomorphism for the product of forms and the shuffle product (by 6.1) just like the Hochschild–Kostant–Rosenberg isomorphism (53). Note further the analogy between (54) and the rescaled map $\hat{p}$ of 4.1.

To prove the isomorphism $Q\Omega_R S \cong \coprod_R S$ we show first that Q has a right adjoint (7.4). In the next lemma we use the symbol T for both the tensor $\mathfrak{Fin}$- and DG-rings.

Lemma 7.1. *Let* $(U,d)\in \mathrm{Ch}^{\geqslant 0}$. *Then there is a natural isomorphism of* $\mathfrak{Fin}$*-rings* $TQU \xrightarrow{\cong} QTU$.

Proof. This is a formal consequence of Theorem 5.2. □

Notation 7.2. The following *DG*-rings shall be considered often in what follows:

$$D(n) := T\mathbb{Z}\langle n, n+1\rangle \supset S(n) := T\mathbb{Z}[n]. \tag{55}$$

Corollary 7.3. *Let I be a set*, $n_i \geqslant 0$. *Then*

$$Q\left(\coprod_{i\in I} D(n_i)\right) = \coprod_{i\in I} QD(n_i).$$

Proof.

$$\begin{aligned}
Q\left(\coprod_{i\in I} D(n_i)\right) &= Q\left(\coprod_{i\in I} T(\mathbb{Z}\langle n_i, n_{i+1}\rangle)\right),\\
&= QT\left(\bigoplus_{i\in I}\mathbb{Z}\langle n_i, n_{i+1}\rangle\right),\\
&= TQ\left(\bigoplus_{i\in I}\mathbb{Z}\langle n_i, n_{i+1}\rangle\right) = T\left(\bigoplus_{i\in I} Q\mathbb{Z}\langle n_i, n_{i+1}\rangle\right),\\
&= \coprod_{i\in I} TQ\mathbb{Z}\langle n_i, n_{i+1}\rangle = \coprod_{i\in I} QD(n_i). \qquad \square
\end{aligned}$$

Proposition 7.4. *Let* Rings *be the category of associative unital rings and DGR^* that of cochain differential graded rings. The functor* $Q : DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$ *has a right adjoint.*

Proof. This is an adaptation of the proof of the dual of Freyd's Special Adjoint Theorem [11, Chapter V, Section 8, Theorem 2]. Let $B\in \mathrm{Rings}^{\mathfrak{Fin}}$. Put

$$DB := \coprod_{n\geqslant 0}\ \coprod_{\hom(QD(n),B)} D(n). \tag{56}$$

If $s\in\hom(QD(n),B)$, write $j_s : D(n)\to DB$ for the corresponding inclusion. Define $\alpha : QDB \to B$ by $\alpha j_s = s$. Consider the two-sided $\mathfrak{Fin}$-ideal

$$DB \rhd K := \sum\{I \lhd DB : \alpha(QI) = 0\}. \tag{57}$$

Set $PB := DB/K$. Because $Q\colon \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ is exact, we have a natural map $\hat{\alpha}$ making the following diagram commute

$$\begin{tikzcd} QDB \arrow[r, "\alpha"] \arrow[d] & B \\ QPB \arrow[ur, "\hat{\alpha}"'] & \end{tikzcd} \tag{58}$$

Hence $(PB, \hat{\alpha})$ is an object of the category $Q \uparrow B$ (notation is as in [11]). We shall see it is final, which proves that P is right adjoint to Q. Let $(R, f) \in Q \uparrow B$. Put

$$ER := \coprod_{n \geqslant 0} \coprod_{\hom(D(n), R)} D(n).$$

If $r : D(n) \to R$ is a homomorphism, write $i_r : D(n) \to ER$ for the corresponding inclusion. Consider the homomorphisms $\pi : ER \to R$, $\pi i_r = r$ and $g : ER \to DB$, $g i_r = j_{fQr}$. We claim that the following diagram commutes:

$$\begin{tikzcd} QER \arrow[r, "Qg"] \arrow[d, "Q\pi"'] & QDB \arrow[d, "\alpha"] \\ QR \arrow[r, "f"'] & B \end{tikzcd} \tag{59}$$

Indeed by 7.3, commutativity can be checked at each "cell" $Q(D(n))$, where it is clear. Using (59) together with the exactness of Q, we get that $g(\ker \pi) \subset K$. Thus g induces a map $\hat{g}$ making the following diagram commute:

$$\begin{tikzcd} ER \arrow[r, "g"] \arrow[d, "\pi"'] & DB \arrow[d] \\ R \arrow[r, "\hat{g}"'] & PB \end{tikzcd} \tag{60}$$

It follows that also the following commutes:

$$\begin{tikzcd} QER \arrow[r, "Qg"] \arrow[d, "Q\pi"'] & QDB \arrow[dr, "\alpha"] & \\ QR \arrow[r, "Q\hat{g}"'] & QPB \arrow[r, "\hat{\alpha}"'] & B \end{tikzcd} \tag{61}$$

Putting together the latter diagram with (58) and (59) we get that $fQ(\pi) = \hat{\alpha}Q(\hat{g})Q(\pi)$. Because π is surjective and Q exact, we conclude $f = \hat{\alpha}Q(\hat{g})$; in other words $\hat{g}$ is a homomorphism $(R, f) \to (PB, \hat{\alpha})$ in $Q \uparrow B$. Let $h : (R, f) \to (PB, \hat{\alpha})$ be another. Lift $\hat{h}$ to a map $h : ER \to DB$. Then by (61),

$$\alpha Q(h) = \hat{\alpha}Q(\hat{h}\pi) = fQ(\pi) = \alpha Q(g).$$

Hence the image of $g - h$ lands in K, and therefore $\hat{g} = \hat{h}$. $\square$

Remark 7.5. Essentially, the same proof as that of the theorem above shows that also $Q: \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ has a right adjoint. One just has to replace $\coprod$ and $D(n)$ for $\oplus$ and $\mathbb{Z}\langle n, n+1\rangle$.

Theorem 7.6. *Let $R \to S$ be a ring homomorphism, $R \uparrow$ Rings the category of R-algebras, $\coprod_R$ the coproduct in $R \uparrow$ Rings, $\coprod_R S$ the $\mathfrak{Fin}$-ring of Section 2 above and $\Omega_R S$ the R-DG-algebra of relative noncommutative differential forms of* [3]. *Then $Q(\Omega_R S) = \coprod_R S$.*

Proof. The $\mathfrak{Fin}$-ring $\coprod_R S$ is characterized by the following property

$$\hom_{(R\uparrow\mathrm{Rings})^{\mathfrak{Fin}}}\left(\coprod\nolimits_R S, C\right) = \hom_{R\uparrow\mathrm{Rings}}(S, C^0). \tag{62}$$

We must show $Q\Omega_R S$ has the same property. On the other hand we have

$$\hom_{R\uparrow DGR^*}(\Omega_R S, X) = \hom_{R\uparrow\mathrm{Rings}}(S, X^0). \tag{63}$$

Here we identify R with the DGR^* concentrated in codimension 0 with trivial derivation. Let P be the right adjoint of $Q: DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$; its existence is guaranteed by Proposition 7.4. Identifying R with the constant $\mathfrak{Fin}$-ring, noting that $QR = R$ and using (63), we obtain

$$\begin{aligned}\hom_{R\uparrow(\mathrm{Rings}^{\mathfrak{Fin}})}(Q(\Omega_R S), C) &= \hom_{R\uparrow DGR^*}(\Omega_R S, PC)\\ &= \hom_{R\uparrow\mathrm{Rings}}(S, PC^0).\end{aligned}$$

Therefore to prove the corollary it suffices to show that $PC^0 = C^0$. We have

$$\begin{aligned}PC^0 &= \hom_{\mathrm{Ch}^{\geqslant 0}}(\mathbb{Z}\langle 0,1\rangle, PC)\\ &= \hom_{DGR^*}(D(0), PC)\\ &= \hom_{\mathrm{Rings}^{\mathfrak{Fin}}}(QD(0), C)\\ &= \hom_{\mathrm{Rings}^{\mathfrak{Fin}}}(TQ\mathbb{Z}\langle 0,1\rangle, C) \ \text{(by 7.1)}\\ &= \hom_{\mathfrak{Ab}^{\mathfrak{Fin}}}(Q\mathbb{Z}\langle 0,1\rangle, C).\end{aligned} \tag{64}$$

By definition

$$Q^n\mathbb{Z}\langle 0,1\rangle = \mathbb{Z}\langle 0,1\rangle \boxtimes TV^n = \mathbb{Z}(1\otimes 1) \oplus \bigoplus_{i=1}^{n} \mathbb{Z}(1\otimes v_i). \tag{65}$$

Put $e_0 = 1\otimes 1$, $e_i = 1\otimes v_i + e_0$ $1 \leqslant i \leqslant n$. It follows from (20) that $\alpha(e_i) = e_{\alpha(i)}$ for all $\alpha: [n] \to [m] \in \mathfrak{Fin}$. Therefore, $Q\mathbb{Z}\langle 0,1\rangle = \oplus\mathbb{Z}$, whence (64) equals

$$= \hom_{\mathfrak{Ab}^{\mathfrak{Fin}}}\left(\bigoplus \mathbb{Z}, C\right) = C^0. \qquad \square$$

Corollary 7.7. *View the* $\mathfrak{Fin}$-*ring* $\coprod_R S$ *as a cyclic module by restriction, and consider its associated normalized mixed complex* $(N\coprod_R S, \mu, B)$. *Then the map* l *of Theorem* 4.2 *is a mixed complex equivalence* $l:(\Omega_R S, 0, d) \to (N\coprod_R S, \mu, B)$.

Remark 7.8. As a particular case of Theorem 7.6 we get a ring isomorphism

$$S\coprod_R S \cong Q^1\Omega_R S = \Omega_R S \boxtimes TV^1 \cong \Omega_R S. \tag{66}$$

Here $\Omega_R S$ is equipped with the product $\circ$ of (48). A similar isomorphism but with a different choice of $\circ$ was proved by Cuntz and Quillen [3, Proposition 1.3], under the stated assumption that $R = \mathbb{C}$. Their choice of $\circ$ actually works whenever 2 is invertible in R, and the rings which arise from $\Omega_R S$ with our product and that of [3] are isomorphic in that case. Hence 7.6 may be viewed as a strong generalization of Cuntz–Quillen's result.

8. Comparison with Nuss' product

In [13], Nuss considers the "twist"

$$\tau : S \otimes_R S \to S \otimes_R S, \quad \tau(s \otimes t) = st \otimes 1 + 1 \otimes st - s \otimes t.$$

It is clear that $\tau^2 = 1$ and that, for the multiplication map $\mu_0 : S \otimes_R S \to S$, we have $\mu_0\tau = \mu_0$. He shows further [13, 1.3] that τ satisfies the Yang–Baxter equation. Using τ, he introduces a ring structure on the $n+1$ fold tensor power $S \otimes_R \cdots \otimes_R S$ for all $n \geqslant 1$, by a standard procedure (use Proposition 2.3 of [2] and induction). We want to reinterpret this product in a different way. For this consider the (Amitsur) cosimplicial R-bimodule

$$\bigotimes_R S : [n] \mapsto \bigotimes_{i=0}^{n} {}_R\, S_l.$$

By definition of $\Omega_R S$, we have $N(\otimes_R S) = \Omega_R S$. Hence the Dold–Kan correspondence gives an isomorphism of cosimplicial R-bimodules

$$\bigotimes_R S \cong K\Omega_R S. \tag{67}$$

On the right hand side we also have product (49). It is noted in [13] that (67) is a ring isomorphism in codimension $\leqslant 1$. The next Proposition shows it is actually a ring isomorphism in all codimensions.

Proposition 8.1. *Equip* $\otimes_R S$ *with the product defined in* [13] *and* $K\Omega_R S$ *with that given by* (49). *Then* (67) *is an isomorphism of* $\mathfrak{Fin}$-*rings.*

Proof. Write $\bullet$ for Nuss' product. Consider the following map

$$\delta_i = \partial_{i+1}^{n-i}\partial_0^i \in \hom_\Delta([0],[n]) \quad (0 \leqslant i \leqslant n).$$

One checks the following identities hold in $\otimes_R^n S$, for $a, b \in S$:

$$\delta_i(a) \bullet \delta_j(b) = \begin{cases} \partial_{j+1}^{n-j}\partial_{i+1}^{j-i-1}\partial_0^i(a \otimes b), & i < j, \\ \delta_i(ab), & i = j, \\ -\delta_j(a) \bullet \delta_i(b) + \delta_i(ab) + \delta_j(ab), & i > j. \end{cases} \tag{68}$$

In particular $\delta_i : S \to \otimes_R^n S$ is a ring homomorphism for $\bullet$. By the universal property of $\coprod_R^n S$, we have a unique ring homomorphism $\alpha^n : \coprod_R^n S \to \otimes_R^n S$ satisfying $\alpha^n \delta_i = \delta_i$ for all i. By (69),

$$\begin{aligned} s_0 \otimes \cdots \otimes s_n &= \delta_0(s_0) \bullet \cdots \bullet \delta_n(s_n) \\ &= \alpha(\delta_0(s_0) \cdots \delta_n(s_n)). \end{aligned}$$

Thus α is surjective. On the other hand the composite of α with the isomorphism $Q\Omega_R S \xrightarrow{\sim} \coprod_R S$ sends $ds \otimes v_i$ to $q_i(s) := \delta_i(s) - \delta_0(s)$. But it follows from (68) that

$$q_i(a) \bullet q_j(b) = \begin{cases} -q_j(a) \bullet q_i(b) & (i \neq j), \\ 0 & (i = j). \end{cases} \tag{69}$$

Thus α descends to a ring homomorphism $\bar{\alpha} : K\Omega_R S \to \otimes_R S$. On the other hand, we have an R-linear map $\beta : \otimes_R S \to K\Omega_R S$, $\beta(s_0 \otimes \cdots \otimes s_n) = \delta_0(s_0) \cdots \delta_n(s_n)$. Clearly $\alpha\beta = 1$. To finish the proof it suffices to show that β is surjective. But we have

$$\begin{aligned} &a_0 da_1 \cdots da_r \otimes v_{i_1} \wedge \cdots \wedge v_{i_r} \\ &\quad = \delta_0(a_0)(\delta_1(a_1) - \delta_0(a_1)) \cdots (\delta_r(a_r) - \delta_0(a_r)) \\ &\quad \equiv \delta_0(a_0) \cdots \delta_r(a_r) \bmod \bigoplus_{i=0}^{r-1} \Omega_R^i S \otimes \Lambda^i V \\ &\quad = \beta(a_0 \otimes \cdots \otimes a_r \otimes 1 \otimes \cdots \otimes 1). \end{aligned}$$

Hence it follows by induction on r, that $\Omega_R^r S \otimes \Lambda^r V$ is included in the image of β. □

9. Dold–Kan equivalence for rings

Definition 9.1. Let $f : R \to S$ be a homomorphism in DGR^*. We say that f is a *weak equivalence* if it induces an isomorphism in cohomology. We call f a *fibration* if each $f^n : R^n \to S^n$ is surjective, and a *cofibration* if it has the left lifting property (LLP) of [14] with respect to those fibrations which are also weak equivalences (trivial fibrations). Similarly, a map $g : A \to B$ of cosimplicial rings is a *weak equivalence* if it induces an isomorphism in cohomotopy, a *fibration* if each $g^n : A^n \twoheadrightarrow B^n$ is surjective and a *cofibration* if it has the LLP with respect to trivial fibrations. It is proved in [8] that the structure just defined makes DGR^* closed model. The next proposition shows that the same is valid for cosimplicial rings.

Proposition 9.2. *With the notions of fibration, cofibration and weak equivalence defined in* 9.1, Rings^{Δ} *is a closed model category.*

Proof. A commutative version of this is given in [16], Theorem 2.1.2. Essentially the same proof works in the noncommutative case; simply substitute the coproduct $\coprod$ of Rings for $\otimes$, which is the coproduct in the category Comm of commutative rings. One only has to check that for all $n \geqslant 0$, the structure maps $\mathbb{Z} \to D(n) := TK\mathbb{Z}\langle n, n+1\rangle \in \mathrm{Rings}^{\Delta}$ induce weak equivalences

$$A \xrightarrow{\sim} A \coprod D(n) \quad (A \in \mathrm{Rings}^{\Delta}). \tag{70}$$

For this we imitate Jardine's argument [8]. We observe that if $A \in \mathrm{Rings}^{\Delta}$ and we write $C(n) = K\mathbb{Z}\langle n, n+1\rangle$ then there is an isomorphism of cosimplicial groups

$$\begin{aligned} A \coprod D(n) &= A[C(n)] \\ &:= A \oplus (A \otimes C(n) \otimes A) \oplus (A \otimes C(n) \otimes A \otimes C(n) \otimes A) \oplus \cdots \end{aligned}$$

with the product defined by

$$\begin{aligned} &(a_1 \otimes c_1 \otimes a_2 \otimes \cdots \otimes c_k \otimes a_{k+1})(a'_1 \otimes c'_1 \otimes a'_2 \otimes \cdots \otimes c'_l \otimes a'_{l+1}) \\ &\quad = (a_1 \otimes c_1 \otimes \cdots \otimes c_k \otimes a_{k+1}a'_1 \otimes \cdots \otimes c'_l \otimes a'_{l+1}) \end{aligned}$$

and cofaces and codegeneracies induced by those of A and $C(n)$. Thus to prove (70) it suffices to show that if C and D are cosimplicial groups and D is contractible, then the inclusion $\imath : C \to C[D]$ is a quasi-isomorphism. But $\operatorname{coker} \imath$ is a sum of cosimplicial groups each of which is isomorphic to one of the form $C \otimes D \otimes \cdots \otimes D \otimes C$. Hence it suffices to show that $D \otimes D'$ is contractible if D is. This latter statement follows from the following property of the cosimplicial path functor (see [16, p. 30]):

$$D^{\Delta[1]} \otimes D' = (D \otimes D')^{\Delta[1]}. \qquad \square$$

Lemma 9.3. (i) *The functor* $\tilde{Q} : DGR^* \xrightarrow{Q} \mathrm{Rings}^{\mathfrak{Fin}} \xrightarrow{forget} \mathrm{Rings}^{\Delta}$ *preserves colimits, finite limits, cofibrations, fibrations, and weak equivalences.*

(ii) *Let* $K : DGR^* \to \mathrm{Rings}^{\Delta}$ *be the functor sending* $A \mapsto KA$ *where* KA *is equipped with the product* (49). *Then there is a natural isomorphism of left derived functors* $\mathbb{L}\tilde{Q} \xrightarrow{\cong} \mathbb{L}K$.

Proof. Limits and colimits in Rings^{Δ} are computed codimensionwise, and the same is true in $\mathrm{Rings}^{\mathfrak{Fin}}$. In particular the forgetful functor preserves limits and colimits. The functor $Q : DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$ preserves colimits by Proposition 7.4. Thus $\tilde{Q}$ preserves colimits. On the other hand limits in $\mathrm{Rings}^{\mathfrak{Fin}}$ can be computed in $\mathfrak{Ab}^{\mathfrak{Fin}}$. As Q: $\mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ is exact and preserves direct sums, it follows that $\tilde{Q}$ preserves finite limits. Similarly, as the forgetful functors $DGR^* \to \mathrm{Ch}^{\geqslant 0}$ and $\mathrm{Rings}^{\Delta} \to \mathfrak{Ab}^{\Delta}$ as well as Q: $\mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ preserve fibrations and weak equivalences, it follows that $\tilde{Q}$ does. One checks, using Lemma 7.1, that $\tilde{Q}$ preserves the basic cofibrations $S(m) \to D(m)$,

$\mathbb{Z} \to D(m)$. Because it also preserves colimits it follows that if $m_i, i \in I$ is a family of positive integers and $e_i : S(m_i) \to X$ $(i \in I)$ a family of maps, then the following maps are cofibrations:

$$\tilde{Q}\left(X \rightarrowtail X \coprod_{\coprod_{i\in I} S(m_i)} \coprod_{i \in I} D(m_i)\right)$$

$$\tilde{Q}\left(X \rightarrowtail X \coprod \coprod_{i \in I} D(m_i)\right).$$

But by the proof of 9.2 and the remark [1, p. 23], every cofibration in DGR^* is a retract of one obtained as a colimit of such cofibrations. Hence $\tilde{Q}$ preserves all cofibrations. Thus (i) is proved. As shown in Section 5, the natural weak equivalence $\hat{p} : \tilde{Q}A \to KA$ of 4.2 is a homomorphism of cosimplicial rings. This proves (ii). □

Remark 9.4. A functor L_* with properties similar to those proved for $\tilde{Q}$ in Lemma 9.3 is considered in [15] for the dual situation of chain DG- and simplicial rings. The authors use the shuffle product to make the normalized chain complex of a simplicial ring into a chain DG-ring, thus obtaining a functor $N_* : \mathrm{Rings}^{\Delta^{op}} \to DGR_*$. The functor L_* is defined as the left adjoint of N_*. Dually, one can equip the normalized complex of a cosimplicial ring with the shuffle product, consider the resulting functor $\tilde{N} : \mathrm{Rings}^{\Delta} \to DGR^*$ and take its left adjoint L^*. However we point out that L^* and $\tilde{Q}$ are not isomorphic. In other words $\tilde{Q}$ is not left adjoint to $\tilde{N}$. To see this, note that, by 7.1, if $A \in \mathrm{Ch}^{\geqslant 0}$, then $\hom_{\mathrm{Rings}^{\Delta}}(\tilde{Q}TA, R) = \hom_{\mathfrak{Ab}^{\Delta}}(QA, R)$, while $\hom_{DGR^*}(TA, \tilde{N}R) = \hom_{\mathrm{Ch}^{\geqslant 0}}(A, NR) = \hom_{\mathfrak{Ab}^{\Delta}}(KA, R)$. Hence if $\tilde{Q}$ were left adjoint to $\tilde{N}$, then K and Q should be isomorphic as functors $\mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\Delta}$, which is clearly false.

Remark 9.5. We have seen in Proposition 7.4 that Q has a right adjoint P. Since the forgetful functor $U : \mathrm{Rings}^{\mathfrak{Fin}} \to \mathrm{Rings}^{\Delta}$ also has a right adjoint ([11], X.3.2), and $\tilde{Q} = UQ$, it follows that $\tilde{Q}$ is the left adjoint of an adjoint pair $(\tilde{Q}, \tilde{P})$. On the other hand, by Lemma 9.3(i), we know that $\tilde{Q}$ preserves cofibrations and weak equivalences, and thus it is the left adjoint of a Quillen adjoint functor pair [7, Definition 1.3.1].

Theorem 9.6. *The adjoint functors* $\tilde{Q} : DGR^* \leftrightarrows \mathrm{Rings}^{\Delta} : \tilde{P}$ *of* 9.3 i) *and* 9.5 *form a Quillen equivalence in the sense of* [7] 1.3.12.

Proof. Let $g : R := \tilde{P}(S)^c \overset{\sim}{\twoheadrightarrow} \tilde{P}(S)$ be the functorial cofibrant replacement obtained by the small object argument. Since the functor $\tilde{Q}$ reflects weak equivalences, it suffices to show that the adjoint map $f : \tilde{Q}R \to S$ is a weak equivalence ([7, Theorem 1.3.16]). We note for future use that by the small object argument and because $\tilde{Q}$ and $\tilde{P}$ are adjoint, the dotted arrow in the diagram below exists whenever the top horizontal arrow is in the image of $\tilde{Q}$: $\hom_{DGR^*}(S(m),R)\to \hom_{Rings^\Delta}(\tilde{Q}S(m),\tilde{Q}R)$.

$$\begin{tikzcd} \tilde{Q}S(m) \arrow[r] \arrow[d] & \tilde{Q}R \arrow[d] \\ \tilde{Q}D(m) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{71}$$

To prove that f is a weak equivalence, we must show that the following map is an isomorphism for all m

$$f: H^m N\tilde{Q}R \overset{\cong}{\to} H^m NS. \tag{72}$$

We first prove that (73) is surjective. If $x \in H^m NS$ is an element, call x the map $\mathbb{Z} \to H^m NS$, $1 \mapsto x$. Choose a cochain homomorphism $\hat{x}: \mathbb{Z}[m] \to NS$ inducing x. We have an exact sequence

$$0 \longrightarrow \mathbb{Z}[m+1] \longrightarrow \mathbb{Z}\langle m, m+1\rangle \longrightarrow \mathbb{Z}[m] \longrightarrow 0$$

Because both N and Q are exact, we have a solid line commutative diagram

$$\begin{tikzcd} NQ\mathbb{Z}[m+1] \arrow[rr, "0"] \arrow[d] & & N\tilde{Q}R \arrow[d, "f"] \\ NQ\mathbb{Z} < m, m+1 > \arrow[r] \arrow[urr, dotted, "h"] & NQ\mathbb{Z}[m] \arrow[r] \arrow[dr] & NS \\ & & \mathbb{Z}[m] \arrow[u, "x"'] \end{tikzcd} \tag{73}$$

To prove that the dotted arrow exists, apply the functor K to obtain a commutative diagram:

$$\begin{tikzcd} Q\mathbb{Z}[m+1] \arrow[r, "Q(0)"] \arrow[d] & \tilde{Q}R \arrow[d] \\ Q\mathbb{Z} < m, m+1 > \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{74}$$

Next use Lemma 7.1 to obtain a diagram of the form (71) in which the top row is in the image of Q, whence the dotted arrow exists in (71), whence also in (74) and (73). Call y the arrow $NQ\mathbb{Z}[m] \to NQR$ induced by h. Then the image of 1 through $\bar{y}: \mathbb{Z} = H^m(NQ\mathbb{Z}[m]) \to H^m N\tilde{Q}R$ maps to x under (72). This proves that (72) is surjective. To show it is also injective, let $x: \mathbb{Z}[m] \to N\tilde{Q}R$ represent an element in the kernel of (72). Then $fx: \mathbb{Z}[m] \to NS$ factors through a map $x': \mathbb{Z}\langle m-1, m\rangle \to NS$. Because $\hat{p}$ is natural we have a commutative diagram

$$\begin{tikzcd} NQ\mathbb{Z}[m] \arrow[r, "x\hat{p}"] \arrow[d] & N\tilde{Q}R \arrow[d, "f"] \\ NQ\mathbb{Z}\langle m-1, m\rangle \arrow[r, "x'\hat{p}"'] & NS \end{tikzcd}$$

Because $\hat{p}$ is an equivalence, it suffices to show that $x\hat{p}$ induces the zero map in cohomology. Next, by virtue of 4.2 there is a homotopy $fx\hat{p} \to fNQ(\overline{x\hat{p}})$. Because $Q\mathbb{Z}[m] \rightarrowtail Q\mathbb{Z}\langle m-1, m\rangle$ is a cofibration this homotopy extends to one between $x'\hat{p}$ and some map y which fits into the following commutative diagram:

$$\begin{tikzcd} NQ\mathbb{Z}[m] \arrow[r, "NQ(\overline{x\hat{p}})"] \arrow[d, tail] & N\tilde{Q}R \arrow[d, "f"] \\ NQ\mathbb{Z} < m-1, m > \arrow[r, "y"'] \arrow[ur, dotted] & NS \end{tikzcd} \tag{75}$$

The same argument used during the course of the proof of the surjectivity of (72) shows that the dotted arrow exists. Hence $x\hat{p}$ induces the zero map in cohomology, since it is homotopic to $NQ(\overline{x\hat{p}})$, and the latter induces zero by (75). □

Corollary 9.7. *The functor* $\mathbb{L}\tilde{Q}: \mathrm{Ho}DGR^* \to \mathrm{HoRings}^{\Delta}$ *of* 9.3 *is an equivalence of categories.*

Proof. Immediate from 9.6 and [7, 1.3.13]. □

Corollary 9.8. *Let* $K: \mathrm{Ch}^{\geqslant 0} \to \mathfrak{Ab}^{\Delta}$ *be the Dold–Kan functor. If* $A \in DGR^*$, *equip* KA *with the product* (49). *Then the left derived functor* $\mathbb{L}K$ *of* $DGR^* \to \mathrm{Rings}^{\Delta}$, $A \mapsto KA$ *is a category equivalence* $\mathrm{Ho}DGR^* \xrightarrow{\sim} \mathrm{HoRings}^{\Delta}$.

Proof. Immediate from 9.7 and 9.3(ii). □

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References

[1] A. Bousfield, V. Gugenheim, On *PL* de Rham theory and rational homotopy type, Mem. AMS 179 (1976) 1–94.
[2] A. Cap, H. Schichl, J. Vanžura, On twisted tensor products of algebras, Comm. Algebra 23 (12) (1995) 4701–4735.
[3] J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995) 251–289.
[4] J. Cuntz, D. Quillen, Cyclic homology and nonsingularity, J. Amer. Math. Soc. 8 (1995) 373–442.
[5] A. Dold, Homology of symmetric products and other functors of complexes, Ann. Math. 68 (1958) 54–80.
[6] J.A. Guccione, J.J. Guccione, J. Majadas, Noncommutative Hochschild homology, unpublished preprint, 1994, in Spanish.
[7] M. Hovey, Model categories, in: Mathematical Surveys and Monographs, Vol. 63, AMS, Providence, RI, 1999.

[8] J.F. Jardine, A Closed Model Structure for Differential Graded Algebras, in: J. Cuntz, M. Khalkhali (Eds.), Cyclic Homology and Noncommutative Geometry, Vol. 17, Fields Institute Communications, AMS, Providence, RI, 1997, pp. 55–58.
[9] M. Karoubi, Correspondence de Dold–Kan et formes differentielles, J. Algebra 198 (1997) 618–626.
[10] J. Loday, Cyclic Homology, Springer, Berlin, Heidelberg, New York, 1992.
[11] S. Mac Lane, Categories for the working mathematician, in: Graduate Texts in Mathematics, Vol. 5, Springer, Berlin, New York, 1971.
[12] S. Mac Lane, Homology. Die Grundlehren der mathematischen Wissenschaften, Bd. 114 Academic Press, Inc., Publishers, New York; Springer, Berlin, Göttingen, Heidelberg, 1963.
[13] P. Nuss, Noncommutative descent and nonabelian cohomology, K-theory 12 (1997) 23–74.
[14] D. Quillen, Homotopical Algebra, in: Lecture Notes in Mathematics, Vol. 43, Springer, Berlin, 1967.
[15] S. Schwede, B. Shipley, Equivalences of monoidal model categories, Algebraic Geom. Topol. 3 (2003) 287–334.
[16] B. Toen, Schematization of homotopy types, http://arXiv.org/abs/math.AG/0012219.
[17] C. Weibel, An Introduction to Homological Algebra, in: Cambridge Studies in Advanced Mathematics, Vol. 38, Cambridge University Press, Cambridge, 1994.