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Pairings in triangular Witt theory

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Abstract

Given a product $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ between triangulated categories with duality, we show that under some conditions there exist naturally two different pairings $W^i(\mathcal{K}) \times W^j(\mathcal{L}) \to W^{i+j}(\mathcal{M})$, where W^* denotes the triangulated Witt functor of Balmer [P. Balmer, K-theory 19 (2000) 311–363]. Our main example of such a situation is the case that $\mathcal{K} = \mathcal{L} = \mathcal{M}$ is the bounded derived category of vector bundles over a scheme X and $\boxtimes$ is the (derived) tensor product. The derived Witt groups of this scheme $W^*(X) := \bigoplus_{i\in\mathbb{Z}} W^i(X)$ become a graded skew-commutative ring with two different product structures which are both equally natural. In the last section we prove then a projection formula for our product and show as an application that a Springer-type theorem is true for the derived Witt groups, too.

0. Introduction

Witt theory for triangulated categories with duality was introduced recently by Balmer [1]. In this theory, we can shift the duality structure on a triangulated category $\mathcal{K}$ and obtain an infinite series of Witt groups $W^i(\mathcal{K})$ ($i \in \mathbb{Z}$), which proves to be 4-periodic: $W^i(\mathcal{K}) \simeq W^{i+4}(\mathcal{K})$ naturally. The triangulated categories environment—as opposed to the Witt theory of exact categories with duality introduced earlier—enables one to prove some fundamental properties of the Witt groups of schemes as, e.g., localization [1]. We want to introduce in this paper further structures on this triangular Witt groups; in particular we want to prove that under some circumstances there exists a product on $W^*(\mathcal{K}) := \bigoplus_{i\in\mathbb{Z}} W^i(\mathcal{K})$ making this a graded algebra.

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We consider in this paper the following situation. Let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be triangulated categories with duality and $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ a bi-exact functor. If we can make the functors $- \boxtimes L$ and $K \boxtimes -$ in a coherent way duality preserving we get a left "action" of $W^*(\mathcal{K})$ which sends $W^*(\mathcal{L})$ into $W^*(\mathcal{M})$ and an analogous right "action". In the special case $\mathcal{K} = \mathcal{L} = \mathcal{M}$ we get a left and a right product on $W^*(\mathcal{K})$. All the axioms we need are satisfied by the main example we have in mind, namely $\mathcal{K} = D^b(\mathcal{P}(X))$ the bounded derived category of vector bundles over a scheme X. Both the left and right product make the derived Witt groups of a scheme a graded, associative and skew-commutative algebra with unit. We prove in our last section a projection formula for this product. As an application we get using a trick of Bayer-Fluckiger and Lenstra [4] a Springer-type theorem for the derived Witt groups. Other interesting applications of this product structure will be found in a forthcoming paper of Balmer [3]. E.g., he generalizes and re-proves there a result of Knebusch [8] about the kernel of the natural map $W(X) \to W(\text{funct. field of } X)$ (X an integral scheme).

Now a short review of the content of this paper. In Sections 1 and 2 we give conditions under which a bi-exact functor $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ between triangulated categories with duality induces a pairing $W^r(\mathcal{K}) \times W^s(\mathcal{L}) \to W^{r+s}(\mathcal{M})$ between the triangulated Witt groups. Our point of departure is that $A \boxtimes -$ and $- \boxtimes B$ for $A \in \mathcal{K}$ and $B \in \mathcal{L}$ should become duality preserving functors with the aid of forms on them. A feature of this approach is that we easily can prove that the connecting homomorphism in the localization sequence for triangular Witt groups [1] is compatible with our product (Theorem 2.11). After that we take a closer look at the pairing/product which gives the (derived) tensor product on the derived and coherent Witt groups.

We have tried to make the first two sections as self contained as possible (except in the examples), in particular we explain all the relevant facts concerning triangular Witt theory, but in the sections about coherent and derived Witt groups of a scheme we assume some familiarity of the reader with the papers [2,6].

Notation and conventions. We denote the translation functor in a triangulated category $\mathcal{K}$ by $T_{\mathcal{K}}$ or just T if not specified otherwise.

We assume throughout that the morphism groups in all the categories under consideration are (uniquely) 2-divisible. In particular we assume that $1/2$ is in the global section of all the schemes we consider.

1. Duality, products and triangulated categories

1.1. Products

We start with the notion of a δ-exact functor ($\delta = \pm 1$). This is an additive covariant (respectively contravariant) functor $F : \mathcal{K} \to \mathcal{L}$ between the triangulated categories $\mathcal{K}$ and $\mathcal{L}$ equipped with an isomorphism of functors $\theta : FT_{\mathcal{K}} \xrightarrow{\simeq} T_{\mathcal{L}}F$ (respectively $T_{\mathcal{L}}^{-1}F \xrightarrow{\simeq} FT_{\mathcal{K}}$) such that for every exact triangle $(A, B, C, \alpha, \beta, \gamma)$ in $\mathcal{K}$ the triangle $(FA, FB, FC, F(\alpha), F(\beta), \delta\theta_A F(\gamma))$ (respectively $(FC, FB, FA, F(\beta), F(\alpha), \delta T_{\mathcal{L}} F(\gamma) T_{\mathcal{L}}(\theta_A)))$ is exact in $\mathcal{L}$.

Remark 1.1. Assume F is covariant. We then have iterated versions of θ (they exists in the contravariant case, too, but we do not need this), namely isomorphisms of functors $\theta^{(i)}: FT_{\mathcal{K}}^{i} \to T_{\mathcal{L}}^{i} F$ for all $i \in \mathbb{Z}$. They are defined as follows.

We set $\theta^{(0)} := \mathrm{id}$, $\theta^{(1)} := \theta$ and $\theta^{(-1)} := (T_{\mathcal{L}}^{-1}(\theta_{T_{\mathcal{K}}^{-1}}))^{-1}$. For $|i| \geqslant 1$ we set:

$$\theta^{(i)} := T_{\mathcal{L}}^{\mathrm{sgn}(i)}\big(\theta^{(i-\mathrm{sgn}(i))}\big) \cdot \theta^{(\mathrm{sgn}(i))}_{T_{\mathcal{K}}^{i-\mathrm{sgn}(i)}},$$

where $\mathrm{sgn}(i) = 1$ if $i > 0$ and $= -1$ if $i < 0$. By induction, we see that

$$\theta^{(i)} = T_{\mathcal{L}}^{i-\mathrm{sgn}(i)}\big(\theta^{(\mathrm{sgn}(i))}\big) \cdot \theta^{(i-\mathrm{sgn}(i))}_{T_{\mathcal{K}}^{\mathrm{sgn}(i)}} \quad \text{for } |i| \geqslant 1.$$

Clearly we could also take this equation for an inductive definition. Note that $\theta^{(j+k)} = T_{\mathcal{L}}^{k}(\theta^{(j)}) \cdot \theta^{(k)}_{T_{\mathcal{K}}^{j}}$ for all $j, k \in \mathbb{Z}$.

We define now a product between triangulated categories.

Definition 1.2. Let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be triangulated categories. A product between $\mathcal{K}$ and $\mathcal{L}$ with codomain $\mathcal{M}$ is a bi-covariant functor

$$\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$$

which is 1-exact in both variables and satisfies the following condition: the functorial isomorphisms

$$\mathfrak{r}_{A,B} : A \boxtimes (T_{\mathcal{L}} B) \xrightarrow{\simeq} T_{\mathcal{M}}(A \boxtimes B) \quad \text{and} \quad \mathfrak{l}_{A,B} : (T_{\mathcal{K}} A) \boxtimes B \xrightarrow{\simeq} T_{\mathcal{M}}(A \boxtimes B)$$

associated with the 1-exact functors $A \boxtimes -$ and $- \boxtimes B$ make the diagram

$$\begin{array}{ccc} (T_{\mathcal{K}} A) \boxtimes (T_{\mathcal{L}} B) & \xrightarrow{\mathfrak{l}_{A,T_{\mathcal{L}} B}} & T_{\mathcal{M}}(A \boxtimes T_{\mathcal{L}} B) \\ \downarrow{\scriptstyle \mathfrak{r}_{T_{\mathcal{K}} A,B}} & & \downarrow{\scriptstyle T_{\mathcal{M}}(\mathfrak{r}_{A,B})} \\ T_{\mathcal{M}}(T_{\mathcal{K}} A \boxtimes B) & \xrightarrow[T_{\mathcal{M}}(\mathfrak{l}_{A,B})]{} & T_{\mathcal{M}}^{2}(A \boxtimes B) \end{array}$$

skew-commutative for any $A \in \mathcal{K}$ and $B \in \mathcal{L}$, i.e., $T_{\mathcal{M}}(\mathfrak{r}_{A,B}) \cdot \mathfrak{l}_{A,T_{\mathcal{L}} B} = -T_{\mathcal{M}}(\mathfrak{l}_{A,B}) \cdot \mathfrak{r}_{T_{\mathcal{K}} A,B}$. If $\mathcal{K} = \mathcal{L} = \mathcal{M}$ we say that the pair $(\mathcal{K}, \boxtimes)$ is a triangulated category with product.

The functorial isomorphisms $\mathfrak{r}$ and $\mathfrak{l}$ involved in the definition above have also shifted versions $\mathfrak{r}^{(i)}$ and $\mathfrak{l}^{(i)}$, which are defined as in Remark 1.1. They satisfy the following relation.

Lemma 1.3. *For all $i, j \in \mathbb{Z}$, $A \in \mathcal{K}$ and $B \in \mathcal{L}$, the following diagram is $(-1)^{ij}$-commutative*:

$$\begin{array}{ccc} (T^i_{\mathcal{K}}A) \boxtimes (T^j_{\mathcal{L}}B) & \xrightarrow{\mathfrak{l}^{(i)}_{A,T^j_{\mathcal{L}}B}} & T^i_{\mathcal{M}}(A \boxtimes T^j_{\mathcal{L}}B) \\ \downarrow{\scriptstyle \mathfrak{r}^{(j)}_{T^i_{\mathcal{K}}A,B}} & & \downarrow{\scriptstyle T^i_{\mathcal{M}}(\mathfrak{r}^{(j)}_{A,B})} \\ T^j_{\mathcal{M}}(T^i_{\mathcal{K}}A \boxtimes B) & \xrightarrow[T^j_{\mathcal{M}}(\mathfrak{l}^{(i)}_{A,B})]{} & T^{i+j}_{\mathcal{M}}(A \boxtimes B) \end{array}$$

i.e., $T^i_{\mathcal{M}}(\mathfrak{r}^{(j)}_{A,B}) \cdot \mathfrak{l}^{(i)}_{A,T^j_{\mathcal{L}}B} = (-1)^{ij} T^j_{\mathcal{M}}(\mathfrak{l}^{(i)}_{A,B}) \cdot \mathfrak{r}^{(j)}_{T^i_{\mathcal{K}}A,B}$.

Proof. By induction (cf. Remark 1.1 above). □

Example 1.4. The main example we have in mind is the following one. Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$. We set $\mathcal{L} = \mathcal{M} = D^b_{fg}(\mathcal{M}(X))$ the bounded derived category of quasi coherent $\mathcal{O}_X$-modules with coherent homology and $\mathcal{K} = D^b(\mathcal{P}(X))$ the bounded derived category of locally free (of finite rank) $\mathcal{O}_X$-modules. Then the (derived) tensor product

$$\otimes : \mathcal{K} \times \mathcal{L} \to \mathcal{L} = \mathcal{M}$$

is a product of $\mathcal{K}$ and $\mathcal{L}$ with codomain $\mathcal{L}$.

To fix our sign convention we recall the definition of the tensor product of two complexes $(P_\bullet, d^P) \in \mathcal{K}$ and $(M_\bullet, d^M) \in \mathcal{L}$. The complex $P_\bullet \otimes M_\bullet$ is given in degree n by:

$$(P_\bullet \otimes M_\bullet)_n = \bigoplus_{i+j=n} P_i \otimes_{\mathcal{O}_X} M_j$$

and the differential $d^{P \otimes M}$ by $d^P_i \otimes \mathrm{id}_{M_j} + (-1)^i \mathrm{id}_{P_i} \otimes d^M_j$ on $P_i \otimes_{\mathcal{O}_X} M_j$. For this choice of signs the natural isomorphism $\mathfrak{l}_{P_\bullet, M_\bullet}$ is essentially the identity and $\mathfrak{r}_{P_\bullet, M_\bullet}$ is given by $(-1)^i$ on $P_i \otimes \cdots$.

Note that $\mathcal{K}$ becomes a triangulated category with product if we restrict the tensor product to the full triangulated subcategory $\mathcal{K}$ of $\mathcal{L}$.

1.2. Duality

Definition 1.5. Let $\mathcal{K}$ be a triangulated category, $\delta_{\mathcal{K}} = \pm 1$, $D_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ a $\delta_{\mathcal{K}}$-exact contravariant functor satisfying the strict condition $D_{\mathcal{K}} T_{\mathcal{K}} = T^{-1}_{\mathcal{K}} D_{\mathcal{K}}$ and $\varpi^{\mathcal{K}} : \mathrm{id} \xrightarrow{\simeq} D_{\mathcal{K}} D_{\mathcal{K}}$ an isomorphism of functors.

(1) (Balmer [1].) The quadruple $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ is called a triangulated category with $\delta_{\mathcal{K}}$-exact duality $D_{\mathcal{K}}$, if $T_{\mathcal{K}}\varpi^{\mathcal{K}}_M = \varpi^{\mathcal{K}}_{T_{\mathcal{K}}M}$ and

$$D_{\mathcal{K}}\big(\varpi^{\mathcal{K}}_M\big) \cdot \varpi^{\mathcal{K}}_{D_{\mathcal{K}}M} = \mathrm{id}_{D_{\mathcal{K}}M}.$$

(2) We say that a pair (A, ϕ) is a j-symmetric form on $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ if $A \in \mathcal{K}$ and ϕ is an isomorphism $A \xrightarrow{\simeq} T^j_{\mathcal{K}} D_{\mathcal{K}} A$ such that

$$(-1)^{j(j+1)/2}\delta^j_{\mathcal{K}} \cdot T^j_{\mathcal{K}} D_{\mathcal{K}}(\phi)\varpi^{\mathcal{K}}_A = \phi.$$

Two j-symmetric forms (A, ϕ) and (B, ψ) are called isometric, if there exists an isomorphism $\gamma : A \xrightarrow{\simeq} B$, such that

$$\phi = T^j_{\mathcal{K}} D_{\mathcal{K}}(\gamma) \cdot \psi \cdot \gamma.$$

Notation 1.6. It is an easy exercise to show that $T^i D_{\mathcal{K}}$ is then a $(-1)^i\delta_{\mathcal{K}}$-exact duality on $\mathcal{K}$ making

$$\mathcal{K}^{(i)} := \big(\mathcal{K}, T^i D_{\mathcal{K}}, (-1)^i\delta_{\mathcal{K}}, (-1)^{i(i+1)/2}\delta^i_{\mathcal{K}}\varpi^{\mathcal{K}}\big)$$

a triangulated category with duality. Clearly $\mathcal{K}^{(0)} = (\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$.

Example 1.7. 1. (Cf. [2].) Let X be a scheme and $\mathcal{K} = D^b(\mathcal{P}(X))$. Then the derived functor $\mathcal{D}_X$ of the exact functor $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ is a 1-exact functor on this triangulated category making it a triangulated category with duality.

2. (Cf. [6].) Assume moreover that X is Gorenstein and has finite Krull dimension. Then the derived functor of $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ makes $D^b_{fg}(\mathcal{M}(X))$ a triangulated category with 1-exact duality. Following [6] we call this duality the canonical duality and denote it by $\mathcal{X}_{fg}$.

It is clear that an exact functor F between triangulated categories with duality does in general not care about the duality structure on this category. In particular if the pair (A, ϕ) is a j-symmetric form there is no reason that the image $F(A)$ carries a form at all. Therefore we introduce duality preserving functors.

Definition 1.8. A duality preserving functor between triangulated categories with duality

$$\big(\mathcal{A}, D_{\mathcal{A}}, \delta_{\mathcal{A}}, \varpi^{\mathcal{A}}\big) \quad \text{and} \quad \big(\mathcal{B}, D_{\mathcal{B}}, \delta_{\mathcal{B}}, \varpi^{\mathcal{B}}\big)$$

is a pair (F, ρ), where F is a 1-exact covariant functor $\mathcal{A} \to \mathcal{B}$ and

$$\rho : F D_{\mathcal{A}} \to D_{\mathcal{B}} F$$

is an isomorphism of functors, such that the diagrams

1.

$$\begin{array}{ccc} F & \xrightarrow{F(\varpi^{\mathcal{A}})} & FD_{\mathcal{A}}D_{\mathcal{A}} \\ {\scriptstyle \varpi^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \rho_{D_{\mathcal{A}}}} \\ D_{\mathcal{B}}D_{\mathcal{B}}F & \xrightarrow[D_{\mathcal{B}}\rho]{} & D_{\mathcal{B}}FD_{\mathcal{A}} \end{array}$$

and

2.

$$\begin{array}{ccc} FT_{\mathcal{A}}D_{\mathcal{A}} & \xrightarrow{\rho_{T_{\mathcal{A}}^{-1}}} & D_{\mathcal{B}}FT_{\mathcal{A}}^{-1} \\ {\scriptstyle (\delta_{\mathcal{A}}\delta_{\mathcal{B}})\cdot\theta_{D_{\mathcal{A}}}}\downarrow & & \downarrow{\scriptstyle D_{\mathcal{B}}T_{\mathcal{B}}^{-1}\theta_{T_{\mathcal{A}}^{-1}}} \\ T_{\mathcal{B}}FD_{\mathcal{A}} & \xrightarrow[T_{\mathcal{B}}\rho]{} & T_{\mathcal{B}}D_{\mathcal{B}}F \end{array}$$

commute, where $\theta : FT_{\mathcal{A}} \xrightarrow{\simeq} T_{\mathcal{B}}F$.

The isomorphism of functors ρ is called a duality transformation for F.

Remark 1.9. For $i \in \mathbb{Z}$, we define the ith shifted duality transformation of F:

$$\rho^{(i)} := (\delta_{\mathcal{A}}\delta_{\mathcal{B}})^i \cdot T_{\mathcal{B}}^i(\rho) \cdot \theta_{D_{\mathcal{A}}}^{(i)} : FT_{\mathcal{A}}^i D_{\mathcal{A}} \to T_{\mathcal{B}}^i D_{\mathcal{B}} F.$$

We leave it to the reader to check that then $(F, \rho^{(i)})$ is a duality preserving functor for the shifted duality: $(F, \rho^{(i)}) : \mathcal{A}^{(i)} \to \mathcal{B}^{(i)}$.

We easily verify the following

Lemma 1.10. *Let in the situation of Definition* 1.8 *above* (A, ϕ) *be a* j*-symmetric form on* $\mathcal{A}$*. Then*

$$(F, \rho)_*(A, \phi) := \big(F(A), \rho_A^{(j)} \cdot F(\phi)\big)$$

is a j*-symmetric form on* $\mathcal{B}$*.*

It should be pointed out that the duality transformation ρ is not forced by the functor F. Moreover there exists functors which could be made in many different ways duality preserving. One example for this comes form the following situation. Let $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$, $(\mathcal{L}, D_{\mathcal{L}}, \delta_{\mathcal{L}}, \varpi^{\mathcal{L}})$ and $(\mathcal{M}, D_{\mathcal{M}}, \delta_{\mathcal{M}}, \varpi^{\mathcal{M}})$ be triangulated categories with duality and assume that we have a product $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$. In this situation we have exact functors $A \boxtimes -$ and $- \boxtimes B$ for every $A \in \mathcal{K}$ and $B \in \mathcal{L}$. We want to make these functors duality preserving in a way which depends on the forms on A, respectively B. For this we need isomorphisms $\eta_{A,B}: D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B \xrightarrow{\simeq} D_{\mathcal{M}}(A \boxtimes B)$ satisfying some properties given by the following

Definition 1.11. In the situation above, we say that the product $\boxtimes: \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is a dualizing pairing between the triangulated categories with duality $\mathcal{K}^{(0)}$ and $\mathcal{L}^{(0)}$ with codomain $\mathcal{M}^{(0)}$, if there are isomorphisms

$$\eta_{A,B}: D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B \to D_{\mathcal{M}}(A \boxtimes B)$$

functorial in A and in B which make the following two diagrams commutative:

1.

$$\begin{array}{ccc} A \boxtimes B & \xrightarrow{\varpi^{\mathcal{K}}_A \boxtimes \varpi^{\mathcal{L}}_B} & D^2_{\mathcal{K}}A \boxtimes D^2_{\mathcal{L}}B \\ {\scriptstyle \varpi^{\mathcal{M}}_{A\boxtimes B}}\downarrow & & \downarrow{\scriptstyle \eta_{D_{\mathcal{K}}A, D_{\mathcal{L}}B}} \\ D^2_{\mathcal{M}}(A \boxtimes B) & \xrightarrow[D_{\mathcal{M}}(\eta_{A,B})]{} & D_{\mathcal{M}}(D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B) \end{array}$$

2.

$$\begin{array}{ccccc} T_{\mathcal{M}}(D_{\mathcal{K}}T_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B) & \xleftarrow{\mathfrak{l}_{D_{\mathcal{K}}T_{\mathcal{K}}A, D_{\mathcal{L}}B}} & D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B & \xrightarrow{\mathfrak{r}_{D_{\mathcal{K}}A, D_{\mathcal{L}}T_{\mathcal{L}}B}} & T_{\mathcal{M}}(D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}T_{\mathcal{L}}B) \\ {\scriptstyle (\delta_{\mathcal{K}}\delta_{\mathcal{M}})\cdot}\downarrow{\scriptstyle T_{\mathcal{M}}(\eta_{T_{\mathcal{K}}A,B})} & & \downarrow{\scriptstyle \eta_{A,B}} & & {\scriptstyle (\delta_{\mathcal{L}}\delta_{\mathcal{M}})\cdot}\downarrow{\scriptstyle T_{\mathcal{M}}(\eta_{A,T_{\mathcal{L}}B})} \\ T_{\mathcal{M}}D_{\mathcal{M}}((T_{\mathcal{K}}A) \boxtimes B) & \xleftarrow[T_{\mathcal{M}}D_{\mathcal{M}}(\mathfrak{l}_{A,B})]{} & D_{\mathcal{M}}(A \boxtimes B) & \xrightarrow[T_{\mathcal{M}}D_{\mathcal{M}}(\mathfrak{r}_{A,B})]{} & T_{\mathcal{M}}D_{\mathcal{M}}(A \boxtimes T_{\mathcal{L}}B)\ . \end{array}$$

If $\mathcal{K}^{(0)} = \mathcal{L}^{(0)} = \mathcal{M}^{(0)}$, we say that $\mathcal{K}^{(0)} = (\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}}, \boxtimes)$ is a triangulated category with duality and product.

We will need the following iterated version of diagram 2 above.

Lemma 1.12. *For all $A \in \mathcal{K}$, $B \in \mathcal{L}$ and all $i \in \mathbb{Z}$ the following diagram commutes*:

$$\begin{array}{ccccc} T^i_{\mathcal{M}}(D_{\mathcal{K}}T^i_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B) & \xleftarrow{\mathfrak{l}^{(i)}_{D_{\mathcal{K}}T^i_{\mathcal{K}}A, D_{\mathcal{L}}B}} & D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B & \xrightarrow{\mathfrak{r}^{(i)}_{D_{\mathcal{K}}A, D_{\mathcal{L}}T^i_{\mathcal{L}}B}} & T^i_{\mathcal{M}}(D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}T^i_{\mathcal{L}}B) \\ {\scriptstyle (\delta_{\mathcal{K}}\delta_{\mathcal{M}})^i\cdot}\downarrow{\scriptstyle T^i_{\mathcal{M}}(\eta_{T^i_{\mathcal{K}}A,B})} & & \downarrow{\scriptstyle \eta_{A,B}} & & {\scriptstyle (\delta_{\mathcal{L}}\delta_{\mathcal{M}})^i\cdot}\downarrow{\scriptstyle T^i_{\mathcal{M}}(\eta_{A,T^i_{\mathcal{L}}B})} \\ T^i_{\mathcal{M}}D_{\mathcal{M}}(T^i_{\mathcal{K}}A \boxtimes B) & \xleftarrow[T^i_{\mathcal{M}}D_{\mathcal{M}}(\mathfrak{l}^{(i)}_{A,B})]{} & D_{\mathcal{M}}(A \boxtimes B) & \xrightarrow[T^i_{\mathcal{M}}D_{\mathcal{M}}(\mathfrak{r}^{(i)}_{A,B})]{} & T^i_{\mathcal{M}}D_{\mathcal{M}}(A \boxtimes T^i_{\mathcal{L}}B)\ . \end{array}$$

Proof. By induction (cf. Remark 1.1). □

Example 1.13. 1. Let X be any scheme and $\mathcal{K} = D^b(\mathcal{P}(X))$. This category is a triangulated category with duality $\mathcal{D}_X = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ (cf. Example 1.7). As we have seen this is also a category with product, given by the (derived) tensor product $\otimes_{\mathcal{O}_X}$. The reader will easily verify that the natural identification $\mathcal{D}_X(P_\bullet) \otimes_{\mathcal{O}_X} \mathcal{D}_X(Q_\bullet) \xrightarrow{\simeq} \mathcal{D}_X(P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet)$ makes it a triangulated category with duality and product.

2. Assume now that X is Gorenstein of finite Krull dimension. Let $\mathcal{K}$ be as above and $\mathcal{L} = \mathcal{M} = D^b_{fg}(\mathcal{M}(X))$. We have on $\mathcal{K}$ the usual duality structure and on $\mathcal{L}$ the canonical duality $\mathcal{X}_{fg}$. Again the canonical isomorphism $\mathcal{D}_X(P_\bullet) \otimes_{\mathcal{O}_X} \mathcal{X}_{fg} M_\bullet \xrightarrow{\simeq} \mathcal{X}_{fg}(P_\bullet \otimes_{\mathcal{O}_X} M_\bullet)$ makes this a dualizing product.

1.3. Making $A \boxtimes -$ and $- \boxtimes B$ duality preserving

Assume now that we have a dualizing pairing $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ as in Definition 1.11. Let (A, ϕ) be a i-symmetric form on $\mathcal{K}$ and (B, ψ) a j-symmetric form on $\mathcal{L}$. We want to use ϕ and ψ to make the 1-exact functors $A \boxtimes -$, respectively $- \boxtimes B$ duality preserving. We do this in the following way:

1. A duality transformation $\mathfrak{L}(A, \phi)$ for $A \boxtimes -$ is defined by:

$$A \boxtimes D_{\mathcal{L}} L \xrightarrow{\phi \boxtimes \mathrm{id}_{D_{\mathcal{L}} L}} T^i_{\mathcal{K}} D_{\mathcal{K}} A \boxtimes D_{\mathcal{L}} L \xrightarrow{\mathfrak{l}^{(i)}_{D_{\mathcal{K}} A, D_{\mathcal{L}} L}} T^i_{\mathcal{M}}(D_{\mathcal{K}} A \boxtimes D_{\mathcal{L}} L)$$
$$\xrightarrow{T^i_{\mathcal{M}}(\eta_{A,L})} T^i_{\mathcal{M}} D_{\mathcal{M}}(A \boxtimes L)$$

   for any $L \in \mathcal{L}$.
2. A duality transformation $\mathfrak{R}(B, \psi)$ for $- \boxtimes B$ is defined by:

$$D_{\mathcal{K}} K \boxtimes B \xrightarrow{\mathrm{id}_{D_{\mathcal{K}} K} \boxtimes \psi} D_{\mathcal{K}} K \boxtimes T^j_{\mathcal{L}} D_{\mathcal{L}} B \xrightarrow{\mathfrak{r}^{(j)}_{D_{\mathcal{K}} K, D_{\mathcal{L}} B}} T^j_{\mathcal{M}}(D_{\mathcal{K}} K \boxtimes D_{\mathcal{L}} B)$$
$$\xrightarrow{T^j_{\mathcal{M}}(\eta_{K,B})} T^j_{\mathcal{M}} D_{\mathcal{M}}(K \boxtimes B)$$

   for any $K \in \mathcal{K}$.

Lemma 1.14. *The pairs*

$$\big(A \boxtimes -, \mathfrak{L}(A, \phi)\big) : \mathcal{L}^{(0)} \to \mathcal{M}^{(i)} \quad \textit{and} \quad \big(- \boxtimes B, \mathfrak{R}(B, \psi)\big) : \mathcal{K}^{(0)} \to \mathcal{M}^{(j)}$$

are duality preserving functors.

Proof. We prove this for $(A \boxtimes -, \mathfrak{L}(A,\phi))$, the proof for $(- \boxtimes B, \mathfrak{R}(B,\psi))$ is analogous. We have to verify the commutativity of the two diagrams in Definition 1.8. The first of them commutes for the following reasons:

$$
\begin{aligned}
&T^i_{\mathcal{M}} D_{\mathcal{M}}\big(\mathfrak{L}(A,\phi)_L\big) \cdot \delta^i_{\mathcal{M}}(-1)^{i(i+1)/2}\,\varpi^{\mathcal{M}}_{A\boxtimes L}\\
&\quad= \delta^i_{\mathcal{M}}(-1)^{i(i+1)/2} T^i_{\mathcal{M}} D_{\mathcal{M}}(\phi \boxtimes \mathrm{id}_{D_{\mathcal{L}}L}) \cdot T^i_{\mathcal{M}} D_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, D_{\mathcal{L}}L}\big) \cdot D_{\mathcal{M}}(\eta_{A,L}) \cdot \varpi^{\mathcal{M}}_{A\boxtimes L}\\
&\quad= \delta^i_{\mathcal{M}}(-1)^{i(i+1)/2} T^i_{\mathcal{M}} D_{\mathcal{M}}(\phi \boxtimes \mathrm{id}_{D_{\mathcal{L}}L}) \cdot T^i_{\mathcal{M}} D_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, D_{\mathcal{L}}L}\big)\\
&\qquad \cdot \eta_{D_{\mathcal{K}}A, D_{\mathcal{L}}L} \cdot \varpi^{\mathcal{K}}_A \boxtimes \varpi^{\mathcal{L}}_L \quad \text{(Definition 1.11)}\\
&\quad= \delta^i_{\mathcal{M}}(-1)^{i(i+1)/2} T^i_{\mathcal{M}} D_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{T^{-i}_{\mathcal{K}}A, D_{\mathcal{L}}L}\big) \cdot \eta_{T^{-i}_{\mathcal{K}}A, D_{\mathcal{L}}L} \cdot T^i_{\mathcal{K}} D_{\mathcal{K}}(\phi) \boxtimes \mathrm{id}_{D^2_{\mathcal{L}}L}\\
&\qquad \cdot \varpi^{\mathcal{K}}_A \boxtimes \varpi^{\mathcal{L}}_L \quad (\eta, \mathfrak{l}^{(i)} \text{ nat.})\\
&\quad= (\delta_{\mathcal{K}}\delta_{\mathcal{M}})^i \cdot T^i_{\mathcal{M}} D_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{T^{-i}_{\mathcal{K}}A, D_{\mathcal{L}}L}\big) \cdot \eta_{T^{-i}_{\mathcal{K}}A, D_{\mathcal{L}}L} \cdot \phi \boxtimes \varpi^{\mathcal{L}}_L \quad (\phi \text{ is } i\text{-form})\\
&\quad= T^i_{\mathcal{M}}(\eta_{A, D_{\mathcal{L}}L}) \cdot \mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, D^2_{\mathcal{L}}L} \cdot \phi \boxtimes \varpi^{\mathcal{L}}_L \quad \text{(Lemma 1.12)}\\
&\quad= \mathfrak{L}(A,\phi)_{D_{\mathcal{L}}L} \cdot \mathrm{id}_A \boxtimes \varpi^{\mathcal{L}}_L.
\end{aligned}
$$

Consider the second diagram in the definition. We have:

$$
\begin{aligned}
&T^{i+1}_{\mathcal{M}} D_{\mathcal{M}}(\mathfrak{r}_{A, T^{-1}_{\mathcal{L}}L}) \cdot \mathfrak{L}(A,\phi)_{T^{-1}_{\mathcal{L}}L}\\
&\quad= (\delta_{\mathcal{L}}\delta_{\mathcal{M}}) T^{i+1}_{\mathcal{M}}(\eta_{A,L}) \cdot T^i_{\mathcal{M}}(\mathfrak{r}_{D_{\mathcal{K}}A, D_{\mathcal{L}}L}) \cdot \mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, T_{\mathcal{L}}D_{\mathcal{L}}L} \cdot \phi \boxtimes \mathrm{id}_{T_{\mathcal{L}}D_{\mathcal{L}}L}\\
&\qquad \text{(Definition 1.11)}\\
&\quad= (-1)^i (\delta_{\mathcal{L}}\delta_{\mathcal{M}}) T^{i+1}_{\mathcal{M}}(\eta_{A,L}) \cdot T_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, D_{\mathcal{L}}L}\big) \cdot \mathfrak{r}_{T^i_{\mathcal{K}}D_{\mathcal{K}}A, D_{\mathcal{L}}L} \cdot \phi \boxtimes \mathrm{id}_{T_{\mathcal{L}}D_{\mathcal{L}}L}\\
&\qquad \text{(Lemma 1.3)}\\
&\quad= (-1)^i (\delta_{\mathcal{L}}\delta_{\mathcal{M}}) T^{i+1}_{\mathcal{M}}(\eta_{A,L}) \cdot T_{\mathcal{M}}\big(\mathfrak{l}^{(i)}_{D_{\mathcal{K}}A, D_{\mathcal{L}}L}\big) \cdot T_{\mathcal{M}}(\phi \boxtimes \mathrm{id}_{D_{\mathcal{L}}L}) \cdot \mathfrak{r}_{A, D_{\mathcal{L}}L}\\
&\qquad (\mathfrak{r} \text{ nat.})\\
&\quad= (-1)^i (\delta_{\mathcal{L}}\delta_{\mathcal{M}}) T_{\mathcal{M}}\big(\mathfrak{L}(A,\phi)_L\big) \cdot \mathfrak{r}_{A, D_{\mathcal{L}}L}.
\end{aligned}
$$

So this diagram commutes, too, and we are done. □

Observe now the following fact, which follows easily from Lemma 1.3.

Lemma 1.15. *The identity* $\mathrm{id}_{A\boxtimes B}$ *is an isometry between*

$$\big(A \boxtimes -,\ \mathfrak{L}(A,\phi)\big)_*(B,\psi) \quad \textit{and} \quad \big(- \boxtimes B,\ \mathfrak{R}(B,\psi)\big)_*\big(A, (-1)^{ij}(\delta_{\mathcal{K}}\delta_{\mathcal{M}})^j(\delta_{\mathcal{L}}\delta_{\mathcal{M}})^i\phi\big).$$

2. Triangular Witt theory

2.1. Triangular Witt groups

In the following $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ is a triangulated category with $\delta_{\mathcal{K}}$-exact duality. We start by recalling Balmer's definition of the Witt groups of $\mathcal{K}$ (cf. [1]). For this we have first to specify which spaces are regarded as trivial. The definition is the following one.

Definition 2.1 (Balmer). Let (V, φ) be a 0-symmetric form on $\mathcal{K}$. It is called neutral, if there exists an exact triangle

$$T_{\mathcal{K}}^{-1} D_{\mathcal{K}} L \xrightarrow{\nu} L \xrightarrow{\iota} V \xrightarrow{D_{\mathcal{L}}(\iota)\varphi} D_{\mathcal{K}} L$$

in $\mathcal{K}$ with $T_{\mathcal{K}}^{-1} D_{\mathcal{K}}(\nu) = \delta_{\mathcal{K}}(\varpi_L^{\mathcal{K}} \cdot \nu)$. In this case $L \xrightarrow{\iota} V$ is called a Lagrangian of the 0-symmetric form (V, φ).

The isometry classes of 0-symmetric forms on $\mathcal{K}$ with the orthogonal sum as operation form a monoid $MW(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$. It is easy to check that the sum of two neutral forms is again neutral, hence the isometry classes $NW(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ of neutral forms are a submonoid.

Definition 2.2 (Balmer). We set

$$W\big(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}}\big) := \frac{MW(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})}{NW(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})}.$$

and define for $i \in \mathbb{Z}$ the ith triangular Witt group of $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ as:

$$W^i(\mathcal{K}) := W\big(\mathcal{K}^{(i)}\big)$$

(cf. Notation 1.6 for the definition of $\mathcal{K}^{(i)}$). Clearly $W^0(\mathcal{K}) = W(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$.

The monoid $W^i(\mathcal{K})$ is in fact a group, since the orthogonal sum $(V, \varphi) \perp (V, -\varphi)$ is neutral. The class of an i-symmetric form (V, φ) in $W^i(\mathcal{K})$ will be denoted by $[V, \varphi]$.

Remark 2.3. Note that $W^s(\mathcal{K}^{(i)}) = W^{s+i}(\mathcal{K}^{(0)})$.

Example 2.4. We use the notation of Example 1.7. $W^i(X) := W^i(D^b(\mathcal{P}(X)), \mathcal{D}_X)$ is the ith derived Witt group of the scheme X and $\widetilde{W}^i(X) := W^i(D^b_{fg}(\mathcal{M}(X)), \mathcal{X}_{fg})$ is the ith coherent Witt group of the Gorenstein scheme (of finite Krull dimension) X.

We have then the following

Theorem 2.5. 1. [2, Theorem 4.3]. *Let $W(X)$ and $W^-(X)$ be the (usual) Witt groups of symmetric, respectively skew-symmetric, spaces of the scheme X (cf. for the definition* [8] *or* [9]*). Then the functor $\mathcal{P}(X) \to D^b(\mathcal{P}(X))$ induces isomorphisms*

$$W(X) \xrightarrow{\simeq} W^0(X) \quad \textit{and} \quad W^-(X) \xrightarrow{\simeq} W^2(X).$$

2. [6, Corollary 2.17]. *If X is a regular scheme of finite Krull dimension, then the natural functor $D^b(\mathcal{P}(X)) \to D^b_{fg}(\mathcal{M}(X))$ is duality preserving and induces (cf. Theorem* 2.6*) an isomorphisms*

$$W^s(X) \xrightarrow{\simeq} \widetilde{W}^s(X)$$

for all $s \in \mathbb{Z}$.

As one could expect, duality preserving functors between triangulated categories induce homomorphisms between the triangulated Witt groups.

Theorem 2.6. *Let $(F, \rho) : (\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}}) \to (\mathcal{L}, D_{\mathcal{L}}, \delta_{\mathcal{L}}, \varpi^{\mathcal{L}})$ be a duality preserving functor. Then F induces a homomorphism*

$$W^i(F, \rho) : W^i(\mathcal{K}) \to W^i(\mathcal{L}), \quad [A, \phi] \mapsto \left[F(A), \rho_A^{(i)} \cdot F(\phi)\right] = \left[(F, \rho)_*(A, \phi)\right]$$

(cf. Lemma 1.10*) for all $i \in \mathbb{Z}$, where (A, ϕ) is an i-symmetric form on $\mathcal{K}$. Moreover, if F is an equivalence of categories, then $W^i(F, \rho)$ is an isomorphism for all $i \in \mathbb{Z}$.*

Proof. By Remark 1.9 it is enough to prove this for $i = 0$. We leave the straightforward verifications to the reader with the hint to use [1, Theorem 3.5] to show that the homomorphism $W^i(F, \rho)$ is injective if F is an equivalence. □

Remark 2.7. It is important to note that the homomorphism induced by a duality preserving functor on the Witt groups depends on the duality transformation ρ. E.g., assume we have a finite dimensional vector space V over a field endowed with two bilinear forms which are not Witt equivalent. Then the exact functor $V \otimes -$ can be made in two different ways duality preserving such that the induced maps on the Witt groups are not the same (cf. Lemma 1.14).

Example 2.8. $(T^2_{\mathcal{K}}, \mathrm{id}) : (\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}}) \to (\mathcal{K}, T^4_{\mathcal{K}} D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ is a duality preserving functor. It induces isomorphisms $W^n(\mathcal{K}) \xrightarrow{\simeq} W^{n+4}(\mathcal{K})$ for all $n \in \mathbb{Z}$, which proves the 4-periodicity of the Witt groups.

2.2. The pairing on the Witt groups

Let now $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ be a dualizing pairing between the triangulated categories with duality $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi^{\mathcal{K}})$ and $(\mathcal{L}, D_{\mathcal{L}}, \delta_{\mathcal{L}}, \varpi^{\mathcal{L}})$ with codomain $(\mathcal{M}, D_{\mathcal{M}}, \delta_{\mathcal{M}}, \varpi^{\mathcal{M}})$. Let further (A,ϕ) be an i-symmetric form on $\mathcal{K}$ and (B,ψ) a j-symmetric form on $\mathcal{L}$. Because

$$\big(A\boxtimes -, \mathfrak{L}(A,\phi)\big):\mathcal{L}^{(0)}\to\mathcal{M}^{(i)} \quad\text{and}\quad \big(-\boxtimes B, \mathfrak{R}(B,\psi)\big):\mathcal{K}^{(0)}\to\mathcal{M}^{(j)}$$

are duality preserving functors (Lemma 1.14) we get by Theorem 2.6 well defined homomorphisms

$$W^m\big(\mathcal{L}^{(0)}\big)\to W^m\big(\mathcal{M}^{(i)}\big),\quad [L,\beta]\mapsto W^m\big(A\boxtimes -, \mathfrak{L}(A,\phi)\big)(L,\beta)$$

and

$$W^m\big(\mathcal{K}^{(0)}\big)\to W^m\big(\mathcal{M}^{(j)}\big),\quad [K,\alpha]\mapsto W^m\big(-\boxtimes B, \mathfrak{R}(B,\psi)\big)(K,\alpha).$$

From Lemma 1.15 we deduce that $W^m(A\boxtimes -, \mathfrak{L}(A,\phi))$ and $W^m(-\boxtimes B, \mathfrak{R}(B,\psi))$ depend only on the Witt class of the forms (A,ϕ) respectively (B,ψ), hence we have

Theorem 2.9. *In the situation above we have a left pairing*

$$\star_l: W^r(\mathcal{K})\times W^s(\mathcal{L})\to W^{r+s}(\mathcal{M}),\quad \big([K,\alpha],[L,\beta]\big)\mapsto W^m\big(K\boxtimes -, \mathfrak{L}(K,\alpha)\big)(L,\beta)$$

and a right pairing

$$\star_r: W^r(\mathcal{K})\times W^s(\mathcal{L})\to W^{r+s}(\mathcal{M}),\quad \big([K,\alpha],[L,\beta]\big)\mapsto W^m\big(-\boxtimes L, \mathfrak{R}(L,\beta)\big)(K,\alpha)$$

for any $r,s\in\mathbb{Z}$. These pairings are related by the following formula:

$$k\star_l l = (-1)^{rs}(\delta_{\mathcal{K}}\delta_{\mathcal{M}})^s(\delta_{\mathcal{L}}\delta_{\mathcal{M}})^r k\star_r l,$$

where $k\in W^r(\mathcal{K})$ and $l\in W^s(\mathcal{L})$. In particular if $\mathcal{K}^{(0)}=\mathcal{L}^{(0)}=\mathcal{M}^{(0)}$ then

$$W^*(\mathcal{K}) := \bigoplus_{m\in\mathbb{Z}} W^m(\mathcal{K})$$

is a graded algebra with a left product and a right product.

2.3. The pairing and the localization sequence

We discuss here only the left pairing. Clearly we have analogous results for the right pairing.

We start by recalling the localization sequence. For this we assume that our triangulated category $\mathcal{L}$ satisfies the enriched octahedron axiom (e.g., [1, Section 1]). This axiom is true in all the triangulated categories of our interest such as the derived categories of exact categories or their full subcategories and localizations.

Let $S \subseteq \operatorname{Mor}\mathcal{L}$ be a localizing class of morphisms compatible with the triangulated structure of $\mathcal{L}$ (cf. [5, p. 251]). Then $S^{-1}\mathcal{L}$ is also a triangulated category and we have an exact sequence of triangulated categories

$$\mathcal{L}_S \xrightarrow{\iota} \mathcal{L} \xrightarrow{q} S^{-1}\mathcal{L},$$

where $\mathcal{L}_S$ is the kernel of the quotient functor $q : \mathcal{L} \to S^{-1}\mathcal{L}$, i.e., the full subcategory of $\mathcal{L}$ of objects M with $qM \simeq 0$.

If $(\mathcal{L}, D_{\mathcal{L}}, \delta_{\mathcal{L}}, \varpi^{\mathcal{L}})$ is a triangulated category with duality and $D_{\mathcal{L}}S = S$ the categories $\mathcal{L}_S$ and $S^{-1}\mathcal{L}$ are also triangulated categories with $\delta_{\mathcal{L}}$-exact duality. Their dualities are induced by $D_{\mathcal{L}}$ and we denote them (following the convention in [1]) by $D_{\mathcal{L}}$, too. We have then a long exact sequence

$$\cdots \to W^m(\mathcal{L}_S) \xrightarrow{W^m(\iota)} W^m(\mathcal{L}) \xrightarrow{W^m(q)} W^m\big(S^{-1}\mathcal{L}\big) \xrightarrow{\partial} W^{m+1}(\mathcal{L}_S) \to \cdots$$

(the functors ι and q become in a canonical way duality preserving, hence we suppress the corresponding duality transformation). From the functorial properties of this localization sequence (e.g., [6, Theorem 2.9]) we get:

Theorem 2.10. *Assume we have a dualizing pairing* $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ *as above and localizing classes of morphisms* $S \subseteq \operatorname{Mor}\mathcal{L}$ *and* $T \subseteq \operatorname{Mor}\mathcal{M}$ *which are compatible with the dualities. Let* (A, ϕ) *be an* i*-symmetric form on* $\mathcal{K}$ *and assume* $\mathrm{id}_A \boxtimes s \in T$ *for all* $s \in S$*. Then we have a commutative diagram with exact rows*:

$$\begin{array}{ccccccccc}
\cdots & W^m(\mathcal{L}) & \longrightarrow & W^m(S^{-1}\mathcal{L}) & \xrightarrow{\partial} & W^{m+1}(\mathcal{L}_S) & \longrightarrow & \cdots \\
& \downarrow{\scriptstyle W^m(A\boxtimes -,\mathfrak{L}(A,\phi))} & & \downarrow{\scriptstyle W^m(A\boxtimes -,\mathfrak{L}(A,\phi))} & & \downarrow{\scriptstyle W^{m+1}(A\boxtimes -,\mathfrak{L}(A,\phi))} & & \\
\cdots & W^{m+i}(\mathcal{M}) & \longrightarrow & W^{m+i}(T^{-1}\mathcal{M}) & \xrightarrow{\partial} & W^{m+1+i}(\mathcal{M}_T) & \longrightarrow & \cdots,
\end{array}$$

where $\mathcal{M}_T$ *and* $T^{-1}\mathcal{M}$ *have the obvious meaning and* $W^*(A \boxtimes -, \mathfrak{L}(A,\phi))$ *denotes the induced duality preserving functors on* $\mathcal{L}_S$*, respectively* $S^{-1}\mathcal{L}$*.*

It follows from Lemma 1.15 that the homomorphism

$$W^m\big(A \boxtimes -, \mathfrak{L}(A,\phi)\big) : W^m\big(S^{-1}\mathcal{L}\big) \to W^{m+i}\big(T^{-1}\mathcal{M}\big)$$

depends on the Witt class of the space (A, ϕ) only. We want to show that the same is true for the restricted homomorphism $W^m(\mathcal{I}_S) \to W^{m+i}(\mathcal{M}_T)$. For this it is enough to show that if (A, ϕ) is neutral, i.e., $[A, \phi] = 0$ in $W^i(\mathcal{L})$, the induced morphism $W^*(A \boxtimes -, \mathfrak{L}(A,\phi)) : W^m(\mathcal{I}_S) \to W^{m+i}(\mathcal{M}_T)$ is the zero map (note that by [1, Theorem 3.6] a form represents zero in the Witt group if and only if it is neutral).

For let $L \xrightarrow{\iota} A$ be a Lagrangian in $\mathcal{L}$ of the neutral form (A, ϕ) and (X, φ) an m-symmetric form in $\mathcal{L}_S$. Then (X, φ) is also a m-symmetric form in $\mathcal{L}$ and $(- \boxtimes X, \mathfrak{R}(X, \varphi))$ a duality preserving functor $\mathcal{K}^{(0)} \to \mathcal{M}^{(m)}$ (Lemma 1.14). We leave it to the reader to check that $L \boxtimes X \xrightarrow{\iota \boxtimes \mathrm{id}_X} A \boxtimes X$ is a Lagrangian for

$$\mathfrak{R}(X,\varphi)_*(A,\phi) = (-1)^{im}(\delta_{\mathcal{K}}\delta_{\mathcal{M}})^m(\delta_{\mathcal{L}}\delta_{\mathcal{M}})^i\,\mathfrak{L}(A,\phi)_*(X,\varphi) \quad \text{(Lemma 1.15)}$$

in $\mathcal{M}$. But by the assumption of Theorem 2.10 above $L \boxtimes X$ is in $\mathcal{M}_T$ and so this is also a Lagrangian in $\mathcal{M}_T$. Our claim follows.

Summarizing we have shown that there are (left) pairings

$$W^i(\mathcal{K}) \times W^m(\mathcal{L}_S) \to W^{m+i}(\mathcal{M}_T) \quad \text{and} \quad W^i(\mathcal{K}) \times W^m\big(S^{-1}\mathcal{L}\big) \to W^{m+i}\big(T^{-1}\mathcal{M}\big).$$

We denote these (left) pairings also by $\star_l$. With this notation we can formulate

Theorem 2.11. *We have for all $i, j \in \mathbb{Z}$ a commutative diagram*

$$\begin{array}{ccc} W^i(\mathcal{K}) \times W^j(S^{-1}\mathcal{L}) & \xrightarrow{\star_l} & W^{i+j}(T^{-1}\mathcal{M}) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\partial} & & \Big\downarrow{\scriptstyle \partial} \\ W^i(\mathcal{K}) \times W^{j+1}(\mathcal{L}_S) & \xrightarrow{\star_l} & W^{i+j+1}(\mathcal{M}_T) \end{array}$$

Proof. Follows from Theorem 2.10. □

3. The pairing between derived and coherent Witt groups

3.1. The product structures on the derived Witt groups

Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$. From Example 1.13 we know that the bounded derived category $D^b(\mathcal{P}(X))$ of locally free $\mathcal{O}_X$-modules (of finite rank) is a triangulated category with product and duality. Hence Theorem 2.9 tells us that we have a left and right product, denoted by $\star_l$ respectively $\star_r$, on

$$W^*(X) := \bigoplus_{i\in\mathbb{Z}} W^i(X)$$

making this a graded algebra. We see at once that the class of the symmetric space $\mathcal{O}_X \xrightarrow{\mathrm{id}} \mathcal{O}_X$ is an unit in this ring and that the natural functor $\mathcal{P}(X) \to D^b(\mathcal{P}(X))$ (cf. Theorem 2.5) induces a ring homomorphism $W(X) \to W^*(X)$.

As well known we have natural isomorphism $P \otimes_{\mathcal{O}_X} (Q \otimes_{\mathcal{O}_X} R) \simeq (P \otimes_{\mathcal{O}_X} Q) \otimes_{\mathcal{O}_X} R$ and $P \otimes_{\mathcal{O}_X} Q \simeq Q \otimes_{\mathcal{O}_X} P$, where P, Q and R are complexes in $D^b(\mathcal{P}(X))$ and $\otimes_{\mathcal{O}_X}$ means the derived tensor product in $D^b(\mathcal{P}(X))$. We leave the straightforward verification to the reader that these induce isometries

$$[P,\phi]\star_l\big([Q,\psi]\star_l[R,\xi]\big)\overset{\simeq}{\to}\big([P,\phi]\star_l[Q,\psi]\big)\star_l[R,\xi]\quad\text{and}$$
$$[P,\phi]\star_l[Q,\psi]\overset{\simeq}{\to}(-1)^{ij}[Q,\psi]\star_l[P,\phi]$$

(and also for the right product $\star_r$), where (P,ϕ) is an i-symmetric form, (Q,ψ) is a j-symmetric form and (R,ξ) is a k-symmetric form on $D^b(\mathcal{P}(X))$. Altogether we have:

Theorem 3.1. *The left $\star_l$ and right product $\star_r$ make $W^*(X)$ a graded, associative and skew-commutative algebra with one, such that the natural identification $W(X)\overset{\simeq}{\to}W^0(X)$ is an algebra isomorphism for both the left and right product which coincide on $W^0(X)$.*

This product structure is functorial in the following way:

Theorem 3.2. *Let $f:X\to Y$ be a morphism between noetherian schemes. Then the induced homomorphism*

$$W^*(f^*):W^*(Y)\to W^*(X)$$

is an algebra homomorphism for the left and the right product, where $f^*:D^b(\mathcal{P}(Y))\to D^b(\mathcal{P}(X))$.

Proof. Recall first that f^* becomes in a natural way duality preserving, hence the suppressing of a duality transformation. We denote the map on forms (cf. Lemma 1.10) just by f^*, too.

Let (P,ϕ) be an i-symmetric form and (Q,ψ) a j-symmetric form on $D^b(\mathcal{P}(Y))$. As well known, we have a natural isomorphism $f^*(P)\otimes_{\mathcal{O}_X}f^*(Q)\overset{\simeq}{\to}f^*(P\otimes_{\mathcal{O}_Y}Q)$. The reader will see at once that this isomorphism induces an isometry

$$\big(f^*(P)\otimes_{\mathcal{O}_X}-,\ \mathfrak{L}\big(f^*(P,\phi)\big)\big)_*\big(f^*(Q,\psi)\big)\overset{\simeq}{\to}f^*\big(\big(P\otimes_{\mathcal{O}_Y}-,\ \mathfrak{L}(P,\phi)\big)_*(Q,\psi)\big),$$

hence our result. □

3.2. The pairing $W^r(X)\times\widetilde{W}^s(X)\to\widetilde{W}^{r+s}(X)$ for a Gorenstein scheme X

We now turn to the situation of Example 1.13. Let X be a Gorenstein scheme of finite Krull dimension. The derived tensor product

$$\otimes_{\mathcal{O}_X}:D^b\big(\mathcal{P}(X)\big)\times D^b_{fg}\big(\mathcal{M}(X)\big)\to D^b_{fg}\big(\mathcal{M}(X)\big)$$

is a dualizing pairing, and hence we get from Theorem 2.9 the existence of a left- and a right pairing

$$\star_l\quad\text{respectively}\quad\star_r:W^r(X)\times\widetilde{W}^s(X)\to\widetilde{W}^{r+s}(X).$$

We observe that $[\mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_X] \in W^0(X)$ acts as identity on $\widetilde{W}^*(X) := \bigoplus_{r\in\mathbb{Z}} \widetilde{W}^r(X)$ and that the canonical isomorphism $(P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet) \otimes_{\mathcal{O}_X} M_\bullet \simeq P_\bullet \otimes_{\mathcal{O}_X} (Q_\bullet \otimes_{\mathcal{O}_X} M_\bullet)$, where $P_\bullet, Q_\bullet \in D^b(\mathcal{P}(X))$ and $M_\bullet \in D^b_{fg}(\mathcal{M}(X))$, gives (by isometry) the equations

$$(\alpha \star_l \beta) \star_l \tau = \alpha \star_l (\beta \star_l \tau) \quad \text{and} \quad (\alpha \star_r \beta) \star_r \tau = \alpha \star_r (\beta \star_r \tau)$$

in $\widetilde{W}^*(X)$, where $\alpha, \beta \in W^*(X)$ and $\tau \in \widetilde{W}^*(X)$. Hence:

Theorem 3.3. *Both the left and right pairing $W^r(X) \times \widetilde{W}^s(X) \to \widetilde{W}^{r+s}(X)$ make $\widetilde{W}^*(X)$ a graded unital left module over the algebra $W^*(X)$.*

Let now $f : X \to Y$ be a flat morphism between two Gorenstein schemes of finite Krull dimension. As well known (cf., e.g., [7]) the pull back functor $f^* : D^b_{fg}(\mathcal{M}(Y)) \to D^b_{fg}(\mathcal{M}(X))$ can be made in a canonical way duality preserving and induces so a homomorphism $\widetilde{W}^*(f^*) : \widetilde{W}^*(Y) \to \widetilde{W}^*(X)$. Using the same arguments as in the proof of Theorem 3.2 we get the following result (we formulate this only for the left pairing which anyway seems to be more natural in this situation).

Theorem 3.4. *The map $\widetilde{W}^*(f^*) : \widetilde{W}^*(Y) \to \widetilde{W}^*(X)$ is a graded module homomorphism, i.e.,*

$$\widetilde{W}^{r+s}(f^*)(\alpha \star_l \beta) = W^r(f^*)(\alpha) \star_l \widetilde{W}^s(f^*)(\beta)$$

for $\alpha \in W^r(Y)$ and $\beta \in \widetilde{W}^s(Y)$.

Remark 3.5. We have analogous results for the Witt groups with support, e.g., a pairing

$$W^r(X) \times \widetilde{W}^s_Z(X) \to \widetilde{W}^{r+s}_Z(X),$$

where $Z \subseteq X$ is a closed subscheme of the Gorenstein scheme of finite Krull dimension X. We leave this to the reader.

4. A projection formula

Let R be a commutative noetherian ring (with 1) and S an R-algebra which is a finite projective R-module. We assume that there exists $\tau \in \mathrm{Hom}_R(S, R)$ such that

$$S \simeq \mathrm{Hom}_S(S, S) \to \mathrm{Hom}_R(S, R), \quad g \mapsto \tau \cdot g$$

is an isomorphism of R-modules. Then for all $Q \in \mathcal{P}(S)$ (i.e., Q is a finitely generated projective S-module) the natural morphism $\mathrm{Hom}_S(Q, S) \to \mathrm{Hom}_R(Q, R)$, $g \mapsto \tau \cdot g$ is an isomorphism of projective R-modules. Clearly this functorial bijection induces a functorial isomorphism of complexes $\rho^\tau_{Q_\bullet} : \mathrm{Hom}_S(Q_\bullet, S) \xrightarrow{\simeq} \mathrm{Hom}_R(Q_\bullet, R)$, where $Q_\bullet \in D^b(\mathcal{P}(S))$. Now S is a projective and finite R-module, hence we have a well defined push forward functor $\mathrm{Tr}_{S/R}: D^b(\mathcal{P}(S)) \to D^b(\mathcal{P}(R))$. The pair $(\mathrm{Tr}_{S/R}, \rho^\tau)$ is then a duality preserving functor which induces homomorphisms

$$\mathrm{Tr}^\tau_{S/R*}: W^i(S) \to W^i(R)$$

for all $i \in \mathbb{Z}$, by Theorem 2.6.

Let now X be a noetherian scheme over R, $X_S = S \times_R X$ and $\pi: X_S \to X$ the projection. The canonical morphisms $X_S \to S$ and $X \to R$ give pullback functors $D^b(\mathcal{P}(S)) \to D^b(\mathcal{P}(X_S))$ respectively $D^b(\mathcal{P}(R)) \to D^b(\mathcal{P}(X))$. We use them to get dualizing pairings $D^b(\mathcal{P}(S)) \times D^b(\mathcal{P}(X_S)) \to D^b(\mathcal{P}(X_S))$ and $D^b(\mathcal{P}(R)) \times D^b(\mathcal{P}(X)) \to D^b(\mathcal{P}(X))$ (we leave here the obvious details to the reader). The by now well known arguments give left pairings

$$W^r(S) \times W^s(X_S) \to W^{r+s}(X_S) \quad \text{and} \quad W^r(R) \times W^s(X) \to W^{r+s}(X),$$

which we also denote $\star_l$ (we hope this will cause no confusion). Clearly $W^*(X_S)$ and $W^*(X)$ become in this way graded $W^*(S)$- respectively $W^*(R)$-modules. The R-linear map τ induces also a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_{X_S} \simeq \mathcal{H}\mathrm{om}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}, \mathcal{O}_{X_S}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\pi_*\mathcal{O}_{X_S}, \mathcal{O}_X)$. Since S is flat over R this is an isomorphism. The arguments above give then for all $i \in \mathbb{Z}$ transfer homomorphisms

$$\mathrm{Tr}^\tau_{X_S/X*}: W^i(X_S) \to W^i(X).$$

Theorem 4.1 (Projection formula). *Let $x \in W^i(X)$ and $s \in W^j(S)$. Then*

$$\mathrm{Tr}^\tau_{X_S/X*}\big(s \star_l \pi^*(x)\big) = \mathrm{Tr}^\tau_{S/R*}(s) \star_l x.$$

Proof. It is easy to check that any element $x \in W^i(X)$ can be represented by an i-symmetric form $(Q_\bullet, \psi)$ on $D^b(\mathcal{P}(X))$, such that $Q_\bullet \xrightarrow{\psi} T^i(\mathcal{D}_X Q_\bullet)$ is an isomorphism of complexes, i.e., not a fraction (here $\mathcal{D}_X$ is the canonical duality in $D^b(\mathcal{P}(X))$ and T the translation functor in this derived category). Let further $s = [P_\bullet, \varphi] \in W^j(S)$.

Then the left side of the equation can be represented on the (r, s)-component $(r + s = n)$ by:

$$\begin{aligned} P_s \otimes_R Q_r &\to \mathcal{H}\mathrm{om}_{\mathcal{O}_{X_S}}(P_{-s+j} \otimes Q_{-r+i}, \mathcal{O}_X), \\ p \otimes q &\mapsto \big\{d \otimes c \mapsto (-1)^{s+ij}\tau\big(\varphi(p)(d)\big) \cdot \psi(q)(c)\big\}. \end{aligned}$$

The reader checks at once that this form represents also the right side of the equation. □

We conclude with a nice application. The proof is just an adaption of the verification of the analogous result for the classical Witt groups given by Bayer-Fluckiger and Lenstra in [4].

Theorem 4.2. *Let X be a scheme over a field K and L a finite extension of K of odd degree. Then the natural homomorphism*

$$W^i(X) \to W^i(X \times_K L)$$

is injective for all $i \in \mathbb{Z}$.

Proof. It is enough to show this for a simple extension L/K. Since L is of odd degree over K there exists a K-linear map $L \xrightarrow{\tau} K$, so that the corresponding transfer morphism $\mathrm{Tr}^{\tau}_{L/K\,*}$ satisfies $\mathrm{Tr}^{\tau}_{L/K\,*}([L, \mathrm{id}]) = [K, \mathrm{id}]$ (e.g., [9, proof of I, Proposition 10.3.1]). Our result follows from the projection formula. □

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