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Projection-Splitting Algorithms for Monotone Variational Inequalities

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Abstract—We consider and analyze some new projection-splitting algorithms for solving monotone variational inequalities by using the technique of updating the solution. Our modification is in the spirit of the extragradient method. The modified methods converge for monotone continuous operators. The new iterative method differs from the existing projection methods. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Variational inequalities, Projection method, Fixed point, Convergence.

1. INTRODUCTION

Variational inequalities theory provides us the most general, natural, simple, unified, and efficient framework for studying a wide class of unrelated problems arising in pure and applied sciences, see, for example, [1–15] and the references therein. In recent years, considerable interest has been shown in developing powerful and efficient numerical techniques for solving variational inequalities and related optimization problems. There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle, linear approximation, Newton's method, and descent framework for solving variational inequalities. Projection method and its variant forms represent an important tool for finding the approximate solutions, the origin of which can be traced back to Lions and Stampacchia [8]. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem by using the concept of projection. This alternative formulation has played a significant and useful part in suggesting and analyzing various iterative methods for solving variational inequalities and complementarity problems, see [2–12,14,16,17]. It is well known that the convergence of the classical projection method requires the operator to be strongly monotone and Lipschitz continuous. These retrictive conditions rule out many applications of the projection method for a class of problems. The extragradient method [2,14] overcomes this difficulty by performing an additional forward step and a projection at each iteration according to the double projection formula. Its convergence requires only that a solution exists and the operator is Lipschitz continuous. Noor [11,12] has modified the extragradient method by performing an additional forward step and projection at each iteration by using the technique of

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updating the solution. As a result of this modification, we have a double projection formula. Using this modification, a number of iterative schemes have been suggested and analyzed for solving monotone variational inequalities. Some of these methods are similar to the splitting methods of Peaceman and Rachford [13] and Douglas and Rachford [18]. In this paper, we use the technique of updating the solution to modify the double projection in the spirit of the extragradient method to suggest a number of projection-splitting methods for solving the monotone variational inequalities. One of our proposed methods is compatible with the θ-scheme of Glowinski and Le Tallec, see [5,6], and the other one can be considered as a Douglas and Rachford type splitting method. For the applications of the splitting methods in partial differential equations, see [19] and the references therein. The convergence analysis of these new methods requires the operator to be only monotone, which is much weaker than the requirements for the convergence of other projection and extragradient methods, see [2,5,6,14,17].

2. FORMULATION

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a closed convex set in H and $T : H \longrightarrow H$ be a nonlinear operator. We now consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K. \tag{2.1}$$

Problem (2.1) is called the variational inequality, which was introduced and studied by Stampacchia [15] in 1964. It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering, and applied sciences can be studied in the unified and general framework of the variational inequality (2.1), see [1–12,14–17] and the references therein.

If $K^* = \{u \in H : \langle u, v \rangle \geq 0, \text{ for all } v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (2.2) is equivalent to finding $u \in K$ such that

$$Tu \in K^* \qquad \text{and} \qquad \langle Tu, u \rangle = 0, \tag{2.2}$$

which are known as the generalized complementarity problems. Such problems have been studied extensively in the literature, see, for example, [1–3,5–14,16,17].

LEMMA 2.1. *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.3}$$

if and only if

$$u = P_K(z),$$

where P_K is the projection of H onto K. Also, the projection operator P_K is nonexpansive.

3. MAIN RESULTS

In this section, we use the projection technique to suggest some iterative methods for solving the variational inequalities. For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

LEMMA 3.1. *The function $u \in K$ is a solution of (2.1) if and only if $u \in H$ satisfies the relation*

$$u = P_K[u - \rho Tu], \tag{3.1}$$

where $\rho > 0$ is a constant.

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative formulation is very important from the numerical analysis point of view. This fixed-point formulation was used to suggest and analyze the following iterative method.

ALGORITHM 3.1. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T u_n], \qquad n = 0, 1, 2, \ldots,$$

where $0 < \rho < 2\alpha/\beta^2$, α is the strongly monotonicity constant and $\beta > 0$ is the Lipschitz continuity constant of the nonlinear operator T, see [2,9].

We note that the projection method requires the restrictive assumption that T must be strongly monotone and Lipschitz continuous for convergence. To overcome this difficulty, a technique of updating u was used to suggest the double formula. Equation (3.1) can be written as

$$u = P_K[u - \rho T P_K[u - \rho T u]. \tag{3.2}$$

This fixed-point formulation enables us to suggest the following iterative method, which is known as the extragradient method, see [2,6,14,17].

ALGORITHM 3.2. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T P_K[u_n - \rho T u_n]], \qquad n = 0, 1, 2, \ldots.$$

We now modify the extragradient method under which the modification entails an additional forward step and a projection step at each iteration. By updating u, we may write equation (3.1) in the form

$$u = P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]] = P_K[I - \rho T] P_K[I - \rho T](u), \tag{3.3}$$

which also involves the double projection. This fixed-point formulation is used to suggest the following.

ALGORITHM 3.3. (See [11].) For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= P_K[P_K[u_n - \rho T u_n] - \rho T P_K[u_n - \rho T u_n]], \\ &= P_K[I - \rho T] P_K[I - \rho T](u_n), \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{3.4}$$

This method is similar to splitting method of Peaceman and Rachford [13]. It consists of two forward-backward steps, where the order of T and P_K has not been changed. For the convergence analysis of Algorithm 3.3, see [11,16].

We now again modify the double projection formula in the spirit of the extrgradient method. By updating the solution u, equation (3.1) can be written as

$$\begin{aligned} u &= P_K[P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]] - \rho T P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]]] \\ &= P_K[I - \rho T] P_K[I - \rho T] P_K[I - \rho T](u). \end{aligned}$$

This fixed-point formulation is used to suggest the following iterative method.

ALGORITHM 3.4. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[I - \rho T] P_K[I - \rho T] P_K[I - \rho T](u_n), \qquad n = 0, 1, 2, \ldots.$$

We remark that Algorithm 3.4 is called the forward-backward projection-splitting method and is comparable with the θ-scheme of Glowinski and Le Tallec, see [5,6], which consists of three forward-backward steps. It has been shown [5] that θ-scheme is efficient and more pratical relative to other splitting type algorithms.

We now define the residue vector by the relation

$$\begin{aligned} R(u) = u - P_K[P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]] \\ - \rho T P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]]]. \end{aligned} \tag{3.5}$$

From Lemma 3.1, it is clear that $u \in K$ is a solution of (2.1) if and only if $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{3.6}$$

For a positive stepsize $\gamma \in (0, 2)$, equation (3.6) can be written as

$$u + \rho Tu = u + \rho Tu - \gamma R(u).$$

This fixed-point formulation enables us to suggest the following new implicit method for solving the variational inequalities (2.1).

ALGORITHM 3.5. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n + \rho Tu_n - \rho Tu_{n+1} - \gamma R(u_n), \qquad n = 0, 1, 2, \dots. \tag{3.7}$$

Note that for $\gamma = 1$, Algorithm 3.4 collapses to the following.

ALGORITHM 3.6. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= (I + \rho T)^{-1}[P_K[P_K[P_K[u_n - \rho Tu_n] - \rho TP_K[u_n - \rho Tu_n]] \\ &\quad - \rho TP_K[P_K[u_n - \rho Tu_n] - \rho TP_K[u_n - \rho Tu_n]]] + \rho Tu_n], \\ &= J_T[P_K[I - \rho T]P_K[I - \rho T]P_K[I - \rho T] + \rho T](u_n), \qquad n = 0, 1, 2, \dots, \end{aligned}$$

where $J_T = (I + \rho T)^{-1}$ is the resolvent operator.

Algorithm 3.6 is called the forward-backward projection-splitting method for solving the variational inequalities (2.1). This method is a new one and can be considered as a Douglas-Rachord type splitting method [18] for solving the variational inequalities. This method is a generalization of a modified forward-backward splitting method of Tseng [17]. One can obtain a number of decomposition methods by invoking appropriately special cases of convex programming and related optimization problems.

For the convergence analysis of Algorithm 3.5, we need the following results, which are proved by modifying the ideas of He [7] and Noor [11,12,16].

LEMMA 3.2. *Let $\bar{u} \in K$ be solution of (2.1). If $T : H \longrightarrow H$ is a monotone operator, then*

$$\langle u - \overline{u} + \rho(Tu - T\overline{u}), R(u)\rangle \geq \|R(u)\|^2, \qquad \text{for all } u \in K. \tag{3.8}$$

PROOF. Let $\overline{u} \in K$ be a solution of (2.1). Then

$$\langle T\overline{u}, v - \overline{u}\rangle \geq 0, \qquad \text{for all } v \in K. \tag{3.9}$$

Taking $v = P_K[P_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]] - \rho TP_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]]]$ in (3.9), we have

$$\begin{aligned} &\rho\langle T\overline{u}, P_K[P_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]] - \rho TP_K[P_K[u - \rho Tu] \\ &\qquad -\rho TP_K[u - \rho Tu]]] - \overline{u}\rangle \geq 0. \end{aligned} \tag{3.10}$$

Letting $u = P_K[P_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]] - \rho TP_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]]]4$, $z = u - \rho Tu$, $v = \overline{u}$ in (2.3); and using (3.5), we have

$$\begin{aligned} &\langle R(u) - \rho Tu, P_K[P_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]] \\ &\qquad -\rho TP_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]]] - \overline{u}\rangle \geq 0. \end{aligned} \tag{3.11}$$

Adding (3.10), (3.11), and using (3.5), we have

$$\langle R(u) - \rho(Tu - T\overline{u}),\ u - \overline{u} - R(u)\rangle \geq 0. \tag{3.12}$$

From (3.12) by rearranging the terms, we have

$$\begin{aligned} \langle u - \overline{u} + \rho(Tu - T\overline{u}), R(u)\rangle &\geq \langle R(u), R(u)\rangle + \rho\langle Tu - T\overline{u}, u - \overline{u}\rangle \\ &\geq \|R(u)\|^2, \qquad \text{using the monotonicity of } T. \end{aligned}$$

LEMMA 3.3. *Let $\overline{u} \in K$ be a solution of (2.1), and u_{n+1} be the approximate solution obtained from Algorithm 3.5, then*

$$\|u_{n+1} - \overline{u} + \rho(Tu_{n+1} - T\overline{u})\|^2 \leq \|u_n - \overline{u} + \rho(Tu_n - T\overline{u})\|^2 - \gamma(2-\gamma)\|R(u)\|^2. \tag{3.13}$$

PROOF. Combining (3.7) and (3.8), we have

$$\begin{aligned}\|u_{n+1} - \overline{u} + \rho(Tu_{n+1} - T\overline{u})\|^2 &= \|u_n - \overline{u} + \rho(Tu_n - T\overline{u}) - \gamma R(u_n)\|^2 \\ &\leq \|u_n - \overline{u} + \rho(Tu_n - T\overline{u})\|^2 - 2\gamma\langle u_n - \overline{u} + \rho(Tu_n - T\overline{u}), R(u_n)\rangle \\ &\quad + \gamma^2\|R(u_n)\|^2 \\ &\leq \|u_n - \overline{u} + \rho(Tu_n - T\overline{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2.\end{aligned}$$

THEOREM 3.1. *Let u_{n+1} be the approximate solution obtained from Algorithm 3.5 and $\overline{u} \in K$ be the solution of (2.1), then $\lim_{n\to\infty} u_n = \overline{u}$.*

PROOF. Let u be a solution of (2.1). Then from (3.13), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|u_0 - u + \rho(Tu_0 - Tu)\|^2,$$

which implies that

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\overline{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_i}\}$ of the sequence $\{u_n\}$ converge to $\overline{u}$. Since $R(u)$ is continuous, it follows that

$$R(\overline{u}) = \lim_{i\to\infty} R(u_{n_i}) = 0,$$

which implies that $\overline{u}$ is a solution of (2.1) by invoking Lemma 3.1. Consequently,

$$\|u_{n+1} - \overline{u} + \rho(Tu_{n+1} - T\overline{u})\|^2 \leq \|u_n - \overline{u} + \rho(Tu_n - T\overline{u})\|^2,$$

and we see that the sequence $\{u_n\}$ has exactly one cluster point. Thus,

$$\lim_{n\to\infty} u_n = \overline{u} \in K,$$

satisfying the variational inequality (2.1).

We remark that to implement Algorithm 3.5, one has to find the approximate solution u_{n+1} implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another method.

For a positive stepsize $\gamma \in (0,2)$, we rewrite equation (3.6) in the form

$$u = u - \gamma R(u).$$

This fixed-point formulation is used to suggest the following.

ALGORITHM 3.7. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - \gamma R(u_n), \qquad n = 0, 1, 2, \ldots. \tag{3.14}$$

Note that for $\gamma = 1$, Algorithm 3.7 collapses to Algorithm 3.4. For the convergence analysis of Algorithm 3.7, we need the following results, which can be proved by the techniques of Lemma 3.2.

LEMMA 3.4. *Let $\overline{u} \in K$ be a solution of (2.1). If $T : H \longrightarrow H$ is a monotone operator, then*

$$\langle u - \overline{u}, R(u) \rangle \geq \|R(u)\|^2, \qquad \text{for all } u \in K. \tag{3.15}$$

PROOF. Its proof is very similar to the one in [11,16].

LEMMA 3.5. *Let $\overline{u} \in K$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.7, then*

$$\|u_{n+1} - \overline{u}\|^2 \leq \|u_n - \overline{u}\|^2 - \gamma(2 - \gamma)\|R(u_n)\|^2.$$

PROOF. From (3.14) and (3.15), we have

$$\|u_{n+1} - \overline{u}\|^2 = \|u_n - \overline{u} - \gamma R(u_n)\|^2 \leq \|u_n - \overline{u}\| - \gamma(2 - \gamma)\|R(u_n)\|^2.$$

Following the technique of Theorem 3.1 and invoking Lemma 3.5, one can easily prove that the approximate solution u_{n+1} obtained from Algorithm 3.7 converges to the exact solution $\overline{u} \in K$ of the monotone variational inequality (2.1).

4. CONCLUSION

In this paper, we have suggested and analyzed a number of new projection-splitting type methods for solving the monotone variational inequalities by using the technique of updating the solution. These methods are suggested in the spirit of the extragradient methods by adding a step forward and a projection step at each iteration. The convergence of these methods requires only the monotonicity of the operator, which is a much weaker condition than the requirements for the convergence of other methods. The development and refinement of these methods need further research efforts. For related work, see [20,21].

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