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Edge Disjoint Placement of Graphs

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1. NOTATION AND INTRODUCTION

We shall use standard graph theory notation. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. All graphs will be assumed to have neither loops nor multiple edges.

We say that graphs G and H, containing an equal number of vertices, are *mutually placeable* (m.p.) if there is a bijection $\sigma : V(H) \to V(G)$ so that $\{i, j\} \in E(H)$ implies $\{\sigma i, \sigma j\} \notin E(G)$. We shall call such a σ a *placement*. Equivalently, G and H are m.p. if there exist edge disjoint copies of G, H as subgraphs of the complete graph on n vertices. In the succeeding sections, we find the following sufficient conditions for G, H to be m.p..

(1) If $|E(G)|, |E(H)| \leqslant n - 2$, then G, H are m.p.

(2) If $|E(G)||E(H)| < \binom{n}{2}$, then G, H are m.p.

(3) Set α equal the maximal degree of the points of G and β equal the maximal degree of the points of H. If $2\alpha\beta < n$, then G, H are m.p.

We also show these results are best possible, or nearly best possible, in certain senses. We note that G, H are m.p. iff $\bar{G}$ contains a copy of H. We use this to show, from condition (3), that if every point of G_1 has degree $\geqslant \frac{3}{4}$, then G_1 contains any prescribed H of maximal degree 2.

Result (1) solves a conjecture of Milner and Welsh [3] which provided the original motivation for this paper.

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2. The case $|E(G)|, |E(H)| \leqslant n - 2$

The object of this section is to prove the following.

THEOREM 1. *If* $|E(G)|, |E(H)| \leqslant n - 2$, *when* $n = |V(H)| = |V(G)|$, *then* G, H *are mutually placeable.*

Proof. We use induction on n. The theorem is straightforward for small n, say $n \leqslant 5$. Now fix G, H on n vertices and assume the theorem true for any smaller number of vertices. It suffices to show G, H are m.p. under the additional hypothesis $|E(G)| = |E(H)| = n - 2$. For then, given any G_1, H_1 satisfying the theorem's hypotheses we extend arbitrarily to G, H with $n - 2$ edges and the bijection σ for G, H works also for G_1, H_1.

By a *tree* in G or H let us mean an isolated component with t vertices and $t - 1$ edges, some t. This includes the cases $t = 1$, isolated point, and $t = 2$, isolated edge. Each t vertex component which is not a tree must have at least t edges. Thus, by an elementary counting argument, G and H must each contain at least two trees. The proof splits into several cases, with case (a) containing the main idea.

Case (a): Neither G nor H contain isolated points or edges. In a tree with $\geqslant 3$ vertices we may always find an end point α of degree 1 joined to a vertex β of degree >1. Let us use the notation $\alpha \to \beta$ if vertex α is joined *only* to vertex β.

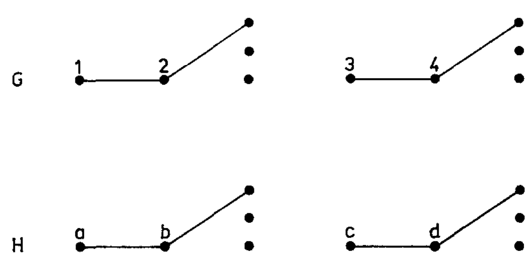

FIG. 1. Case (a).

In G we find in the two trees vertices 1, 2, 3, 4 such that $1 \to 2$, $3 \to 4$, 2 and 4 have degrees >1 and $\{2, 3\} \notin E(G)$ (since they are in different components). In H we find vertices a, b, c, d, $a \to b$, $c \to d$, and the analogous properties. Let

$$G^* = G \,|(V(G) - \{1, 2, 3, 4\}), \qquad H^* = H \,|(V(H) - \{a, b, c, d\}).$$

Then $|E(G^*)| \leqslant |E(G)| - 4 = |V(G^*)| - 2$ and $|E(H^*)| \leqslant |E(H)| - 4 = |V(H^*)| - 2$, since in G, the vertices $\{1, 2, 3, 4\}$ are on at least 4 edges, and similarly for H. Hence we find a placement

$$\sigma^* : V(G^*) \to V(H^*).$$

Extend σ^* to

$$\sigma : V(G) \to V(H)$$

defined by

$$\sigma(x) = \sigma^*(x), \qquad x \neq 1, 2, 3, 4,$$
$$\sigma(1) = b, \qquad \sigma(3) = d,$$
$$\sigma(2) = c, \qquad \sigma(4) = a.$$

We claim σ is a placement. For suppose $\{i, j\} \in E(G)$. If $i, j \notin \{1, 2, 3, 4\}$, then $\{\sigma_i, \sigma_j\} \notin E(H)$ since σ^* is a placement. Suppose $i = 1$. Then $\{i, j\} \in E(G)$ only if $j = 2$ but $\{\sigma 1, \sigma 2\} = \{b, c\} \notin E(H)$. Suppose $i = 2$. Then if $E(H)$ contains $\{\sigma_i, \sigma_j\} = \{c, \sigma_j\}$ we must have $\sigma j = d$, so $j = 3$ but $\{2, 3\} \notin E(G)$. The cases $i = 3, 4$ are analogous.

Case (b): There is a component of size 2 in either G or H. We may assume $\{1, 2\}$ is a two-point component in G. The remaining $n - 2$ vertices have $n - 3$ edges so we find a point $3 \in V(G)$ of degree $\geqslant 2$.

Subcase (b1): H has no isolated point. Take a tree component and find $a, b \in V(H)$, $a \to b$. Pick $c \in V(H)$, $\{b, c\} \notin E(H)$, c of maximal degree. (Some c, $\{b, c\} \notin E(H)$, exists as there are only $n - 2$ edges.) If $\deg(b) = 1$, then $\deg(c) \geqslant 2$. Otherwise, we find $\deg(c) \geqslant 1$ unless all edges of H contain b. In any case $\{a, b, c\}$ are in at least 3 edges. Now we find a placement σ^*: $V(G) - \{1, 2, 3\} \to V(H) - \{a, b, c\}$ and extend to σ by $\sigma(1) = b$, $\sigma(2) = c$, $\sigma(3) = a$.

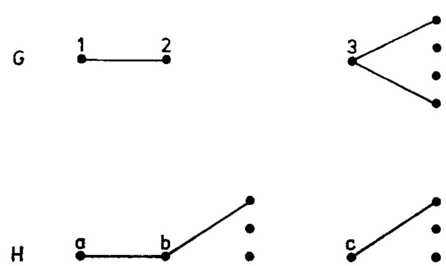

FIG. 2. Case (b1)

Subcase (b2): H has an isolated point, say $a \in V(H)$. Pick $b \in V(H)$ of maximal degree and $c \in V(H)$ of maximal degree so that $c \neq a$, $\{b, c\} \notin E(H)$.

Subsubcase (b21): If no such c exists H must be an isolated point a plus a star with center b of degree $n - 2$. In this case set $\sigma(1) = b$, $\sigma(2) = a$ rest arbitrary.

Continuation of (b2): By elementary counting $\deg(b) \geqslant 2$ and, excluding case (b21), $\deg(c) \geqslant 1$. We find a placement $\sigma^* : V(G) - \{1, 2, 3\} \to V(H) - \{a, b, c\}$ and extend by $\sigma(1) = b$, $\sigma(2) = c$, $\sigma(3) = a$.

Case (c): There are no two point components but there are isolated points. Suppose $1 \in V(G)$ is an isolated point. Pick $2 \in V(G)$ with $\deg(2) \geqslant 2$.

Subcase (c1): H has an isolated point a. Pick $b \in V(H)$, $\deg(b) \geqslant 2$. Find a placement $\sigma^* : V(G) - \{1, 2\} \to V(H) - \{a, b\}$ and extend by $\sigma(1) = b$, $\sigma(2) = b$.

Subcase (c2): H has no isolated points. If H has two-point components

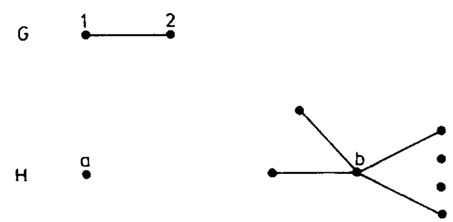

FIG. 3. Case (b21)

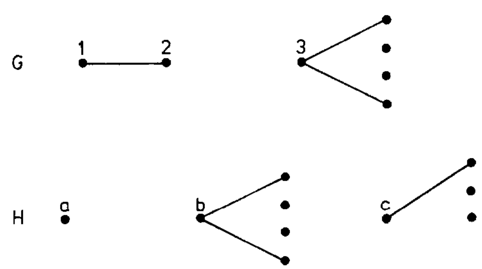

FIG. 4. Case (b2)

we use case (b). Otherwise we find a component with points $a, b \in V(H)$, $a \to b$, $\deg(b) \geqslant 2$. Find a placement with points $a, b \in V(H)$, $a \to b$, deg (b) $\geqslant 2$. Find a placement $\sigma^* : V(G) - \{1, 2\} \to V(H) - \{a, b\}$ and extend by $\sigma(1) = b$, $\sigma(2) = a$. Q.E.D.

EXAMPLE. Theorem 1 is, in a certain sense, best possible. Let G be a star,

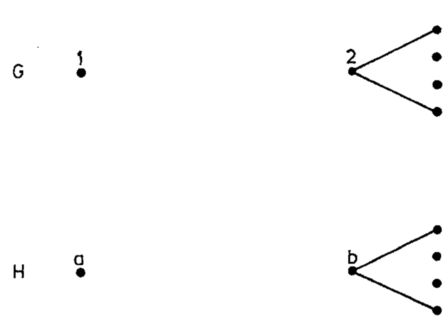

FIG. 5. Case (c1)

with $n - 1$ edges, and H be a graph with no isolated points and $\{\frac{1}{2}\}$ edges. Then G, H are not mutually placeable.

The object of this section is to prove the following:

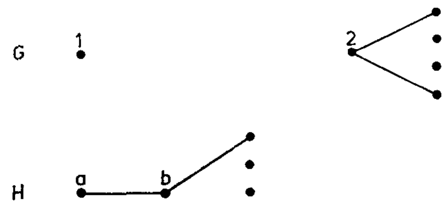

FIG. 6. Case (c2)

THEOREM 2. *If* $|E(G)||E(H)| < \binom{n}{2}$ *then* G, H *are mutually placeable.*

We note that for n even the example of the previous section shows that Theorem 2 is best possible.

Proof. Fix G, H satisfying the conditions of Theorem 2. We use the *probabilistic method.* Let σ be a *random* bijection from $V(G)$ to $V(H)$. More precisely, consider the probability space whose $n!$ points are the possible bijections σ, each with probability $n!^{-1}$. For any $e = \{i, j\} \in E(G), f = \{a, b\} \in E(H)$, let A_{ef} denote the event $\sigma(e) = f$. Then

$$\text{Prob}[A_{ef}] = 2(n-2)!/n! = \binom{n}{2}^{-1}.$$

Let $A = \bigvee A_{ef}$, the disjunction over all $e \in E(G), f \in E(H)$. Then

$$\begin{aligned} \text{Prob}[A] &= \text{Prob}[\bigvee A_{ef}] \\ &\leqslant \sum \text{Prob}[A_{ef}] \\ &= |E(G)|\,|E(H)| \binom{n}{2}^{-1} < 1. \end{aligned}$$

Hence for some fixed σ the event A does not hold. That is, σ is a placement. Q.E.D.

COROLLARY 1. *If n even, $n = 2m$, and $|E(G)| + |E(H)| < 3m - 2$, then G, H are mutually placeable.*

If n odd, $n = 2m - 1$, and $|E(G)| + |E(H)| \leqslant 3m$, then G, H are mutually placeable.

These results follow from application of Theorems 1 and 2. They are best possible by Example 1.

4. Maximal Degree Conditions

THEOREM 3.[1] *Let* $|V(G)| = |V(H)| = n$. *Let* α *be the maximal degree of the vertices of* G *and* β *the maximal degree of the vertices of* H. *Assume*

$$2\alpha\beta < n.$$

Then G, H *are mutually placeable.*

Proof. Let $\sigma: V(G) \to V(H)$ be the bijection which minimizes $|\sigma(E(G)) \cap E(H)|$. Assume $\{a, c\} \in \sigma(E(G)) \cap E(H)$. We shall derive a contradiction by finding σ^* with smaller intersection. We examine those $b \in V(H)$ such that either

(i) $b = a$,

(ii) $\{a, b\} \in \sigma(E(G)) \cap E(H)$,

(iii) for some $x \in V(H)$, $\{a, x\} \in \sigma(E(G))$, $\{x, b\} \in E(H)$,

or

(iv) for some $y \in V(H)$, $\{a, y\} \in E(H)$, $\{y, b\} \in \sigma(E(G))$.

There are at most $\alpha\beta$ b's satisfying (iii) and, similarly (iv). More precisely, if t b's satisfy (ii) at most $\alpha\beta - t$ satisfy (iii) and similarly (iv). Hence at most $1 + t + 2(\alpha\beta - t) \leqslant 2\alpha\beta$ points satisfy one of these conditions. Fix $b \in V(H)$ satisfying *none* of (i)–(iv). Define σ^* by

$$\sigma^*(i) = \begin{cases} \sigma(i), & \text{if } \sigma(i) \neq a, b, \\ b, & \text{if } \sigma(i) = a, \\ a, & \text{if } \sigma(i) = b. \end{cases}$$

That is, "flip" a and b in the map σ^*. Let

$$\{x, y\} \in E(H) \cap \sigma^*(E(G));$$

if $\{x, y\} \cap \{a, b\} = \varnothing$, then $\{x, y\}$ was in $E(H) \cap \sigma(E(G))$. But no $\{a, z\}$ or $\{b, z\}$ can be in $E(H) \cap \sigma^*(E(G))$. For if, say, $\{a, z\}$ was, then $\{a, z\} \in E(H)$, $\{b, z\} \in \sigma(E(G))$, but then b would satisfy (iv). The other cases are similar. Hence

$$|E(H) \cap \sigma^*(E(G))| < |E(G) \cap \sigma(E(G))|$$

implying the theorem. Q.E.D.

Now we outline a proof that Theorem 3 is nearly best possible. We show

[1] Theorem 3 was proven independently by Paul Catlin.

that there exist G, H with $\alpha \sim 2n^{1/2}$, $\beta \sim 2n^{1/2} \ln n$ so that G, Z are not mutually placeable. Let G be a fixed regular graph of degree $2n^{1/2}$ and, therefore, $n^{3/2}$ edges. Let **H** be a random graph on $V(H)$ where each edge $\{i, j\} \in E(\mathbf{H})$ with independent probability $p = n^{-1/2} \ln n$. Now fix $\sigma: V(G) \to V(H)$. Then

$$\text{Prob}[E(\mathbf{H}) \cap \sigma(E(G)) = \varnothing]$$
$$= \text{Prob}[\varLambda\{i, j\} \notin E(\mathbf{H})] \qquad (\text{over } \{i, j\} \in \sigma(E(G)))$$
$$= \prod \text{Prob}[\{i, j\} \notin E(\mathbf{H})] \qquad (\text{over } \{i, j\} \in \sigma(E(G)))$$

(since each edge of **H** is chosen *independently*)

$$= (1 - p)^{n^{3/2}} \sim e^{-pn^{3/2}} \sim n^{-n}.$$

Now

$$\text{Prob}[H, G \text{ mutually placeable}]$$
$$\leqslant \sum_{\sigma} \text{Prob}[E(H) \cap \sigma(E(G)) = \varnothing]$$
$$< n!\, n^{-n} \sim e^{-n},$$

so "almost always" H and G are not m.p. However, any $a \in V(H)$ has degree given by the binomial distribution $B(n - 1, p)$ and so is less than $np(1 - \epsilon) = (1 + \epsilon)\, n^{1/2} \ln n$ all but $o(n^{-1})$ of the time and so the maximal degree $\beta \leqslant (1 + \epsilon)\, n^{1/2} \ln n$ "almost always." Thus we may find a specific H with $\beta \leqslant (1 + \epsilon)\, n^{1/2} \ln n$ and G, H not m.p.

5. Circuits and Triangles

Consider the statement: If G has property A and H has property B, then G, H are m.p. We may rewrite this as follows: If G has property A and H^c has property B then H contains an isomorphic copy of G. Here H^c is the complementary graph to H. The observation is merely that a placement $\sigma: V(G) \to \sigma(H)$ gives $\sigma(E(G)) \subseteq \sigma(E(H^c))$.

We express Theorem 3 with $\alpha = 2$, $\beta < \frac{1}{4}n$ in this form.

Corollary 2. *Let G be any graph on n points with maximal degree $\leqslant 2$. Let H be a graph on n points with all points of valence $> \frac{3}{4}n$. Then H contains an isomorphic copy of G.*

Let γ represent the minimal degree of the points of H. If G is a circuit on n points, Corollary 2 is improved by

THEOREM (Dirac [2]). *If $\gamma > \frac{1}{2}n$, then H contains a Hamiltonian circuit.*

If $n = 3m$ and G consists of m disjoint triangles, then Corollary 2 is improved by

THEOREM (Corradi, Hajnal [1]). *If $\gamma \geq \frac{2}{3}$, then H contains $\frac{1}{3}n$ vertex disjoint triangles.*

Conjecture. Let G be any graph on n points with maximal degree ≤ 2. If H is a graph on n points with minimal degree $> \frac{2}{3}n$ then H contains an isomorphic copy of G.

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