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Convex subordination chains and injective mappings in $\mathbb{C}^n$

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ABSTRACT

In this paper we continue the work related to convex subordination chains in $\mathbb{C}$ and $\mathbb{C}^n$, and prove that if $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ is a holomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ such that $\sum_{k=2}^{\infty} k^2 \|A_k\| \leqslant 1$, $a : [0,1] \to [0,\infty)$ is a function of class C^2 on $(0,1)$ and continuous on $[0,1]$, such that $a(1) = 0$, $a(t) > 0$, $ta'(t) > -1/2$ for $t \in (0,1)$, and if $a(\cdot)$ satisfies a differential equation on $(0,1)$, then $f(z,t) = a(t^2)Df(tz)(tz) + f(tz)$ is a convex subordination chain over $(0,1]$ and the mapping $F(z) = a(\|z\|^2)Df(z)(z) + f(z)$ is injective on B^n. We also present certain coefficient bounds which provide sufficient conditions for univalence, quasiregularity and starlikeness for the chain $f(z,t)$. Finally we give some examples of convex subordination chains over $(0,1]$.

1. Introduction and preliminaries

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n \colon \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n is denoted by B^n. In the case of one complex variable, B_r^1 is denoted by U_r and the unit disc U_1 is denoted by U. The closed unit ball in $\mathbb{C}^n$ and the boundary of B^n are denoted respectively by $\overline{B}^n$ and ∂B^n. Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. If $f \in H(B^n)$, we say that f is convex if f is biholomorphic on B^n and $f(B^n)$ is a convex domain in $\mathbb{C}^n$. If $f \in H(B^n)$ and $f(0) = 0$, we say that f is starlike if f is biholomorphic on B^n and $f(B^n)$ is a starlike domain in $\mathbb{C}^n$ with respect to zero. If $f \in H(B^n)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n. Also let $K(B^n)$ (resp. $S^*(B^n)$) be the subset of $S(B^n)$ consisting of convex (resp. starlike) mappings on B^n. The classes $S(B^1)$, $K(B^1)$ and $S^*(B^1)$ are denoted by S, K and S^*. Several properties of mappings in $S(B^n)$, $S^*(B^n)$ and $K(B^n)$ can be found in [2,5,13,16,20].

If $f \in H(B^n)$ is normalized, then f has the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$, $z \in B^n$, where $A_k = \frac{1}{k!} D^k f(0)$ is the k-th Fréchet derivative of f at $z = 0$. It is understood that for $v \in \mathbb{C}^n$, $D^k f(0)(v^k) = D^k f(0)(\underbrace{v, \ldots, v}_{k\text{-times}})$.

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If $f, g \in H(B^n)$, we say that f is subordinate to g ($f \prec g$) if there exists a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leqslant \|z\|$, $z \in B^n$) such that $f = g \circ v$.

Various applications of this notion may be found in [12]. Recently, Graham, Hamada, Kohr and Pfaltzgraff [4] have introduced the notion of a convex subordination chain (c.s.c.) in several complex variables. This notion was introduced by Ruscheweyh [17] in the case of one complex variable. Various applications of this notion can be found in [14,17,19] (for $n = 1$) and in [4,10] (in the case of several complex variables).

Definition 1. Let J be an interval in $\mathbb{R}$. A mapping $f = f(z,t) : B^n \times J \to \mathbb{C}^n$ is called a convex subordination chain (c.s.c.) over J if the following conditions hold:

(i) $f(0,t) = 0$ and $f(\cdot,t)$ is convex (biholomorphic) for $t \in J$.
(ii) $f(\cdot,t_1) \prec f(\cdot,t_2)$ for $t_1, t_2 \in J$, $t_1 \leqslant t_2$.

Graham, Hamada, Kohr and Pfaltzgraff [4] obtained necessary and sufficient conditions for a mapping $f(z,t)$ to be a convex subordination chain and gave several examples of c.s.c. over an interval $J \subseteq [0,\infty)$. Among other results, they proved the following sufficient criterion for a mapping to be a c.s.c. over $(0,1)$, by using a basic separation theorem in convexity theory. For other applications of this result, see [4].

Lemma 2. *Let $f = f(z,t) : \overline{B}^n \times [0,1) \to \mathbb{C}^n$ be a continuous mapping such that $f(\cdot,t)$ is convex on B^n for $t \in (0,1)$, $f(0,t) = f(z,0) = 0$ for $z \in B^n$ and $t \in (0,1)$. For $w \in \partial B^n$, let G_w be the function defined by*

$$G_w(z) = \begin{cases} \Re\langle f(\frac{z}{\|z\|}, \|z\|), w\rangle, & z \in B^n \setminus \{0\}, \\ 0, & z = 0. \end{cases}$$

If G_w has no maximum in B^n_r, for all $r \in (0,1)$ and $w \in \partial B^n$, then $f(z,t)$ is a c.s.c. over $(0,1)$. Moreover, if the mapping $f(\cdot,t)$ is injective on $\overline{B}^n$ for $t \in (0,1)$, then the mapping $F : B^n \to \mathbb{C}^n$ given by

$$F(z) = \begin{cases} f(\frac{z}{\|z\|}, \|z\|), & z \in B^n \setminus \{0\}, \\ 0, & z = 0 \end{cases}$$

is injective on B^n.

We say that a mapping $f \in H(B^n)$ is K-quasiregular, $K \geqslant 1$, if

$$\|Df(z)\|^n \leqslant K|\det Df(z)|, \quad z \in B^n.$$

A mapping $f \in H(B^n)$ is called quasiregular if f is K-quasiregular for some $K \geqslant 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 3. Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|D(f;x)\|^m \leqslant K|\det D(f;x)| \quad \text{a.e. } x \in G,$$

where $D(f;x)$ denotes the real Jacobian matrix of f at x and K is a constant.

Note that a K-quasiregular biholomorphic mapping is K^2-quasiconformal.

Remark 4. (i) Ruscheweyh [17, Theorem 2.41] proved that if $f \in K$, then

$$f(\zeta,t) = \frac{1-t^2}{1+t^2} t\zeta f'(t\zeta) + f(t\zeta)$$

is a c.s.c. over $(0,1]$ on the unit disc U.

(ii) Kohr, Mocanu and Şerb [11, Theorem 10] proved that if $f \in K$ and $a : [0,1] \to [0,\infty)$ is a function of class C^1 on $(0,1)$ and continuous on $[0,1]$ such that $a(1) = 0$, $a(t) > 0$ and $ta'(t) > -1/2$ for $t \in (0,1)$, then

$$f(\zeta,t) = a(t^2)t\zeta f'(t\zeta) + f(t\zeta), \quad |\zeta| < 1,\ t \in (0,1],$$

is a c.s.c. over $(0,1]$ on the unit disc.

(iii) Moreover, if $a(\cdot)$ is C^1 on $[0,1)$, then the function $F(\zeta) = a(|\zeta|^2)\zeta f'(\zeta) + f(\zeta)$ is injective on the unit disc by [11, Theorem 7].

Remark 5. It is not difficult to deduce that the following functions satisfy the conditions in Remark 4(ii):

(i) $a(t) = \frac{1}{c}(\frac{1-t^c}{1+t^c})$, $t \in [0,1]$, $c > 0$;
(ii) $a(t) = \ln(\frac{2}{1+t})$, $t \in [0,1]$;
(iii) $a(t) = e^{-kt} - e^{-k}$, $t \in [0,1]$, $k > 0$.

Let SK be the set of normalized holomorphic functions f on U which satisfy the condition $|f''(\zeta)/f'(\zeta)| \leqslant 1$ for $|\zeta| < 1$. Clearly $SK \subset K$.

If M is a subset of $\mathbb{C}$, let $\overline{co}(M)$ be the closed convex hull of M. Also, if $f, g \in H(U)$, let $f * g$ be the Hadamard product (convolution) of f and g.

The following result is due to Ruscheweyh (see [17, Theorem 2.4]).

Lemma 6. *Let $f \in K$, $g \in S^*$, and let $F : U \to \mathbb{C}$ be a holomorphic function. Then*

$$\frac{f * gF}{f * g}(U) \subseteq \overline{co}\big(F(U)\big).$$

Also, it is known that if $f, g \in SK$ then $f * g \in SK$ (see [17, p. 57]; compare [18, Theorem 2.1]).

The following lemma of independent interest is an improved version of the above result.

Lemma 7. *Let $f \in K$ and $g \in SK$. Then $f * g \in SK$.*

Proof. Let $h = f * g$. It is elementary to obtain the following relations:

$$h'(z) = \frac{1}{z}\big(f(z) * zg'(z)\big) \quad \text{and} \quad h''(z) = \frac{1}{z^2}\big(f(z) * z^2 g''(z)\big), \quad z \in U.$$

Hence

$$\frac{zh''(z)}{h'(z)} = \frac{f(z) * zg'(z)\frac{zg''(z)}{g'(z)}}{f(z) * zg'(z)}, \quad z \in U.$$

Since $g \in SK$, it follows that $q(z) = zg'(z) \in S^*$ and $r(z) = zg''(z)/g'(z)$ is a holomorphic function on U. Hence

$$\frac{zh''(z)}{h'(z)} \in \overline{co}\left\{\frac{\zeta g''(\zeta)}{g'(\zeta)}\colon \zeta \in U\right\}, \quad z \in U,$$

by Lemma 6. On the other hand, since $g \in SK$, it follows that

$$\overline{co}\left\{\frac{\zeta g''(\zeta)}{g'(\zeta)}\colon \zeta \in U\right\} \subseteq \overline{U},$$

and hence $|h''(z)/h'(z)| \leqslant 1$ for $z \in U$. Thus, $h \in SK$, as desired. □

In this paper we continue the work begun in [4] and [11] and prove that if $c > 0$ and $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ is a holomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ such that $\sum_{k=2}^{\infty} k^2 \|A_k\| \leqslant 1$, then

$$f(z,t) = \frac{1}{c}\left(\frac{1-t^{2c}}{1+t^{2c}}\right) Df(tz)(tz) + f(tz)$$

is a convex subordination chain over $(0,1]$ and the mapping

$$F(z) = \frac{1}{c}\left(\frac{1-\|z\|^{2c}}{1+\|z\|^{2c}}\right) Df(z)(z) + f(z)$$

is injective on B^n. If $c = 1$, we obtain [4, Theorem 2.11 and Corollary 2.17]. In the case of one complex variable, see [17]. We also present certain coefficient bounds which provide sufficient conditions for univalence, quasiregularity and starlikeness for the chain

$$f(z,t) = a\big(t^2\big) Df(tz)(tz) + f(tz), \quad z \in B^n,\ t \in [0,1],$$

where $a : [0,1] \to [0,\infty)$ is a function which satisfies the assumptions of Remark 4(ii). Finally we give some examples of c.s.c. over $(0,1]$.

2. Main results

We begin this section with the following sufficient criterion for a mapping to be a c.s.c. over $(0,1]$. This result is a generalization of [4, Theorem 2.11 and Corollary 2.17]. It would be interesting to see if this result remains valid for any mapping $f \in K(B^n)$. In the case of one complex variable, see [11] and [17].

Theorem 8. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a normalized holomorphic mapping on B^n such that*

$$\sum_{k=2}^{\infty} k^2 \|A_k\| \leqslant 1. \tag{1}$$

Also let $a : [0,1] \to [0,\infty)$ be a function of class C^2 on $(0,1)$ and continuous on $[0,1]$ such that $a(1) = 0$, $a(t) > 0$ and $ta'(t) > -1/2$ for $t \in (0,1)$. Assume that $a(\cdot)$ satisfies the differential equation

$$ta''(t)a(t) + a'(t)a(t) - 2t\big(a'(t)\big)^2 - a'(t) = 0, \quad t \in (0,1). \tag{2}$$

Further, let

$$f(z,t) = a\big(t^2\big)Df(tz)(tz) + f(tz), \quad z \in B^n,\ t \in [0,1]. \tag{3}$$

Then $f(z,t)$ is a c.s.c. over $(0,1]$ and the mapping $F : B^n \to \mathbb{C}^n$ given by

$$F(z) = a\big(\|z\|^2\big)Df(z)(z) + f(z) \tag{4}$$

is injective on B^n.

Proof. We shall use arguments similar to those in the proofs of [4, Theorem 2.11 and Corollary 2.17]. We divide the proof into the following steps:

Step I. If $f(z) \equiv z$ then $f(z,t) = (a(t^2)+1)tz$ is a c.s.c. over $(0,1]$. Indeed, $f(\cdot,t)$ is convex and it is easy to see that $(a(s^2)+1)sz \prec (a(t^2)+1)tz$ for $z \in B^n$ and $0 < s \leqslant t \leqslant 1$, by the fact that $a(t) > 0$ and $ta'(t) > -1/2$ for $t \in (0,1)$. Hence, without loss of generality, we may assume that $f(z) \not\equiv z$.

We remark that the condition (1) yields that $f \in K(B^n)$ by [16, Theorem 2.1]. Let

$$\beta_k(t) = t^k\big(ka\big(t^2\big)+1\big) \quad \text{and} \quad g_k(t) = \beta_k(t)/\beta_1(t)$$

for $k \in \mathbb{N}$ and $t \in (0,1]$. Then $g_k(1) = 1$ since $a(1) = 0$, and an elementary computation yields that

$$g_k'(t) = (k-1)t^{k-2}\frac{(ka(t^2)+1)(a(t^2)+1) + 2t^2a'(t^2)}{(a(t^2)+1)^2}, \quad t \in (0,1).$$

Since $ta'(t) > -1/2$ for $t \in (0,1)$, it follows that $g_k'(t) > 0$ for $k \geqslant 2$, $t \in (0,1)$, and hence $g_k(t) < 1$ for $k \geqslant 2$, $t \in (0,1)$. Therefore $\beta_k(t) < \beta_1(t)$ for $k \geqslant 2$, $t \in (0,1)$. Since

$$f(z,t) = \beta_1(t)z + \sum_{k=2}^{\infty}\beta_k(t)A_k\big(z^k\big),$$

and $f(z) \not\equiv z$, we deduce that

$$\sum_{k=2}^{\infty} k^2\beta_k(t)\|A_k\| < \beta_1(t)\sum_{k=2}^{\infty} k^2\|A_k\| \leqslant \beta_1(t),$$

by the condition (1). Then $f(\cdot,t)$ is convex on B^n by [16] and extends as a homeomorphism to $\overline{B}^n$ for $t \in (0,1)$ by [8]. On the other hand, it is clear that the mapping $f(z,t)$ is continuous on $\overline{B}^n \times [0,1)$.

Next, let $z, w \in \partial B^n$ and

$$F_{z,w}(\zeta) = \Re\big\langle a\big(|\zeta|^2\big)Df(\zeta z)(\zeta z) + f(\zeta z), w\big\rangle, \quad \zeta \in U. \tag{5}$$

Then $F_{z,w}$ is of class C^2 on $U \setminus \{0\}$ and is continuous on U. Using elementary computations, we obtain that

$$\zeta\frac{\partial F_{z,w}}{\partial\zeta} + \bar\zeta\frac{\partial F_{z,w}}{\partial\bar\zeta} = 2\Re\big\langle a'\big(|\zeta|^2\big)|\zeta|^2Df(\zeta z)(\zeta z), w\big\rangle + \Re\big\langle a\big(|\zeta|^2\big)\big[D^2f(\zeta z)(\zeta z,\zeta z) + Df(\zeta z)(\zeta z)\big]$$
$$+ Df(\zeta z)(\zeta z), w\big\rangle, \quad \zeta \in U \setminus \{0\},$$

and

$$\frac{\partial^2 F_{z,w}}{\partial\zeta\partial\bar{\zeta}} = \Re\big\langle a''\big(|\zeta|^2\big)|\zeta|^2 Df(\zeta z)(\zeta z), w\big\rangle + \Re\big\langle a'\big(|\zeta|^2\big)\big[D^2 f(\zeta z)(\zeta z,\zeta z) + 2Df(\zeta z)(\zeta z)\big], w\big\rangle, \quad \zeta \in U\setminus\{0\}.$$

Hence, in view of (2) and the above relations, we deduce that $F_{z,w}$ satisfies the following elliptic equation on $U\setminus\{0\}$:

$$\frac{\partial^2 H}{\partial\zeta\partial\bar{\zeta}} - \frac{a'(|\zeta|^2)}{a(|\zeta|^2)}\left(\zeta\frac{\partial H}{\partial\zeta} + \bar{\zeta}\frac{\partial H}{\partial\bar{\zeta}}\right) = 0. \tag{6}$$

Let G_w be the function constructed using $f(z,t)$ given by (3), i.e.

$$G_w(z) = \Re\big\langle a\big(\|z\|^2\big)Df(z)(z) + f(z), w\big\rangle, \quad z\in B^n.$$

Fix $r\in(0,1)$ and $w\in\partial B^n$. Suppose that the function G_w has a maximum in B_r^n.

(i) If this maximum occurs at $z=0$, then $G_w(z)\leqslant G_w(0)=0$, i.e.

$$\Re\big\langle a\big(\|z\|^2\big)Df(z)(z) + f(z), w\big\rangle \leqslant 0, \quad z\in B_r^n.$$

Then

$$\Re\big\langle a\big(t^2\big)Df(tw)(tw) + f(tw), w\big\rangle \leqslant 0, \quad t\in[0,r),$$

and hence

$$\Re\left\langle a\big(t^2\big)Df(tw)(w) + \frac{f(tw)}{t}, w\right\rangle \leqslant 0, \quad t\in(0,r).$$

Letting $t\to 0$ in the above relation and using the fact that $Df(0)=I_n$, we obtain that $(a(0)+1)\|w\|^2\leqslant 0$, i.e. $a(0)+1\leqslant 0$. However, this is impossible.

(ii) If the maximum of G_w occurs at a point $z_0\in B_r^n\setminus\{0\}$, then in view of the above arguments, we deduce that $G_w(z_0)\neq 0$. Let $\tilde{z}=z_0/\|z_0\|$ and $\zeta_0=\|z_0\|$. Considering the function $F_{\tilde{z},w}(\zeta)$ given by (5), we deduce that $F_{\tilde{z},w}$ satisfies the elliptic equation (6) on $U_r\setminus\{0\}$. Clearly, $F_{\tilde{z},w}$ is of class C^2 on $U_r\setminus\{0\}$ and is continuous on the closed disc $\overline{U}_r$. On the other hand, since

$$G_w(z_0) = \max_{z\in B_r^n} G_w(z),$$

we obtain that

$$F_{\tilde{z},w}(\zeta_0) = \max_{|\zeta|<r} F_{\tilde{z},w}(\zeta).$$

Taking into account the strong maximum principle for elliptic equations (see e.g. [1, p. 332]), we conclude that $F_{\tilde{z},w}(\zeta) = F_{\tilde{z},w}(\zeta_0) = G_w(z_0)\neq 0$ for $0<|\zeta|<r$. However, letting $\zeta\to 0$ in the above equality and using the fact that $F_{\tilde{z},w}(0)=0$, we obtain a contradiction.

In view of the above arguments, we deduce that the function G_w cannot have a maximum on B_r^n, and since $r\in(0,1)$ and $w\in\partial B^n$ are arbitrary, we conclude by Lemma 2 that $f(z,t)$ is a c.s.c. over the interval $(0,1)$. Next, applying a version of the Carathéodory convergence theorem in several complex variables (see [9, Theorem 2.1]), we deduce that $f(z,t)$ is a c.s.c. over $(0,1]$.

Step II. We next prove that the mapping F given by (4) is injective on B^n. Taking into account Lemma 2, we deduce that the mapping $f(z/\|z\|,\|z\|)=F(z)$ is injective on $B^n\setminus\{0\}$. Finally, since $f(z/\|z\|,\|z\|)\neq 0$ for $z\in B^n\setminus\{0\}$, by the injectivity of $f(\cdot,t)$ on $\overline{B}^n$ and the fact that $f(0,t)=0$ for $t\in(0,1)$, we deduce that F is injective on B^n. This completes the proof. □

Remark 9. Using elementary computations, it is not difficult to deduce that the general solutions $a(t)$ of Eq. (2), which are of class C^2 on $(0,1)$ and continuous on $[0,1]$, and satisfy the conditions $a(1)=0$, $a(t)>0$ and $ta'(t)>-1/2$ for $t\in(0,1)$, are the following:

$$a(t) = \frac{1}{c}\left(\frac{1-t^c}{1+t^c}\right), \quad t\in[0,1], \text{ where } c>0. \tag{7}$$

Proof. Indeed, in view of (2) it is easy to see that

$$t\left(-\frac{1}{a(t)}\right)'' = \left(-\frac{1}{a(t)}\right)'\left(\frac{1}{a(t)}-1\right), \quad t\in(0,1).$$

Let $b(t) = -1/a(t)$ for $t \in (0,1)$. Then $(tb'(t))' = -\frac{1}{2}(b^2(t))'$ for $t \in (0,1)$, and hence there is $c_1 \in \mathbb{R}$ such that

$$tb'(t) = -\frac{1}{2}b^2(t) + \frac{c_1}{2}, \quad t \in (0,1).$$

Therefore

$$ta'(t) = -\frac{1}{2} + \frac{c_1 a^2(t)}{2}, \quad t \in (0,1). \tag{8}$$

Since $ta'(t) > -1/2$ for $t \in (0,1)$, we deduce that $c_1 > 0$. Using again (8), we deduce that

$$ta'(t) = \frac{1}{2}\left(\frac{a^2(t) - c_2^2}{c_2^2}\right), \quad t \in (0,1),$$

where $c_2 = 1/\sqrt{c_1}$.

We next prove that $a(t) \neq c_2$ for $t \in (0,1]$. Suppose that there exists $t_0 \in (0,1)$ such that $a(t_0) = c_2 > 0$. Let

$$A = \{t \in [t_0, 1]:\ a(t) = c_2\}.$$

Then A is a nonempty compact set, which contains the maximal element t_1 such that $t_1 \geqslant t_0$, $t_1 \neq 1$, and $a(t_1) = c_2$. From the maximality of t_1, it is clear that $a(t) < c_2$ for $t \in (t_1, 1]$. Let k_0 be a positive integer such that $t_1 + 1/k_0 < 1$. It follows for $k \geqslant k_0$ that $t_1 + 1/k < 1$ and

$$\int_{t_1+1/k}^{1} \frac{a'(t)}{a^2(t) - c_2^2}\,dt = \int_{t_1+1/k}^{1} \frac{1}{2c_2^2 t}\,dt.$$

The above relation implies that

$$\frac{c_2 - a(t_1 + 1/k)}{c_2 + a(t_1 + 1/k)} = (t_1 + 1/k)^{1/c_2}.$$

However, this is a contradiction for k large enough. Hence $a(t) \neq c_2$ for $t \in (0,1]$, as claimed.

In view of the above arguments, we obtain that

$$\frac{a'(t)}{a^2(t) - c_2^2} = \frac{1}{2c_2^2 t}, \quad t \in (0,1).$$

Integrating the above equality on $[t,1]$, and using the fact that $a(1) = 0$ and $a(t) < c_2$ for $t \in (0,1]$, we deduce that

$$\frac{1}{c} - a(t) = \left(a(t) + \frac{1}{c}\right)t^c, \quad t \in (0,1),$$

where $c = 1/c_2 > 0$. Hence we obtain the relation (7), as desired. □

Taking into account Theorem 8 and Remark 9, we obtain the following consequence. Note that in the case $c = 1$, Theorem 10 reduces to [4, Theorem 2.11], that is the n-dimensional version of [17, Theorem 2.41].

Theorem 10. *Let $c > 0$ and $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a normalized holomorphic mapping on B^n which satisfies the condition* (1)*. Then*

$$f(z,t) = \frac{1}{c}\left(\frac{1 - t^{2c}}{1 + t^{2c}}\right)Df(tz)(tz) + f(tz)$$

is a convex subordination chain over $(0,1]$ and the mapping $F : B^n \to \mathbb{C}^n$ given by

$$F(z) = \frac{1}{c}\left(\frac{1 - \|z\|^{2c}}{1 + \|z\|^{2c}}\right)Df(z)(z) + f(z)$$

is injective on B^n.

From Theorem 10 we obtain the following subordination result.

Corollary 11. *Let $c > 0$ and $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a normalized holomorphic mapping on B^n which satisfies the condition* (1)*. Then*

$$\frac{1}{c}\left(\frac{1-t^{2c}}{1+t^{2c}}\right)Df(tz)(tz) + f(tz) \prec f(z), \quad z \in B^n, \ t \in (0,1].$$

We close this section with the following coefficient bounds which provide sufficient conditions for univalence, quasiregularity and starlikeness for the mapping $f(z,t)$ given by (3).

Theorem 12. *Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping such that $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ for $z \in B^n$. Assume that there exists $c \in [0,1]$ such that*

$$\sum_{k=2}^{\infty} k\|A_k\| \leqslant c. \tag{9}$$

Let $a : [0,1] \to [0,\infty)$ be a function which satisfies the assumptions of Remark 4(ii)*. Also let $f(z,t) : B^n \times [0,1] \to \mathbb{C}^n$ be the mapping given by* (3)*. Then $f(\cdot,t)$ is biholomorphic on B^n for $t \in (0,1]$. Moreover, if $c < 1$, then $f(\cdot,t)$ is quasiregular on B^n and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself for $t \in (0,1]$.*

Proof. We remark that the condition (9) yields that f is biholomorphic by [3, Lemma 2.2]. If $c < 1$, then f is quasiregular on B^n and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself by [3, Lemma 2.2] (see also [8, Corollary 4.5], [7, Theorem 4.1]). Let

$$\beta_k(t) = t^k\big(ka\big(t^2\big)+1\big) \quad \text{and} \quad g_k(t) = \beta_k(t)/\beta_1(t)$$

for $k \in \mathbb{N}$ and $t \in (0,1]$. As in the proof of Theorem 8, we deduce that $g_k(t) < 1$ for $k \geqslant 2$ and $t \in (0,1)$. Since

$$f(z,t) = \beta_1(t)z + \sum_{k=2}^{\infty} \beta_k(t)A_k\big(z^k\big), \quad z \in B^n,$$

we deduce that

$$\sum_{k=2}^{\infty} k\beta_k(t)\|A_k\| \leqslant \beta_1(t)\sum_{k=2}^{\infty} k\|A_k\| \leqslant c\beta_1(t), \quad t \in (0,1),$$

by the condition (9). Hence $f(\cdot,t)$ is biholomorphic in view of [3]. If $c < 1$, then $f(\cdot,t)$ is quasiregular on B^n and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself for $t \in (0,1]$, in view of [3]. This completes the proof. □

Remark 13. It would be interesting to see if the mapping F given by (4) is injective on B^n, under the same assumptions as in Theorem 12.

Theorem 14. *Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping such that $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ for $z \in B^n$. Assume that*

$$\sum_{k=2}^{\infty} (2k-1)\|A_k\| \leqslant 1. \tag{10}$$

Let $a : [0,1] \to [0,\infty)$ be a function which satisfies the assumptions of Remark 4(ii)*. Also let $f(z,t) : B^n \times [0,1] \to \mathbb{C}^n$ be the mapping given by* (3)*. Then $f(\cdot,t)$ is starlike and quasiregular on B^n and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself for $t \in (0,1]$.*

Proof. Since $\sum_{k=2}^{\infty} k\|A_k\| \leqslant 2/3$, in view of (10), we deduce that $f(\cdot,t)$ is quasiregular on B^n and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself for $t \in (0,1]$ by Theorem 12. On the other hand, taking into account the condition (10) and [3, Theorem 2.4], it suffices to use arguments similar to those in the proof of Theorem 12, to deduce that $f(\cdot,t)$ is starlike for $t \in (0,1]$. This completes the proof. □

3. Examples of c.s.c. over $(0,1]$ on B^n

In view of Theorem 10, we obtain the following example of a convex subordination chain over $(0,1]$ and injective mapping on B^n.

Example 15. Let $c>0$ and $A:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ be a symmetric bilinear operator such that $\|A\|\leqslant 1/4$. Also let

$$f(z,t)=\frac{t(1+c)+t^{2c+1}(c-1)}{c(1+t^{2c})}z+\frac{t^2(2+c)+t^{2c+2}(c-2)}{c(1+t^{2c})}A\big(z^2\big).$$

Then $f(z,t)$ is a c.s.c. over $(0,1]$. Moreover, the mapping $F:B^n\to\mathbb{C}^n$ given by

$$F(z)=\frac{1+c+\|z\|^{2c}(c-1)}{c(1+\|z\|^{2c})}z+\frac{2+c+\|z\|^{2c}(c-2)}{c(1+\|z\|^{2c})}A\big(z^2\big)$$

is injective on B^n.

Proof. It suffices to apply Theorem 10 for $f(z)=z+A(z^2)$. $\square$

Before to give other examples of c.s.c. over $(0,1]$ on B^n, we recall that if $f_j\in K$ for $j=1,\dots,n$, then the mapping $F:B^n\to\mathbb{C}^n$ given by $F(z)=(f_1(z_1),\dots,f_n(z_n))$ is not necessarily a convex mapping in dimension $n\geqslant 2$ (see [15] and [16]). Indeed, $f(\zeta)=\zeta/(1-\zeta)\in K$, however, the mapping $F:B^n\to\mathbb{C}^n$ given by

$$F(z)=\left(\frac{z_1}{1-z_1},\dots,\frac{z_n}{1-z_n}\right),\quad z=(z_1,\dots,z_n)\in B^n,$$

is not convex in dimension $n\geqslant 2$ (see [15,16,2]). Thus, if $f_j\in K$, $j=1,\dots,n$, and $a:[0,1]\to[0,\infty)$ is a function which satisfies the assumptions of Remark 4(ii), then

$$f(z,t)=\big(a\big(t^2\big)tz_1f_1'(tz_1)+f_1(tz_1),\dots,a\big(t^2\big)tz_nf_n'(tz_n)+f_n(tz_n)\big)\tag{11}$$

is not necessarily a c.s.c. over $(0,1]$. We shall prove that if $|f_j''(\zeta)/f_j'(\zeta)|\leqslant 1$ for $|\zeta|<1$ and $j=1,\dots,n$, then $f(z,t)$ given by (11) is a c.s.c. over $(0,1]$.

Theorem 16. *Let $f_j\in SK$ for $j=1,\dots,n$ and let $a:[0,1]\to[0,\infty)$ be a function which satisfies the assumptions of Remark* 4(ii)*. Then*

$$f(z,t)=\big(a\big(t^2\big)tz_1f_1'(tz_1)+f_1(tz_1),\dots,a\big(t^2\big)tz_nf_n'(tz_n)+f_n(tz_n)\big)$$

is a c.s.c. over $(0,1]$*. Moreover, if $a(\cdot)$ is of class C^1 on $[0,1)$, then the mapping $F:B^n\to\mathbb{C}^n$ given by*

$$F(z)=\big(a\big(|z_1|^2\big)z_1f_1'(z_1)+f_1(z_1),\dots,a\big(|z_n|^2\big)z_nf_n'(z_n)+f_n(z_n)\big)$$

is injective on B^n.

Proof. Let $f_j(z_j,t)=a(t^2)tz_jf_j'(tz_j)+f_j(tz_j)$ for $|z_j|<1$, $j=1,\dots,n$ and $t\in(0,1]$. Then $f_j(z_j,t)$ is a c.s.c. over $(0,1]$ by Remark 4(ii) and

$$f_j(z_j,t)=\frac{a(t^2)tz_j+1-tz_j}{(1-tz_j)^2}*f_j(z_j),\quad j=1,\dots,n.$$

Let $h_j(z_j,t)=(a(t^2)tz_j+1-tz_j)/(1-tz_j)^2$. Then $h_j(\cdot,t)$ is a (non-normalized) convex function on U for $t\in(0,1]$, by [11, Lemma 9]. Let

$$n_j(z_j,t)=\frac{h_j(z_j,t)-1}{t(a(t^2)+1)},\quad |z_j|<1,\ j=1,\dots,n,\ t\in(0,1].$$

Then $n_j(\cdot,t)\in K$ for $t\in(0,1]$ and $j=1,\dots,n$. Also let

$$p_j(z_j,t)=n_j(z_j,t)*f_j(z_j)=\frac{1}{t(a(t^2)+1)}f_j(z_j,t),\quad j=1,\dots,n.$$

Taking into account Lemma 7, we deduce that $p_j(\cdot,t)\in SK$, and thus

$$\left|\frac{f_j''(z_j,t)}{f_j'(z_j,t)}\right|\leqslant 1,\quad |z_j|<1,\ j=1,\dots,n,\ t\in(0,1].$$

Next, it is not difficult to deduce that

$$\big\|\big[Df(z,t)\big]^{-1}D^2f(z,t)(v,v)\big\|^2=\sum_{j=1}^n|v_j|^4\left|\frac{f_j''(z_j,t)}{f_j'(z_j,t)}\right|^2\leqslant\sum_{j=1}^n|v_j|^2=1,$$

for $z=(z_1,\dots,z_n)\in B^n$, $v=(v_1,\dots,v_n)\in\partial B^n$ and $t\in(0,1]$. Then $f(\cdot,t)$ is a convex mapping on B^n for each $t\in(0,1]$, in view of [6, Theorem 3.4] (see also [13, Theorem 4.1] and [21, Corollary 1]). On the other hand, since $f_j(z_j,s)\prec f_j(z_j,t)$ by [11, Theorem 10], there exists a Schwarz function $v_j=v_j(z_j,s,t)$ such that $f_j(z_j,s)=f_j(v_j(z_j,s,t),t)$ for $|z_j|<1$, $0<s\leqslant t\leqslant 1$, $j=1,\dots,n$. Let $v(z,s,t)=(v_1(z_1,s,t),\dots,v_n(z_n,s,t))$ for $z=(z_1,\dots,z_n)\in B^n$ and $0<s\leqslant t\leqslant 1$. Then $v(\cdot,s,t)\in H(B^n)$ and

$$\|v(z,s,t)\|^2=\sum_{j=1}^{n}\left|v_j(z_j,s,t)\right|^2\leqslant\sum_{j=1}^{n}|z_j|^2=\|z\|^2,\quad z=(z_1,\dots,z_n)\in B^n.$$

Hence $v(\cdot,s,t)$ is a Schwarz mapping. Moreover, since $f(z,s)=f(v(z,s,t),t)$ for $z\in B^n$ and $0<s\leqslant t\leqslant 1$, we deduce that $f(\cdot,s)\prec f(\cdot,t)$ for $0<s\leqslant t\leqslant 1$. Taking into account the above arguments, we deduce that $f(z,t)$ is a c.s.c. over $(0,1]$, as desired. Finally, if $a(\cdot)$ is of class C^1 on $[0,1)$, then the function $F_j(z_j)=a(|z_j|^2)z_jf_j'(z_j)+f_j(z_j)$ is injective on U for $j=1,\dots,n$, by Remark 4(iii). Thus the mapping $F(z)=(F_1(z_1),\dots,F_n(z_n))$ is injective on B^n. This completes the proof. □

Taking into account Theorem 16, we obtain the following examples of c.s.c. over $(0,1]$ on the unit ball B^n.

Example 17. Let $\lambda_j\in\mathbb{C}$ be such that $0<|\lambda_j|\leqslant 1$ for $j=1,\dots,n$. Also let $a:[0,1]\to[0,\infty)$ be a function which satisfies the assumptions of Remark 4(ii). Then $f(z,t)$ given by

$$f(z,t)=\left(a\bigl(t^2\bigr)tz_1e^{\lambda_1tz_1}+\frac{e^{\lambda_1tz_1}-1}{\lambda_1},\dots,a\bigl(t^2\bigr)tz_ne^{\lambda_ntz_n}+\frac{e^{\lambda_ntz_n}-1}{\lambda_n}\right)$$

is a convex subordination chain over $(0,1]$.

Proof. Let $f_j(\zeta)=(e^{\lambda_j\zeta}-1)/\lambda_j$ for $j=1,\dots,n$. Then $|f_j''(z_j)/f_j'(z_j)|=|\lambda_j|\leqslant 1$ for $j=1,\dots,n$, and the result follows by Theorem 16. □

Example 18. Let $f_1\in SK$ and let $a:[0,1]\to[0,\infty)$ be a function which satisfies the assumptions of Remark 4(ii). Then

$$f(z,t)=\bigl(a\bigl(t^2\bigr)tz_1f_1'(tz_1)+f_1(tz_1),\bigl(a\bigl(t^2\bigr)+1\bigr)t\tilde{z}\bigr),\quad z=(z_1,\tilde{z})\in B^n,$$

is a c.s.c. over $(0,1]$.

Example 19. Let $a:[0,1]\to[0,\infty)$ be a function which satisfies the assumptions of Remark 4(ii). Then $f(z,t)=(a(t^2)+1)tz$ is a c.s.c. over $(0,1]$ and the mapping $F(z)=(a(\|z\|^2)+1)z$ is injective on B^n.

Proof. It is obvious that $f(z,t)$ is a c.s.c. over $(0,1]$ in view of Example 18. Since $a(t)>0$ and $ta'(t)>-1/2$ for $t\in(0,1)$, it follows that the function $q(t)=t(a(t^2)+1)$ is strictly increasing on $(0,1)$. Then it is easy to see that $F(z)=(a(\|z\|^2)+1)z$ is injective on B^n, as desired. □

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