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The Unitary Implementation of a Locally Compact Quantum Group Action

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In this paper we study actions of locally compact quantum groups on von Neumann algebras and prove that every action has a canonical unitary implemen-

tum groups in action. We will use it in this paper to study subfactors and inclusions of von Neumann algebras. When α is an action of the locally compact quantum group (M, Δ) on the von Neumann algebra N we can give necessary and sufficient conditions under which the inclusion $N^\alpha \subset N \hookrightarrow M_\alpha \ltimes N$ is a basic construction. Here N^α denotes the fixed point algebra and $M_\alpha \ltimes N$ is the crossed product. When α is an outer and integrable action on a factor N we prove that the inclusion $N^\alpha \subset N$ is irreducible, of depth 2 and regular, giving a converse to the results of M. Enock and R. Nest (1996, *J. Funct. Anal.* **137**, 466–543; 1998, *J. Funct. Anal.* **154**, 67–109). Finally we prove the equivalence of minimal and outer actions and we generalize the main theorem of Yamanouchi (1999, *Math. Scand.* **84**, 297–319): every integrable outer action with infinite fixed point algebra is a dual action.

INTRODUCTION

Building on the work of Kac and Vainerman [30], Enock and Schwartz [6], Baaj and Skandalis [1], Woronowicz [32] and Van Daele [31], a precise definition of a locally compact quantum group was recently introduced by J. Kusterman and the author in [19], see [18] and [20] for an overview. For an overview of the historical development of the theory we refer to [20] and the introduction of [19]. This theory provides a topological framework to study quantum groups and it unifies locally compact groups, compact quantum groups and Kac algebras.

Because classical groups are usually defined to act on a space it is very natural to make a quantum group act on a quantum space, which will be

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an algebra. In an algebraic framework the study of Hopf algebras acting on algebras has been very useful.

On the other hand, actions of locally compact groups on von Neumann algebras have always been an important topic in operator algebra theory, see e.g. [2] and [23]. In these works the importance of Haagerup's results on the canonical implementation of locally compact group actions and his results on the dual weight construction, cannot be overestimated. It is simply used all the time, without noticing it. See [10] and [11].

Hence it seems natural to study more generally actions of locally compact quantum groups on von Neumann algebras and to try to develop the same machinery of canonical implementation and dual weight construction. This is what is done in the first half of this paper. We strongly believe that this will serve as an important tool in several applications of locally compact quantum groups. We already give some applications in the second half of this paper. Other applications are given by Kustermans in [17] and by Vainerman and the author in [29a]. This will be explained below.

The special case of Kac algebra actions has been studied by Enock and Schwartz in [4] and [5]. They obtained important results on crossed products, with the biduality theorem as a major achievement. But they never obtained a unitary implementation for an arbitrary action and also Haagerup's theory of dual weights on the crossed product could not be completely generalized. For instance, it remained an open problem whether the crossed product with a Kac algebra action on a von Neumann algebra is in standard position on its natural Hilbert space. It should be mentioned that in [1] also Baaj and Skandalis obtain a biduality theorem for crossed products with multiplicative unitaries.

A first attempt to obtain the unitary implementation of a Kac algebra action was made by J.-L. Sauvageot in [25]. Unfortunately his proof is wrong, and for this we refer to the discussion in the beginning of Section 3.

So in this paper we will define actions of a locally compact quantum group on a von Neumann algebra and we will construct its unitary implementation. We will also give a construction for the dual weight on the crossed product and prove analogous results as those about group actions obtained by Haagerup in [10] and [11]. In particular we prove that the crossed product is in standard position on its natural Hilbert space and we identify the associated modular objects. Hence we do not only give a right proof for the results of Sauvageot, but also we prove new results on the dual weights which make then a workable and applicable tool, and we work in the more general setting of locally compact quantum groups.

In the second half of the paper we will give some applications of these results in the theory of subfactors and inclusions of von Neumann algebras. It has been proved by Enock and Nest in their beautiful papers [7] and [8] that every irreducible, depth 2 inclusion of factors satisfying the regularity condition, can be obtained as $N^\alpha \subset N$ where α is an outer action of a locally compact quantum group on the factor N and N^α is the fixed point algebra. We show in this paper that the action α is always integrable and that, conversely, for every outer and integrable action α on a factor N the inclusion $N^\alpha \subset N$ is irreducible, of depth 2 and regular. So we obtain a converse to the results of Enock and Nest. The same result is stated for the special case of a dual Kac algebra action in [7, 11.14], but not proved. While doing this, we study more generally the problem when the inclusion $N^\alpha \subset N \hookrightarrow M\,{}_\alpha\!\ltimes N$ is a basic construction, and here $M\,{}_\alpha\!\ltimes N$ denotes the crossed product.

As a final application of our results we prove the equivalence of outerness and minimality of a locally compact quantum group action, under the integrability condition. We also generalize the main theorem of Yamanouchi [33] to actions of arbitrary locally compact quantum groups: when working on separable Hilbert spaces, we prove that every integrable outer action with infinite fixed point algebra is a dual action.

It should also be mentioned that our results on the unitary implementation of a locally compact quantum group action are already applied in a recent paper by Kustermans (see [17]) in which he constructs induced corepresentations of locally compact quantum groups. Taking into account the importance of induced representations of locally compact groups, it is clear that the results of Kustermans serve as a major motivation of our work.

DEFINITIONS AND NOTATIONS

The whole of this paper will rely heavily on the modular theory of von Neumann algebras. Throughout the text we will not make efforts to give references to the original papers, but we will use [26] as a general reference.

When θ is a normal, semifinite and faithful (we say n.s.f.) weight on a von Neumann algebra N, one can make the so-called GNS-construction $(K_\theta, \pi_\theta, \Lambda_\theta)$, where K_θ is a Hilbert space, π_θ is a normal representation of N on K_θ and Λ_θ: $\mathcal{N}_\theta \rightarrow K_\theta$ is a linear map satisfying $\pi_\theta(x)\,\Lambda_\theta(y) = \Lambda_\theta(xy)$ for all $x \in N$ and $y \in \mathcal{N}_\theta$. Further $\Lambda_\theta(\mathcal{N}_\theta)$ is dense in K_θ. Here $\mathcal{N}_\theta$ is the left ideal in N defined by $\{x \in N \mid \theta(x^*x) < \infty\}$. The representation π_θ is faithful and often we will identify N and $\pi_\theta(N)$. Then we will use the expression: let us represent N on the GNS-space of θ such that $(K_\theta, \iota, \Lambda_\theta)$ is a GNS-construction.

We will use several standard notations and results of modular theory. We write

$$\mathcal{M}_\theta^+ = \{x \in N^+ \mid \theta(x) < \infty\}$$

and we denote with $(\sigma_t^\theta)_t$ the modular automorphism group of θ. Further we denote with $\mathcal{T}_\theta$ the Tomita algebra defined by

$$\mathcal{T}_\theta = \{x \in N \mid x \text{ is analytic with respect to } (\sigma^\theta) \text{ and}$$
$$\sigma_z^\theta(x) \in \mathcal{N}_\theta \cap \mathcal{N}_\theta^* \text{ for all } z \in \mathbb{C}\}.$$

Given a GNS-construction $(K_\theta, \pi_\theta, \Lambda_\theta)$ we define as usual the modular conjugation J_θ and the modular operator ∇_θ. Recall that

$$J_\theta \nabla_\theta^{1/2} \Lambda_\theta(x) = \Lambda_\theta(x^*)$$

for all $x \in \mathcal{N}_\theta \cap \mathcal{N}_\theta^*$ and $\Lambda_\theta(\mathcal{N}_\theta \cap \mathcal{N}_\theta^*)$ is a core for $\nabla_\theta^{1/2}$.

When θ_1 and θ_2 are n.s.f. weights on N we denote with $([D\theta_1 : D\theta_2]_t)_{t \in \mathbb{R}}$ the Connes cocycle as defined in e.g. [26, 3.1].

Often we will make use of operator valued weights. When N is a von Neumann algebra we denote with N_{ext}^+ the extended positive part of N as introduced by Haagerup in [13], see e.g. [26, 11.1]. For the notion of operator valued weights we refer to [13] or [26, 11.5]. We will denote with $\langle \cdot, \cdot \rangle$ the composition of elements of N_{ext}^+ and N_*^+. When T is an operator valued weight we denote with $\mathcal{N}_T$ the left ideal of elements x such that $T(x^*x)$ is bounded.

All tensor products in this paper are either von Neumann algebra tensor products or tensor products of Hilbert spaces. This will always be clear from the context. We will use σ to denote the flip map on a tensor product $A \otimes B$ of von Neumann algebras and Σ to denote the flip map on a tensor product $H \otimes K$ of Hilbert spaces. When K is a Hilbert space and $\xi \in K$ we denote with θ_ξ the operator in $B(\mathbb{C}, K)$ given by $\theta_\xi(\lambda) = \lambda\xi$ for all $\lambda \in \mathbb{C}$. When H is a Hilbert space and $\xi, \eta \in H$ we denote with $\omega_{\xi,\eta}$ the usual vector functional in $B(H)_*$ given by $\omega_{\xi,\eta}(x) = \langle x\xi, \eta \rangle$. We use ω_ξ as a shorter notation for $\omega_{\xi,\xi}$. We will denote with $\mathcal{D}(T)$ the domain of a (usually densely defined) map T.

Throughout this paper the pair (M, Δ) will denote a (von Neumann algebraic) locally compact quantum group. This means that

- M is a von Neumann algebra and $\Delta: M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying coassociativity: $(\Delta \otimes \iota)\,\Delta = (\iota \otimes \Delta)\,\Delta$.
- There exist n.s.f. weights φ and ψ on M such that

  — φ is left invariant in the sense that $\varphi((\omega \otimes \iota)\,\Delta(x)) = \varphi(x)\,\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$.

  — ψ is right invariant in the sense that $\psi((\iota \otimes \omega)\,\Delta(x)) = \psi(x)\,\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

We refer to [19, 21] and [18] for the theory of locally compact quantum groups in either the von Neumann algebra or C*-algebra language. It should be stressed that in [19] there is given a definition of a locally compact quantum group in the C*-algebra framework, and it is proven that one can associate with this a locally compact quantum group in the von Neumann algebra framework. In [21] the above definition of a von Neumann algebraic locally compact quantum group is given and it is shown how to associate with it a C*-algebraic locally compact quantum group.

One can then prove the existence of a σ-strong* closed map S on M, called the antipode, such that for all a, $b \in \mathcal{N}_\varphi$ we have

$$(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \in \mathcal{D}(S)$$

and

$$S((\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))) = (\iota \otimes \varphi)((1 \otimes a^*)\, \Delta(b)).$$

Moreover, such elements $(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))$ span a σ-strong* core for S. Then there exists a unique one-parameter group $(\tau_t)_{t \in \mathbb{R}}$ of automorphisms of M and a unique $*$-anti-automorphism R of M such that

$$S = R\tau_{-i/2} \qquad R^2 = \iota \qquad R\tau_t = \tau_t R \qquad \text{for all} \quad t \in \mathcal{R}$$

We call τ the scaling group of (M, Δ) and R the unitary antipode. One refers to the expression $S = R\tau_{-i/2}$ as the polar decomposition of the antipode.

Next one can prove that $\Delta R = \sigma(R \otimes R)\, \Delta$, where σ denotes the flip map on $M \otimes M$. One can also prove that the left and right invariant weights on (M, Δ) are unique up to a positive scalar. So ψ and φR are proportional and we can suppose from the beginning that $\psi = \varphi R$. We denote with $(\sigma_t)_{t \in \mathbb{R}}$ the modular group of φ. Then there exists a unique self-adjoint, strictly positive operator δ affiliated with M and satisfying $\sigma_t(\delta) = \nu^t \delta$ and $\psi = \varphi_\delta$, where $\nu > 0$ is a real number. Formally this means that $\psi(x) = \varphi(\delta^{1/2} x \delta^{1/2})$ and for an exact definition of φ_δ we refer to [29, 1.5]. We call δ the modular element of (M, Δ). The number $\nu > 0$ is called the scaling constant of (M, Δ).

Let us represent M on the GNS-space H of φ such that (H, ι, Λ) is a GNS-construction for φ. Let (H, ι, Γ) be the canonical GNS-construction for $\psi = \varphi_\delta$ as defined in [19, 7.2]. Then one can define unitaries W and V in $B(H \otimes H)$ such that

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \qquad \text{for all} \quad a, b \in \mathcal{N}_\varphi$$

$$V(\Gamma(a) \otimes \Gamma(b)) = (\Gamma \otimes \Gamma)(\Delta(a)(1 \otimes b)) \qquad \text{for all} \quad a, b \in \mathcal{N}_\psi.$$

Here $\Lambda \otimes \Lambda$ and $\Gamma \otimes \Gamma$ denote the canonical GNS-maps for the tensor product weights $\varphi \otimes \varphi$ and $\psi \otimes \psi$. Then W and V are multiplicative unitaries, which means that they satisfy the pentagonal equation

$$W_{12} W_{13} W_{23} = W_{23} W_{12}.$$

The unitaries W and V will be used throughout the paper. Denoting with $^-$ the σ-strong* closure we have that

$$M = \{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}^- \qquad \text{and}$$

$$\Delta(x) = W^*(1 \otimes x) W \qquad \text{for all} \quad x \in M.$$

We will denote with J and ∇ the modular conjugation and modular operator of φ in the GNS-construction (H, ι, Λ).

Finally we describe how to define the dual locally compact quantum group $(\hat{M}, \hat{\Delta})$. Define the von Neumann algebra $\hat{M}$ as follows, where again $^-$ denotes the σ-strong* closure.

$$\hat{M} = \{(\omega \otimes \iota)(W) \mid \omega \in M_*\}^-.$$

Then one can define a comultiplication $\hat{\Delta}$ on $\hat{M}$ by

$$\hat{\Delta}(y) = \Sigma W(y \otimes 1) W^* \Sigma \qquad \text{for all} \quad y \in \hat{M}$$

where Σ denotes the flip map on $H \otimes H$. When $\omega \in M_*$ we define $\lambda(\omega) = (\omega \otimes \iota)(W)$. Of course M_* should be thought of as the L^1-functions on the quantum group (M, Δ), and then λ is the left regular representation. We also define

$$\mathcal{I} = \{\omega \in M_* \mid \text{there exists } \eta \in H \text{ such that}$$
$$\omega(x^*) = \langle \eta, \Lambda(x) \rangle \text{ for all } x \in \mathcal{N}_\varphi\}.$$

Such a $\eta \in H$ is necessarily uniquely determined and will be denoted with $\xi(\omega)$. Then there exists a unique n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ with GNS-construction $(H, \iota, \hat{\Lambda})$ such that $\lambda(\mathcal{I}) \subset \mathcal{N}_{\hat{\varphi}}$, $\lambda(\mathcal{I})$ is a σ-strong*-norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega)$ for all $\omega \in \mathcal{I}$.

Then $(\hat{M}, \hat{\Delta})$ will be a locally compact quantum group, and having fixed the GNS-construction $(H, \iota, \hat{\Lambda})$ for $\hat{\varphi}$ we can now repeat the story about (M, Δ) and obtain $(\hat{\sigma}_t)_{t \in \mathbb{R}}$, $\hat{\delta}$, $\hat{W}$, $\hat{V}$, $\hat{J}$ and $\hat{\nabla}$. For all kinds of formulas relating these objects we refer to [21]. We only mention that

$$\hat{J} J = \nu^{i/4} J \hat{J}$$

$$\hat{\nabla}^{it} x \hat{\nabla}^{-it} = \tau_t(x) \qquad \text{and} \qquad \hat{J} x^* \hat{J} = R(x) \qquad \text{for all} \quad x \in M, \quad t \in \mathbb{R}.$$

Finally we denote with $(M, \Delta)^{\mathrm{op}}$ the opposite locally compact quantum group $(M, \Delta^{\mathrm{op}})$ where $\Delta^{\mathrm{op}} = \sigma\Delta$. Further we define $(M, \Delta)' = (M', \Delta')$ where

$$\Delta'(x) = (J \otimes J)\, \Delta(JxJ)(J \otimes J)$$

for all $x \in M'$ and we call $(M, \Delta)'$ the commutant locally compact quantum group. Then one can prove that

$$(M, \Delta)^{\mathrm{op}\,\hat{}} = (M, \Delta)^{\hat{}\,\prime}$$

and for this we refer to [21].

1. ACTIONS OF LOCALLY COMPACT QUANTUM GROUPS

In this section we define actions of locally compact quantum groups on von Neumann algebras and we construct an important tool: the canonical operator valued weight from the von Neumann algebra on which we act to the fixed point algebra, obtained by integrating out the action.

Definition 1.1. Let N be a von Neumann algebra. A normal, injective and unital $*$-homomorphism $\alpha\colon N \to M \otimes N$ will be called a left action of (M, Δ) on N when $(\iota \otimes \alpha)\,\alpha = (\Delta \otimes \iota)\,\alpha$.

A normal, injective and unital $*$-homomorphism $\alpha\colon N \to N \otimes M$ will be called a right action of (M, Δ) on N when $(\alpha \otimes \iota)\,\alpha = (\iota \otimes \Delta)\,\alpha$.

In this paper we will only work with left actions and so we drop the predicate *left*. When α is a right action, $\sigma\alpha$ will be a left action of $(M, \Delta^{\mathrm{op}})$ on N, where σ denotes the flip map from $N \otimes M$ to $M \otimes N$ and Δ^{op} denotes the opposite comultiplication. So it is not a real restriction to work only with left actions. It should be observed that in [4] and [5] they work with right actions.

Definition 1.2. When $\alpha\colon N \to M \otimes N$ is an action of (M, Δ) on N we define the fixed point algebra N^{α} as

$$N^{\alpha} = \{x \in N \mid \alpha(x) = 1 \otimes x\}.$$

It is clear that N^{α} is a von Neumann subalgebra of N.

Recall that N^{+}_{ext} denotes the extended positive part of N.

Proposition 1.3. *Let N be a von Neumann algebra and α an action of (M, Δ) on N. For every $x \in N^{+}$ the element $T_{\alpha}(x) = (\psi \otimes \iota)\,\alpha(x)$ of N^{+}_{ext}*

belongs to $(N^{\alpha})^{+}_{\text{ext}}$. *Further* T_{α} *is a normal, faithful operator valued weight from* N *to* N^{α}.

Proof. Let $x \in N^{+}$ and $\omega \in (M \otimes N)^{+}_{*}$. Denote with $\langle \cdot, \cdot \rangle$ the composition of an element in N^{+}_{ext} and an element in N^{+}_{*}. Then by definition of the operator valued weight $\psi \otimes \iota$ we get

$$\begin{aligned}\langle T_{\alpha}(x), \omega\alpha \rangle = \langle (\psi \otimes \iota)\, \alpha(x), \omega\alpha \rangle &= \psi((\iota \otimes \omega\alpha)\, \alpha(x)) \\ &= \psi((\iota \otimes \omega)(\Delta \otimes \iota)\, \alpha(x)) \\ &= \langle (\psi \otimes \iota \otimes \iota)((\Delta \otimes \iota)\, \alpha(x)), \omega \rangle.\end{aligned}$$

By the right invariant version of [21, 3.1] we get that

$$\langle T_{\alpha}(x), \omega\alpha \rangle = \langle 1 \otimes (\psi \otimes \iota)\, \alpha(x), \omega \rangle = \langle 1 \otimes T_{\alpha}(x), \omega \rangle.$$

From this it follows that $\alpha(T_{\alpha}(x)) = 1 \otimes T_{\alpha}(x)$. So we get that $T_{\alpha}(x) \in (N^{\alpha})^{+}_{\text{ext}}$.

If $x \in N^{+}$ and $a \in N^{\alpha}$ we have

$$\begin{aligned}\langle T_{\alpha}(axa^{*}), \omega \rangle &= \langle (\psi \otimes \iota)((1 \otimes a)\, \alpha(x)(1 \otimes a^{*})), \omega \rangle \\ &= \psi((\iota \otimes a^{*}\omega a)\, \alpha(x)) = \langle T_{\alpha}(x), a^{*}\omega a \rangle.\end{aligned}$$

So we get that T_{α} is indeed an operator valued weight. Because both α and $\psi \otimes \iota$ are faithful and normal, also T_{α} is faithful and normal. ∎

One should observe that the same result is stated and used in [5] for Kac algebra actions. Their proof contains a gap because they do not have an invariance formula like [21, 3.1], which is indispensable. For Kac algebra actions this was repaired by Zsidó (see [34], see also [26, 18.18 and 18.23]). Also in the case of a group action this was a non-trivial problem (see [11]). The simple proof of Zsidó for this invariance formula only works in the Kac algebra setting, where the scaling group is trivial. I would like to thank prof. Enock for the discussion on this topic.

Definition 1.4. An action α of (M, Δ) on a von Neumann algebra N is called integrable when the operator valued weight T_{α} defined in Proposition 1.3 is semifinite.

We will now introduce the well known concept of cocycle equivalent actions (cfr. [4, I.6]).

Definition 1.5. Let α be an action of (M, Δ) on the von Neumann algebra N. A unitary $U \in M \otimes N$ is called an α-cocycle if

$$(\Delta \otimes \iota)(U) = U_{23}(\iota \otimes \alpha)(U).$$

It is clear that in this case the formula

$$\beta(x) = U\alpha(x)\, U^* \qquad \text{for all} \quad x \in N$$

defines an action β of (M, Δ) on N.

Two actions α and β of (M, Δ) on N are called cocycle equivalent if there exists an α-cocycle U such that β is given by the formula above.

It is easy to see that U^* is a β-cocycle when U is an α-cocycle and when $\beta(x) = U\alpha(x)\, U^*$ for all $x \in N$.

2. CROSSED PRODUCTS, THE DUAL ACTION, AND THE DUALITY THEOREM

In this section we fix an action α of a locally compact quantum group (M, Δ) on a von Neumann algebra N. We will define the crossed product $M_\alpha \ltimes N$ in a similar way as in [4]. We will also state some classical theorems concerning crossed products, the biduality theorem being the major one, but we will omit proofs because they are completely analogous to the proofs of [4] and [5]. See also [9], where the results of [4] and [5] are generalized to actions of Woronowicz algebras.

Definition 2.1. We define the crossed product of N with respect to the action α of (M, Δ) on N as the von Neumann subalgebra of $B(H) \otimes N$ generated by $\alpha(N)$ and $\hat{M} \otimes \mathbb{C}$. We denote this crossed product with $M_\alpha \ltimes N$. So we have

$$M_\alpha \ltimes N = (\alpha(N) \cup \hat{M} \otimes \mathbb{C})''.$$

We will now define in the usual way the dual action, which will be an action of $(\hat{M}, \hat{\Delta}^{\text{op}})$ on $M_\alpha \ltimes N$.

Proposition 2.2. *There exists a unique action $\hat{\alpha}$ of $(\hat{M}, \hat{\Delta}^{\text{op}})$ on $M_\alpha \ltimes N$ such that*

$$\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x) \qquad \textit{for all} \quad x \in N$$

$$\hat{\alpha}(a \otimes 1) = \hat{\Delta}^{\text{op}}(a) \otimes 1 \qquad \textit{for all} \quad a \in \hat{M}.$$

Moreover when we denote with $\tilde{W}$ the unitary $(J \otimes J)\, \Sigma W \Sigma (J \otimes J)$ we have

$$\hat{\alpha}(z) = (\tilde{W} \otimes 1)(1 \otimes z)(\tilde{W}^* \otimes 1) \qquad \textit{for all} \quad z \in M_\alpha \ltimes N.$$

As we already mentioned, Enock and Schwartz deal with right actions in [4] and [5]. Hence they also give a slightly different definition for the crossed product, but in fact our definition agrees with theirs. When α is a right action of (M, Δ) on N they define $N \rtimes_\alpha M$ to be $(\alpha(N) \cup \mathbb{C} \otimes \hat{M}')''$. This is in accordance with our definition, because $\sigma\alpha$ is a left action of (M, Δ^{op}) on N. The dual of (M, Δ^{op}) is $(\hat{M}', \hat{\Delta}')$, which gives

$$M_{\sigma\alpha} \ltimes N = (\sigma\alpha(N) \cup \hat{M}' \otimes \mathbb{C})''.$$

So we have $N \rtimes_\alpha M = \sigma(M_{\sigma\alpha} \ltimes N)$, which shows that both definitions in fact agree.

Let us introduce the following concept, which will be needed later on. See also [5, III.1].

Definition 2.3. Let ρ be a self-adjoint, strictly positive operator affiliated with M. Then a n.s.f. weight θ on N is called ρ-invariant if

$$\theta((\omega_{\xi,\xi} \otimes \iota)\, \alpha(x)) = \|\rho^{1/2}\xi\|^2\, \theta(x)$$

for all $x \in \mathscr{M}_\theta^+$ and $\xi \in \mathscr{D}(\rho^{1/2})$.

We will always work with δ^{-1}-invariant weights, where δ is the modular element of the locally compact quantum group in action.

Then the following result can be proved as in [9, 2.9]. For the last statement of the next proposition observe that $\tau_t(\delta) = \delta$ and so the self-adjoint operators δ and $\hat{\nabla}$ commute strongly. Hence their product $\delta\hat{\nabla}$ is closable.

Proposition 2.4. *When θ is a n.s.f. δ^{-1}-invariant weight on N with GNS-construction $(H_\theta, \pi_\theta, \Lambda_\theta)$, then there exists a unique unitary $V_\theta \in M \otimes B(H_\theta)$ such that for all $\xi \in \mathscr{D}(\delta^{1/2})$, $\eta \in H$ and $x \in \mathscr{N}_\theta$*

$$(\omega_{\xi,\eta} \otimes \iota)(V_\theta)\, \Lambda_\theta(x) = \Lambda_\theta((\omega_{\delta^{1/2}\xi,\eta} \otimes \iota)\, \alpha(x)).$$

Denote with J_θ and ∇_θ the modular conjugation and modular operator of θ. Then V_θ satisfies

$$(\Delta \otimes \iota)(V_\theta) = V_{\theta 23} V_{\theta 13}$$

$$(\iota \otimes \pi_\theta)\, \alpha(x) = V_\theta(1 \otimes \pi_\theta(x))\, V_\theta^* \qquad \textit{for all} \quad x \in N$$

$$V_\theta(\hat{J} \otimes J_\theta) = (\hat{J} \otimes J_\theta)\, V_\theta^*$$

$$V_\theta(Q \otimes \nabla_\theta) = (Q \otimes \nabla_\theta)\, V_\theta \qquad \textit{where } Q \textit{ is the closure of } \delta\hat{\nabla}.$$

The following result is crucial (see [9, 2.8]).

PROPOSITION 2.5. • *Let* α *be an integrable action of* (M, Δ) *on* N *and denote with* T_α *the operator valued weight defined in Proposition* 1.3. *Let* μ *be a n.s.f. weight on* N^α. *Then* $\mu \circ T_\alpha$ *is a* δ^{-1}*-invariant weight on* N.

- *Every dual action is integrable.*

With these results at hand one can copy the proofs of [5] to obtain the well known biduality theorem. Before we state this theorem we have to clarify some terminology. The dual action $\hat{\alpha}$ is an action of $(M, \Delta)^{\wedge \mathrm{op}}$ on $M_\alpha \ltimes N$. So we can make the double crossed product $\hat{M}_{\hat{\alpha}} \ltimes (M_\alpha \ltimes N)$ in $B(H \otimes H) \otimes N$ and on this double crossed product there is an action $\hat{\hat{\alpha}}$ of $(M, \Delta)^{\wedge \mathrm{op} \wedge \mathrm{op}}$. Now $(M, \Delta)^{\wedge \mathrm{op} \wedge \mathrm{op}} = (M, \Delta)'^{\mathrm{op}}$ and we can define an isomorphism of locally compact quantum groups

$$\mathscr{J}: (M, \Delta) \to (M, \Delta)'^{\mathrm{op}}$$

given by $\mathscr{J}(x) = \hat{J} J x J \hat{J}$ for all $x \in M$.

THEOREM 2.6 (Biduality theorem). 1. *We have* $B(H) \otimes N = (B(H) \otimes \mathbb{C} \cup \alpha(N))''$.

2. *The map* Φ *from* $B(H) \otimes N$ *to* $B(H \otimes H) \otimes N$ *defined by*

$$\Phi(z) = (W \otimes 1)(\iota \otimes \alpha)(z)(W^* \otimes 1)$$

defines a $*$*-isomorphism from* $B(H) \otimes N$ *onto* $\hat{M}_{\hat{\alpha}} \ltimes (M_\alpha \ltimes N)$, *satisfying*

$$\Phi(\alpha(x)) = 1 \otimes \alpha(x) \qquad \text{for all} \quad x \in N$$

$$\Phi(b \otimes 1) = \hat{\Delta}^{\mathrm{op}}(b) \otimes 1 \qquad \text{for all} \quad b \in \hat{M}$$

$$\Phi(y \otimes 1) = y \otimes 1 \otimes 1 \qquad \text{for all} \quad y \in M'.$$

In particular $\Phi(M_\alpha \ltimes N) = \hat{\alpha}(M_\alpha \ltimes N)$.

3. *When we define*

$$\mu = (\sigma \otimes \iota)(\iota \otimes \alpha): B(H) \otimes N \to M \otimes B(H) \otimes N$$

then μ *is an action of* (M, Δ) *on* $B(H) \otimes N$. *The unitary* $\Sigma V^* \Sigma \otimes 1$ *is a* μ*-cocycle and the action* γ *of* (M, Δ) *on* $B(H) \otimes N$ *defined by*

$$\gamma(z) = (\Sigma V^* \Sigma \otimes 1)\, \mu(z) (\Sigma V \Sigma \otimes 1) \qquad \text{for all} \quad z \in B(H) \otimes N$$

is isomorphic to the bidual action $\hat{\hat{\alpha}}$ *of* $(M, \Delta)^{\wedge \mathrm{op} \wedge \mathrm{op}}$ *on* $\hat{M}_{\hat{\alpha}} \ltimes ((M_\alpha \ltimes N)$ *in the following way*:

$$\hat{\hat{\alpha}}(\Phi(z)) = (\mathscr{J} \otimes \Phi)\, \gamma(z) \qquad \text{for all} \quad z \in B(H) \otimes N.$$

With the help of the biduality theorem Enock and Schwartz were able to prove the following crucial results, which remain true for actions of locally compact quantum groups.

THEOREM 2.7. *We have*

$$(M_{\alpha} \ltimes N)^{\hat{\alpha}} = \alpha(N)$$

$$\alpha(N) = \{z \in M \otimes N \mid (\iota \otimes \alpha)(z) = (\Delta \otimes \iota)(z)\}.$$

3. THE UNITARY IMPLEMENTATION OF A LOCALLY COMPACT QUANTUM GROUP ACTION

In this section we will define in a canonical way the unitary implementation of a locally compact quantum group action. This will be a unitary corepresentation of the quantum group, implementing the action and satisfying some other properties. A same kind of result was obtained for Kac algebra actions by Sauvageot in [25], but the proof of the fact that the implementation is a corepresentation, is wrong. More precisely, Sauvageot's crucial Lemma 4.1 is false. I would like to thank prof. Sauvageot for the discussions on this topic.

We will use a different technique to prove that the implementation is a corepresentation. In the same time we will obtain some interesting results concerning the dual weight on the crossed product $M_{\alpha} \ltimes N$ given a weight on N. We will also settle a problem which was left open in [25].

For integrable actions—and in particular for dual actions—we already obtained an implementation in Proposition 2.4, as it was done by Enock and Schwartz. Nevertheless it is desirable to have an implementation without the integrability condition, first of all for reasons of elegance. But, more importantly, one will need this general implementation result in several applications. We refer to the introduction for a discussion.

Fix an action α of a locally compact quantum group (M, Δ) on a von Neumann algebra N. In Definition 2.1 and Proposition 2.2 we defined the crossed product $M_{\alpha} \ltimes N$ and the dual action $\hat{\alpha}: M_{\alpha} \ltimes N \rightarrow \hat{M} \otimes (M_{\alpha} \ltimes N)$. We already observed in Proposition 2.5 that $\hat{\alpha}$ is integrable. So we can define the n.s.f. operator valued with T from $M_{\alpha} \ltimes N$ to $(M_{\alpha} \ltimes N)^{\hat{\alpha}}$ by

$$T(z) = (\hat{\varphi} \otimes \iota \otimes \iota)\, \hat{\alpha}(z) \qquad \text{for all} \quad z \in (M_{\alpha} \ltimes N)^{+}.$$

For this, observe that $\hat{\alpha}$ is an action of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ and that $\hat{\varphi}$ is the right invariant weight on $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$. By Theorem 2.7 we know that $(M_{\alpha} \ltimes N)^{\hat{\alpha}} = \alpha(N)$. So T is an operator valued weight from $M_{\alpha} \ltimes N$ to $\alpha(N)$.

With this operator valued weight at hand, we can easily define the dual weights on $M_\alpha \ltimes N$. Nevertheless, to make dual weights a workable tool, we need a concrete GNS-construction for them. The structure of this section is then as follows. First we will restrict the dual weight to a weight for which we can give a GNS-construction (Definition 3.4), then we use the restricted weight to obtain the unitary implementation for the action (Definition 3.6 and Proposition 3.7) and finally we prove that the restricted weight is in fact not a restriction, but equal to the original dual weight (Proposition 3.10).

DEFINITION 3.1. Let α be an action of (M, Δ) on N. Denote with T the n.s.f. operator valued weight from $M_\alpha \ltimes N$ to $\alpha(N)$ given by the formula above. For every n.s.f. weight θ on N, we define the dual weight $\tilde{\theta}$ on $M_\alpha \ltimes N$ by the formula:

$$\tilde{\theta} = \theta \circ \alpha^{-1} \circ T.$$

For the rest of this section we fix a n.s.f. weight θ on N. One can prove easily the following lemma.

LEMMA 3.2. *For all $a \in \mathcal{N}_{\hat{\varphi}}$ and $x \in N$ we have*

$$\tilde{\theta}(\alpha(x^*)(a^*a \otimes 1)\,\alpha(x)) = \theta(x^*x)\;\hat{\varphi}(a^*a).$$

Proof. We have

$$\hat{\alpha}(\alpha(x^*)(a^*a \otimes 1)\,\alpha(x)) = (1 \otimes \alpha(x^*))(\hat{\Delta}^{\mathrm{op}}(a^*a) \otimes 1)(1 \otimes \alpha(x)).$$

Choose $\omega \in (M_\alpha \ltimes N)_*^+$. Define $\mu \in \hat{M}_*^+$ by $\mu(b) = \omega(\alpha(x^*)(b \otimes 1)\,\alpha(x))$ for all $b \in \hat{M}$. Then

$$\begin{aligned} \langle T(\alpha(x^*)(a^*a \otimes 1)\,\alpha(x)), \omega \rangle &= \hat{\varphi}((\iota \otimes \mu)\,\hat{\Delta}^{\mathrm{op}}(a^*a)) \\ &= \hat{\varphi}(a^*a)\,\mu(1) = \hat{\varphi}(a^*a)\,\omega(\alpha(x^*x)) \end{aligned}$$

by invariance of $\hat{\varphi}$. So we may conclude that

$$T(\alpha(x^*)(a^*a \otimes 1)\,\alpha(x)) = \hat{\varphi}(a^*a)\,\alpha(x^*x).$$

Then the result of the lemma follows immediately. ∎

From now on we will suppose that N acts on the GNS-space of the n.s.f. weight θ, such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction for θ. We will restrict the weight $\tilde{\theta}$ in the sense of Proposition 7.4 of the appendix in order to obtain a concrete GNS-construction. Fix a GNS-construction (K_1, π_1, Λ_1) for $\tilde{\theta}$. Because of the previous lemma we can define a unique isometry

$$\mathscr{V}: H \otimes K \to K_1 \qquad \text{such that} \quad \mathscr{V}(\hat{\Lambda}(a) \otimes \Lambda_\theta(x)) = \Lambda_1((a \otimes 1)\, \alpha(x))$$

for all $a \in \mathscr{N}_{\hat{\varphi}}$ and $x \in \mathscr{N}_\theta$.

Further we define

$$\mathscr{D}_0 = \operatorname{span}\{(a \otimes 1)\, \alpha(x) \mid a \in \mathscr{N}_{\hat{\varphi}}, x \in \mathscr{N}_\theta\}.$$

Because we have the isometry $\mathscr{V}$ at our disposal there is a well defined linear map

$$\tilde{\Lambda}_0: \mathscr{D}_0 \to H \otimes K: \tilde{\Lambda}_0((a \otimes 1)\, \alpha(x)) = \hat{\Lambda}(a) \otimes \Lambda_\theta(x) \qquad \text{for all} \quad a \in \mathscr{N}_{\hat{\varphi}}, \quad x \in \mathscr{N}_\theta.$$

Because Λ_1 is σ-strong*-norm closed, we can close $\tilde{\Lambda}_0$ for the σ-strong*-norm topology, and then we obtain a linear map $\tilde{\Lambda}: \mathscr{D} \to H \otimes K$ satisfying $\mathscr{D} \subset \mathscr{N}_{\tilde{\theta}}$ and $\mathscr{V} \tilde{\Lambda}(z) = \Lambda_1(z)$ for all $z \in \mathscr{D}$.

In order to apply Proposition 7.4, we need the following lemma.

Lemma 3.3. 1. *$\mathscr{D}$ is a weakly dense left ideal in $M_\alpha \ltimes N$.*

2. *For all $z \in M_\alpha \ltimes N$ and $y \in \mathscr{D}$ we have $\tilde{\Lambda}(zy) = z\tilde{\Lambda}(y)$.*

Proof. Choose $\xi \in H$ and $b \in \mathscr{T}_\varphi$. Let $(e_i)_{i \in I}$ be an orthonormal basis for H. Choose $x \in N$. Because $(\Delta \otimes \iota)\, \alpha(x) = (\iota \otimes \alpha)\, \alpha(x)$ we have

$$(1 \otimes \alpha(x))(W \otimes 1) = (W \otimes 1)(\iota \otimes \alpha)\, \alpha(x).$$

Hence applying $\omega_{\xi, \Lambda(b)} \otimes \iota \otimes \iota$ gives

$$\alpha(x)(\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1) = \sum_{i \in I} (\lambda(\omega_{e_i, \Lambda(b)}) \otimes 1)\, \alpha((\omega_{\xi, e_i} \otimes \iota)\, \alpha(x))$$

in the σ-strong* topology.

Choose now $y \in \mathscr{N}_\theta$. For every finite subset $I_0 \subset I$ we have by Proposition 7.1 that the element

$$z_{I_0} := \sum_{i \in I_0} (\lambda(\omega_{e_i, \Lambda(b)}) \otimes 1)\, \alpha((\omega_{\xi, e_i} \otimes \iota)\, \alpha(x)\; y)$$

belongs to $\mathscr{D}_0$ and

$$\begin{aligned}\tilde{\Lambda}_0(z_{I_0}) &= \sum_{i\in I_0} \hat{\Lambda}(\lambda(\omega_{e_i, \Lambda(b)})) \otimes (\omega_{\xi, e_i} \otimes \iota)\, \alpha(x)\, \Lambda_\theta(y) \\ &= \sum_{i\in I_0} J\sigma_{i/2}(b)\, Je_i \otimes (\omega_{\xi, e_i} \otimes \iota)\, \alpha(x)\, \Lambda_\theta(y) \\ &= (J\sigma_{i/2}(b)\, J \otimes 1)(P_{I_0} \otimes 1)\, \alpha(x)(\xi \otimes \Lambda_\theta(y)),\end{aligned}$$

where P_{I_0} is the projection on $\operatorname{span}\{e_i \mid i \in I_0\}$. So we get that the net (z_{I_0}) converges σ-strong* to the element

$$z := \alpha(x)(\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \alpha(y)$$

and the net $(\tilde{\Lambda}(z_{I_0}))$ converges in norm to

$$\begin{aligned}(J\sigma_{i/2}(b)\, J \otimes 1)\, \alpha(x)(\xi \otimes \Lambda_\theta(y)) &= \alpha(x)(J\sigma_{i/2}(b)\, J\xi \otimes \Lambda_\theta(y)) \\ &= \alpha(x)(\hat{\Lambda}(\lambda(\omega_{\xi, \Lambda(b)})) \otimes \Lambda_\theta(y)) \\ &= \alpha(x)\, \tilde{\Lambda}((\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \alpha(y)).\end{aligned}$$

Then we may conclude that $z \in \mathscr{D}$ and

$$\tilde{\Lambda}(z) = \alpha(x)\, \tilde{\Lambda}((\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \alpha(y)).$$

Because the considered elements $\lambda(\omega_{\xi, \Lambda(b)})$ form a σ-strong*-norm core for $\hat{\Lambda}$ we conclude that for every $x \in N$ and $z \in \mathscr{D}$ we have $\alpha(x)\, z \in \mathscr{D}$ and $\tilde{\Lambda}(\alpha(x)\, z) = \alpha(x)\, \tilde{\Lambda}(z)$.

It is easy to prove that for every $a \in \hat{M}$ and $z \in \mathscr{D}$ we have $(a \otimes 1)\, z \in \mathscr{D}$ and $\tilde{\Lambda}((a \otimes 1)\, z) = (a \otimes 1)\, \tilde{\Lambda}(z)$. From this follows the lemma. ∎

We can now apply Proposition 7.4.

Definition 3.4. There is a unique n.s.f. weight $\tilde{\theta}_0$ on $M_\alpha \ltimes N$ such that $\mathscr{N}_{\tilde{\theta}_0} = \mathscr{D}$ and such that $(H \otimes K, \iota, \tilde{\Lambda})$ is a GNS-construction for $\tilde{\theta}_0$.

Later on we will prove that in fact $\tilde{\theta}_0 = \tilde{\theta}$. This question was left open in the Kac algebra case considered by Sauvageot. In applications the equality $\tilde{\theta}_0 = \tilde{\theta}$ is indispensable, e.g. Proposition 5.7 cannot be proved without knowing the GNS-construction of $\tilde{\theta}$, which amounts to the equality $\tilde{\theta}_0 = \tilde{\theta}$.

In [29a] Vainerman and the author study the bicrossed product construction and extensions locally compact quantum groups. In this situation the Haar weight will be a dual weight and in order to prove invariance and to compute the multiplicative unitary, we need a concrete GNS-construction for this weight.

Let us fix some modular notations.

Definition 3.5. We denote with $\tilde{J}$ and $\tilde{\nabla}$ the modular conjugation and modular operator of $\tilde{\theta}_0$ in the GNS-construction $(H \otimes K, \iota, \tilde{\Lambda})$. We denote with $\tilde{\sigma}$ the modular automorphism group of $\tilde{\theta}_0$ and we put $\tilde{T} = \tilde{J}\tilde{\nabla}^{1/2}$.

We denote with J_θ and ∇_θ the modular conjugation and modular operator of θ in the GNS-construction $(K, \iota, \Lambda_\theta)$, and with σ^θ the modular automorphism group of θ.

With this notations at hand we will now define the unitary implementation of the action α. Of course this terminology will only be justified after the proofs of 3.7, 3.12 and 4.4.

DEFINITION 3.6. Define $U = \tilde{J}(\hat{J} \otimes J_\theta)$. Then U is a unitary in $B(H \otimes K)$ and it is called the unitary implementation of α.

We will first prove the following result.

PROPOSITION 3.7. *We have the following formulas*:

1. $\alpha(x) = U(1 \otimes x) U^*$ *for all* $x \in N$.
2. $\tilde{\sigma}_t \circ \alpha = \alpha \circ \sigma^\theta_t$ *for all* $t \in \mathbb{R}$.
3. $U(\hat{J} \otimes J_\theta) = (\hat{J} \otimes J_\theta) U^*$.

Before we can prove this proposition we need the following lemma.

LEMMA 3.8. *For all* $y \in \mathcal{D}(\sigma^\theta_{i/2})$ *we have* $\alpha(y) \in \mathcal{D}(\tilde{\sigma}_{i/2})$ *and*

$$\tilde{J}\tilde{\sigma}_{i/2}(\alpha(y))^* \tilde{J} = 1 \otimes J_\theta \sigma^\theta_{i/2}(y)^* J_\theta.$$

Proof. Choose $a \in \mathcal{N}_{\hat{\varphi}}$ and $x \in \mathcal{N}_\theta$. Then $xy \in \mathcal{N}_\theta$ and hence $(a \otimes 1)\alpha(x)\alpha(y) \in \mathcal{N}_{\tilde{\theta}_0}$ with

$$\begin{aligned}\tilde{\Lambda}((a \otimes 1)\alpha(x)\alpha(y)) &= \hat{\Lambda}(a) \otimes \Lambda_\theta(xy)\\ &= (1 \otimes J_\theta \sigma^\theta_{i/2}(y)^* J_\theta)\tilde{\Lambda}((a \otimes 1)\alpha(x)).\end{aligned}$$

Because $\mathcal{D}_0$ is a σ-strong*-norm core for $\tilde{\Lambda}$ we may conclude that for every $z \in \mathcal{N}_{\tilde{\theta}_0}$ we have $z\alpha(y) \in \mathcal{N}_{\tilde{\theta}_0}$ and

$$\tilde{\Lambda}(z\alpha(y)) = (1 \otimes J_\theta \sigma^\theta_{i/2}(y)^* J_\theta)\tilde{\Lambda}(z).$$

Then the lemma follows immediately. ∎

Proof of Proposition 3.7. Because $\sigma^\theta_{i/2}(y)^* = \sigma^\theta_{-i/2}(y^*)$ it follows from the previous lemma that for every $y \in \mathcal{D}(\sigma^\theta_{-i/2})$ we have $\alpha(y) \in \mathcal{D}(\tilde{\sigma}_{-i/2})$ and

$$\tilde{\sigma}_{-i/2}(\alpha(y)) = U(1 \otimes \sigma^\theta_{-i/2}(y)) U^*.$$

Taking the adjoint we may replace $-i/2$ by $i/2$ in the formula above.

Let now $y \in \mathcal{D}(\sigma^{\theta}_{-i})$. Then we have $\alpha(y) \in \mathcal{D}(\tilde{\sigma}_{-i/2})$ and

$$\tilde{\sigma}_{-i/2}(\alpha(y)) = U(1 \otimes \sigma^{\theta}_{-i/2}(y))\, U^*.$$

Because $\sigma^{\theta}_{-i}(y) \in \mathcal{D}(\sigma^{\theta}_{i/2})$ we also have $\alpha(\sigma^{\theta}_{-i}(y)) \in \mathcal{D}(\tilde{\sigma}_{i/2})$ and

$$\tilde{\sigma}_{i/2}(\alpha(\sigma^{\theta}_{-i}(y))) = U(1 \otimes \sigma^{\theta}_{i/2}(\sigma^{\theta}_{-i}(y)))\, U^* = U(1 \otimes \sigma^{\theta}_{-i/2}(y))\, U^*.$$

So we get $\tilde{\sigma}_{-i/2}(\alpha(y)) = \tilde{\sigma}_{i/2}(\alpha(\sigma^{\theta}_{-i}(y)))$ and so $\alpha(y) \in \mathcal{D}(\tilde{\sigma}_{-i})$ with $\tilde{\sigma}_{-i}(\alpha(y)) = \alpha(\sigma^{\theta}_{-i}(y))$. It now follows from the results of [13, 4.3 and 4.4] that $\tilde{\sigma}_t \circ \alpha = \alpha \circ \sigma^{\theta}_t$ for every $t \in \mathbb{R}$.

But then it follows that for all $y \in \mathcal{D}(\sigma^{\theta}_{-i/2})$ we have $\tilde{\sigma}_{-i/2}(\alpha(y)) = \alpha(\sigma^{\theta}_{-i/2}(y))$. Combining this with the formula above we get

$$\alpha(\sigma^{\theta}_{-i/2}(y)) = \tilde{\sigma}_{-i/2}(\alpha(y)) = U(1 \otimes \sigma^{\theta}_{-i/2}(y))\, U^*.$$

By the density of such elements $\sigma^{\theta}_{-i/2}(y)$ we get that $\alpha(x) = U(1 \otimes x)\, U^*$ for all $x \in N$.

From the definition of U follows immediately the final formula we had to prove. ∎

Now we have gathered enough material to prove that $\tilde{\theta}_0 = \tilde{\theta}$. For this we need the following lemma (cfr. [5, VI.4]).

LEMMA 3.9. *Let α be an action of (M, Δ) on N. Let θ_1 and θ_2 be two δ^{-1}-invariant n.s.f. weights on N. Then $[D\theta_2 : D\theta_1]_t \in N^{\alpha}$ for all $t \in \mathbb{R}$.*

Proof. Denote with M_2 the von Neumann algebra of 2×2-matrices over $\mathbb{C}$. Denote with e_{ij} the matrix units. Define

$$\gamma: N \otimes M_2 \rightarrow M \otimes N \otimes M_2 : \gamma = \alpha \otimes \iota.$$

Then γ is an action of (M, Δ) on $N \otimes M_2$. Denote with θ the balanced weight on $N \otimes M_2$ (see e.g. [26, 3.1]) given by

$$\theta \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \theta_1(x_{11}) + \theta_2(x_{22}).$$

It is immediately clear that θ is δ^{-1}-invariant for the action γ.

Let $t \in \mathbb{R}$. Denote with μ_t the automorphism of M defined by $\mu_t = \sigma'_t \circ \sigma_{-t} \circ \tau_t$. Here $(\sigma'_t)_{t \in \mathbb{R}}$ denotes the modular automorphism group of ψ.

Then μ_t is implemented by $Q^{it} = \delta^{it} \hat{\nabla}^{it}$. It follows from Proposition 2.4 that $\gamma \circ \sigma_t^\theta = (\mu_t \otimes \sigma_t^\theta) \circ \gamma$ for all $t \in \mathbb{R}$. In particular we have

$$\begin{aligned}\alpha([D\theta_2 : D\theta_1]_t) \otimes e_{21} &= \gamma([D\theta_2 : D\theta_1]_t \otimes e_{21}) = \gamma(\sigma_t^\theta(1 \otimes e_{21})) \\ &= (\mu_t \otimes \sigma_t^\theta)\, \gamma(1 \otimes e_{21}) = (\mu_t \otimes \sigma_t^\theta)(1 \otimes 1 \otimes e_{21}) \\ &= 1 \otimes [D\theta_2 : D\theta_1]_t \otimes e_{21}.\end{aligned}$$

So we get $[D\theta_2 : D\theta_1]_t \in N^\alpha$ for all $t \in \mathbb{R}$. ∎

Now we can prove the following interesting result. It is important for technical reasons and we will need it in Section 5.

PROPOSITION 3.10. *Let θ be a n.s.f. weight on N. Then the weights $\tilde{\theta}$ and $\tilde{\theta}_0$ on $M_\alpha \ltimes N$, defined in* 3.1 *and* 3.4 *are equal.*

Proof. Recall that the dual action $\hat{\alpha}$ is an action of $(\hat{M}, \hat{\Delta}^{\text{op}})$ on $M_\alpha \ltimes N$. We claim that the weight $\tilde{\theta}_0$ is $\hat{\delta}$-invariant. Observe that $\hat{\delta}^{-1}$ is the modular element of $(\hat{M}, \hat{\Delta}^{\text{op}})$ and that is the reason to have $\hat{\delta}$-invariance rather than $\hat{\delta}^{-1}$-invariance.

To prove our claim, choose $a \in \mathcal{N}_{\hat{\varphi}}$, $x \in \mathcal{N}_\theta$, $\xi \in \mathcal{D}(\hat{\delta}^{1/2})$ and $\eta \in H$. Then define

$$z := (\omega_{\xi,\eta} \otimes \iota \otimes \iota)\, \hat{\alpha}((a \otimes 1)\, \alpha(x)) = ((\omega_{\xi,\eta} \otimes \iota)\, \hat{\Delta}^{\text{op}}(a) \otimes 1)\, \alpha(x).$$

It follows from Proposition 7.2 of the appendix that

$$(\omega_{\xi,\eta} \otimes \iota)\, \hat{\Delta}^{\text{op}}(a) = (\iota \otimes \omega_{\xi,\eta})\, \hat{\Delta}(a) \in \mathcal{N}_{\hat{\varphi}} \qquad \text{and}$$
$$\hat{\Lambda}((\omega_{\xi,\eta} \otimes \iota)\, \hat{\Delta}^{\text{op}}(a)) = (\iota \otimes \omega_{\hat{\delta}^{1/2}\xi,\eta})(\hat{V})\, \hat{\Lambda}(a).$$

So we may conclude that $z \in \mathcal{N}_{\tilde{\theta}_0}$ and

$$\begin{aligned}\tilde{\Lambda}(z) &= ((\iota \otimes \omega_{\hat{\delta}^{1/2}\xi,\eta})(\hat{V}) \otimes 1)(\hat{\Lambda}(a) \otimes \Lambda_\theta(x)) \\ &= ((\iota \otimes \omega_{\hat{\delta}^{1/2}\xi,\eta})(\hat{V}) \otimes 1)\, \tilde{\Lambda}((a \otimes 1)\, \alpha(x)).\end{aligned}$$

Because $\mathcal{D}_0$ is a σ-strong*-norm core for $\tilde{\Lambda}$ we conclude that $(\omega_{\xi,\eta} \otimes \iota \otimes \iota)\, \hat{\alpha}(y) \in \mathcal{N}_{\tilde{\theta}_0}$ for all $y \in \mathcal{N}_{\tilde{\theta}_0}$ and

$$\tilde{\Lambda}((\omega_{\xi,\eta} \otimes \iota \otimes \iota)\, \hat{\alpha}(y)) = ((\iota \otimes \omega_{\hat{\delta}^{1/2}\xi,\eta})(\hat{V}) \otimes 1)\, \tilde{\Lambda}(y).$$

Because $\hat{V}$ is unitary we immediately get that $\tilde{\theta}_0$ is $\hat{\delta}$-invariant.

From Proposition 2.5 it follows that $\tilde{\theta}$ is $\hat{\delta}$-invariant. Then we conclude from Lemma 3.9 that $[D\tilde{\theta}_0 : D\tilde{\theta}]_t \in (M_\alpha \ltimes N)^{\hat{\alpha}}$ for all $t \in \mathbb{R}$. So by Theorem 2.7 we can take unitaries $u_t \in N$ such that $[D\tilde{\theta}_0 : D\tilde{\theta}]_t = \alpha(u_t)$ for all $t \in \mathbb{R}$. From the theory of operator valued weights we know that $\sigma_t^{\tilde{\theta}} \circ \alpha = \alpha \circ \sigma_t^{\theta}$. Because $([D\tilde{\theta}_0 : D\tilde{\theta}]_t)$ is a $\sigma^{\tilde{\theta}}$-cocycle, we get that (u_t) is a σ^{θ}-cocycle. By [26, 5.1] we can take a (uniquely determined) n.s.f. weight ρ on N such that $[D\rho : D\theta]_t = u_t$ for all $t \in \mathbb{R}$. With ρ we can define the n.s.f. weight $\tilde{\rho}$ on $M_\alpha \ltimes N$ in the sense of Definition 3.1. Then it follows from the theory of operator valued weights that

$$[D\tilde{\rho} : D\tilde{\theta}]_t = \alpha([D\rho : D\theta]_t) = \alpha(u_t) = [D\tilde{\theta}_0 : D\tilde{\theta}]_t$$

for all $t \in \mathbb{R}$. So $\tilde{\rho} = \tilde{\theta}_0$. Because $\tilde{\theta}_0$ is a restriction of $\tilde{\theta}$ we get that $\tilde{\rho}$ is a restriction of $\tilde{\theta}$.

Fix $a \in \mathcal{M}_{\hat{\varphi}}^+$ with $\hat{\varphi}(a) = 1$. Choose $x \in \mathcal{N}_\rho$. Then it follows from Lemma 3.2 that $\alpha(x^*)(a \otimes 1)\,\alpha(x) \in \mathcal{M}_{\tilde{\rho}}^+$ and

$$\tilde{\rho}(\alpha(x^*)(a \otimes 1)\,\alpha(x)) = \rho(x^*x).$$

Because $\tilde{\rho}$ is a restriction of $\tilde{\theta}$ we get that $\alpha(x^*)(a \otimes 1)\,\alpha(x) \in \mathcal{M}_{\tilde{\theta}}^+$ and

$$\tilde{\theta}(\alpha(x^*)(a \otimes 1)\,\alpha(x)) = \rho(x^*x).$$

Then it follows from Lemma 3.2 that $\theta(x^*x) = \rho(x^*x)$. This means that ρ is a restriction of θ.

Further we have, using the theory of operator valued weights in the first equality and Proposition 3.7 in the last one,

$$\alpha \circ \sigma_t^{\rho} = \sigma_t^{\tilde{\rho}} \circ \alpha = \sigma_t^{\tilde{\theta}_0} \circ \alpha = \alpha \circ \sigma_t^{\theta}.$$

So $\sigma_t^{\rho} = \sigma_t^{\theta}$ for all $t \in \mathbb{R}$. Because ρ is a restriction of θ we may conclude that $\rho = \theta$ and then $\tilde{\theta} = \tilde{\rho} = \tilde{\theta}_0$. ∎

We want to conclude this section with the proof of the fact that $U \in M \otimes B(K)$. First we state the following lemma, which is easily proved because $\tilde{\Lambda}$ is the closure of $\tilde{\Lambda}_0$. Recall that $\tilde{T} = \tilde{J}\,\tilde{\nabla}^{1/2}$.

Lemma 3.11. *Defining $\hat{T} = \hat{J}\,\hat{\nabla}^{1/2}$, we have that the linear space*

$$\operatorname{span}\{\alpha(x^*)(\eta \otimes \Lambda_\theta(y)) \mid x, y \in \mathcal{N}_\theta, \eta \in \mathcal{D}(\hat{T})\}$$

is a core for $\tilde{T}$ and

$$\tilde{T}\alpha(x^*)(\eta \otimes \Lambda_\theta(y)) = \alpha(y^*)(\hat{T}\eta \otimes \Lambda_\theta(x))$$

for all $x, y \in \mathcal{N}_\theta$ and $\eta \in \mathcal{D}(\hat{T})$.

PROPOSITION 3.12. *We have* $U \in M \otimes B(K)$.

Proof. Let $t \in \mathbb{R}$. Because $\hat{\nabla}^{it}$ implements the automorphism τ_t on M we get that $\operatorname{Ad} \hat{\nabla}^{it}$ will also leave M' invariant. So we can define the automorphism group (μ_t) on M by

$$\mu_t(x) = J \hat{\nabla}^{it} J x J \hat{\nabla}^{-it} J \qquad \text{for all} \quad x \in M, \quad t \in \mathbb{R}.$$

So, for every $a \in \mathcal{D}(\mu_{-i/2})$ we have $JaJ \hat{\nabla}^{1/2} \subset \hat{\nabla}^{1/2} J \mu_{-i/2}(a) J$. Further we have

$$\begin{aligned} \mu_t(R(a)) &= J \hat{\nabla}^{it} J \hat{J} a^* \hat{J} J \hat{\nabla}^{-it} J = J \hat{\nabla}^{it} \hat{J} J a^* J \hat{J} \hat{\nabla}^{-it} J \\ &= \hat{J} \mu_t(a^*) \hat{J} = R(\mu_t(a)) \end{aligned}$$

for all $t \in \mathbb{R}$ and $a \in M$. Here we used the formula $\hat{J} J = v^{i/4} J \hat{J}$ stated in the beginning of the paper.

Let now $a \in \mathcal{D}(\mu_{i/2})$, $x, y \in \mathcal{N}_\theta$ and $\eta \in \mathcal{D}(\hat{T})$, where $\hat{T} = \hat{J} \hat{\nabla}^{1/2}$. Then

$$\begin{aligned} (JaJ \otimes 1) \tilde{T} \alpha(x^*)(\eta \otimes \Lambda_\theta(y)) &= (JaJ \otimes 1) \alpha(y^*)(\hat{T}\eta \otimes \Lambda_\theta(x)) \\ &= \alpha(y^*)(JaJ\hat{J} \hat{\nabla}^{1/2} \eta \otimes \Lambda_\theta(x)) \\ &= \alpha(y^*)(\hat{J} J R(a^*) J \hat{\nabla}^{1/2} \eta \otimes \Lambda_\theta(x)). \end{aligned}$$

Now $a^* \in \mathcal{D}(\mu_{-i/2})$ and R and μ_t commute. So $R(a^*) \in \mathcal{D}(\mu_{-i/2})$ and $\mu_{-i/2}(R(a^*)) = R(\mu_{i/2}(a)^*)$. Then we get

$$JR(a^*) J \hat{\nabla}^{1/2} \subset \hat{\nabla}^{1/2} JR(\mu_{i/2}(a)^*) J.$$

Hence we may conclude that $JR(\mu_{i/2}(a)^*) J\eta \in \mathcal{D}(\hat{T}^{1/2})$ and

$$\begin{aligned} (JaJ \otimes 1) \tilde{T} \alpha(x^*)(\eta \otimes \Lambda_\theta(y)) &= \alpha(y^*)(\hat{T} JR(\mu_{i/2}(a)^*) J\eta \otimes \Lambda_\theta(x)) \\ &= \tilde{T} \alpha(x^*)(JR(\mu_{i/2}(a)^*) J\eta \otimes \Lambda_\theta(y)) \\ &= \tilde{T}(JR(\mu_{i/2}(a)^*) J \otimes 1) \alpha(x^*)(\eta \otimes \Lambda_\theta(y)). \end{aligned}$$

Because of the previous lemma we get

$$(JaJ \otimes 1) \tilde{T} \subset \tilde{T}(JR(\mu_{i/2}(a)^*) J \otimes 1) \tag{3.1}$$

for all $a \in \mathcal{D}(\mu_{i/2})$. By taking the adjoint we get

$$(JR(\mu_{i/2}(a)) J \otimes 1) \tilde{T}^* \subset \tilde{T}^*(Ja^*J \otimes 1)$$

for all $a \in \mathscr{D}(\mu_{i/2})$. So for all $a \in \mathscr{D}(\mu_{-i})$

$$(JaJ \otimes 1)\,\tilde{\nabla} = (JaJ \otimes 1)\,\tilde{T}^*\tilde{T} \subset \tilde{T}^*(JR(\mu_{-i/2}(a)^*)\,J \otimes 1)\,\tilde{T}$$
$$\subset \tilde{\nabla}(J\mu_{-i}(a)\,J \otimes 1).$$

Denoting with γ_t the automorphism $\operatorname{Ad}\tilde{\nabla}^{it}$ of $B(H \otimes K)$ we get that for every $a \in \mathscr{D}(\mu_{-i})$ we have $JaJ \otimes 1 \in \mathscr{D}(\gamma_i)$ and $\gamma_i(JaJ \otimes 1) = J\mu_{-i}(a)\,J \otimes 1$. Then the results of [13, 4.3 and 4.4] allow us to conclude that $\gamma_t(JaJ \otimes 1) = J\mu_t(a)\,J \otimes 1$ for every $t \in \mathbb{R}$ and $a \in M$. This gives

$$(JaJ \otimes 1)\,\tilde{\nabla}^{-1/2} \subset \tilde{\nabla}^{-1/2}(J\mu_{i/2}(a)\,J \otimes 1)$$

for all $a \in \mathscr{D}(\mu_{i/2})$. Combining this with Eq. 3.1 we get for every $a \in \mathscr{D}(\mu_{i/2})$

$$(JaJ \otimes 1)\,\tilde{T}\,\tilde{\nabla}^{1/2} \subset \tilde{T}(JR(\mu_{i/2}(a)^*)\,J \otimes 1)\,\tilde{\nabla}^{-1/2}$$
$$\subset \tilde{T}\,\tilde{\nabla}^{-1/2}(JR(a^*)\,J \otimes 1) \subset \tilde{J}(J\hat{J}a\hat{J}J \otimes 1).$$

So we get

$$(JaJ \otimes 1)\,\tilde{J} = \tilde{J}(J\hat{J}a\hat{J}J \otimes 1) = \tilde{J}(\hat{J}JaJ\hat{J} \otimes 1)$$

for every $a \in \mathscr{D}(\mu_{i/2})$, and hence for every $a \in M$. Rewriting this we get $(JaJ \otimes 1)\,U = U(JaJ \otimes 1)$ for every $a \in M$. This gives $U \in M \otimes B(K)$. ∎

Finally we want to prove that U is a unitary corepresentation of (M, Δ), namely $(\Delta \otimes \iota)(U) = U_{23}U_{13}$. This will be done in an indirect way in the next section. Nevertheless the results we use to prove that U is a corepresentation are interesting in themselves.

4. THE UNITARY IMPLEMENTATION IS A COREPRESENTATION

The main aim of this section is to prove that the unitary implementation U is a corepresentation (Theorem 4.4). On our way towards the proof of Theorem 4.4 we will solve three problems which appear naturally in applications (see Section 5 and [17]). First we will see what happens when we choose a different weight θ on N, next we will show how U changes when the action α is deformed with an α-cocycle and finally we will show that in the presence of a δ^{-1}-invariant weight our implementation agrees with the one of Enock and Schwartz given by Proposition 2.4.

In the proof of the first proposition we will make use of Connes' relative modular theory (see e.g. [26, 3.11, 3.12 and 3.16]). When θ_i are n.s.f.

weights on N with GNS-constructions (K_i, π_i, Λ_i) $(i=1,2)$, we denote with $J_{2,1}$ the relative modular conjugation, which is a anti-unitary from H_1 to H_2. Recall that $J_{1,2}=J_{2,1}^*$. If we denote with J_i the modular conjugation of the weight θ_i we have $J_{2,1}J_1=J_2J_{2,1}$ and we denote this unitary with u. Then u is the unique unitary from K_1 to K_2 which satisfies $u\pi_1(x)\,u^*=\pi_2(x)$ for all $x\in N$ and which maps the positive cone of K_1 (determined by the GNS-construction (K_1, π_1, Λ_1)) onto the positive cone of K_2. We will say that u intertwines the two standard representations of N.

Finally we introduce the one-parameter group $\sigma^{2,1}$ of isometries of N given by

$$\sigma_t^{2,1}(x)=[D\theta_2 : D\theta_1]_t\,\sigma_t^{\theta_1}(x)$$

for all $x\in N$ and $t\in\mathbb{R}$.

PROPOSITION 4.1. *Let α be an action of (M,Δ) on N. Let θ_i be n.s.f. weights on N with GNS-constructions $(K_i, \pi_i, \Lambda_i)(i=1,2)$. Let u be the unitary from K_1 to K_2 intertwining the two standard representations of N. Denote for every $i=1,2$ with $\tilde{\theta}_i$ the dual weight of θ_i on $M_\alpha\ltimes N$, with GNS-construction $(H\otimes K_i, \iota\otimes\pi_i, \tilde{\Lambda}_i)$. Denote with $U_i\in M\otimes B(K_i)$ the unitary implementation of α obtained with θ_i, as defined in Definition* 3.6.

Then $1\otimes u$ is the unitary intertwining the two standard representations of $M_\alpha\ltimes N$. In particular

$$U_2=(1\otimes u)\;U_1(1\otimes u^*).$$

Proof. Let $a\in\mathcal{N}_{\hat{\varphi}}$ and $x\in\mathcal{N}_{\theta_1}$. Let $y\in\mathcal{D}(\sigma^{2,1}_{-i/2})$. Then, by [26, 3.12], $xy^*\in\mathcal{N}_{\theta_2}$ and

$$\Lambda_2(xy^*)=J_{2,1}\pi_1(\sigma^{2,1}_{-i/2}(y))\,J_1\Lambda_1(x).$$

So $(a\otimes 1)\,\alpha(x)\,\alpha(y)^*=(a\otimes 1)\,\alpha(xy^*)\in\mathcal{N}_{\tilde{\theta}_2}$ and

$$\begin{aligned}\tilde{\Lambda}_2((a\otimes 1)\,\alpha(x)\,\alpha(y)^*)&=\hat{\Lambda}(a)\otimes\Lambda_2(xy^*)\\&=(1\otimes J_{2,1}\pi_1(\sigma^{2,1}_{-i/2}(y))\,J_1)(\hat{\Lambda}(a)\otimes\Lambda_1(x))\\&=(1\otimes J_{2,1}\pi_1(\sigma^{2,1}_{i-/2}(y))\,J_1)\,\tilde{\Lambda}_1((a\otimes 1)\,\alpha(x)).\end{aligned}$$

Because the elements $(a\otimes 1)\,\alpha(x)$ span a core for $\tilde{\Lambda}_1$ and because $\tilde{\Lambda}_2$ is closed (both in the σ-strong*-norm topology), we have for all $z\in\mathcal{N}_{\tilde{\theta}_1}$ that $z\alpha(y)^*\in\mathcal{N}_{\tilde{\theta}_2}$ and

$$\tilde{\Lambda}_2(z\alpha(y)^*)=(1\otimes J_{2,1}\pi_1(\sigma^{2,1}_{-i/2}(y))\,J_1)\,\tilde{\Lambda}_1(z).$$

Denoting with $\tilde{J}_{2,1}$ and $(\tilde{\sigma}_t^{2,1})$ the relative modular apparatus of the weights $\tilde{\theta}_2$ and $\tilde{\theta}_1$, it follows from [26, 3.12] that $\alpha(y) \in \mathscr{D}(\tilde{\sigma}^{2,1}_{-i/2})$ and

$$\tilde{J}_{2,1}(\iota \otimes \pi_1)(\tilde{\sigma}^{2,1}_{-i/2}(\alpha(y)))\, \tilde{J}_1 = 1 \otimes J_{2,1}\pi_1(\sigma^{2,1}_{-i/2}(y))\, J_1.$$

Because $[D\tilde{\theta}_2 : D\tilde{\theta}_1]_t = \alpha([D\theta_2 : D\theta_1]_t)$ for every $t \in \mathbb{R}$ we see that $\tilde{\sigma}_t^{2,1} \circ \alpha = \alpha \circ \sigma_t^{2,1}$. So we have $\tilde{\sigma}^{2,1}_{-i/2}(\alpha(y)) = \alpha(\sigma^{2,1}_{-i/2}(y))$. Combining this with the equation above we get

$$(\iota \otimes \pi_1)\, \alpha(\sigma^{2,1}_{-i/2}(y)) = \tilde{J}^*_{2,1}(\hat{J} \otimes J_{2,1})(1 \otimes \pi_1(\sigma^{2,1}_{-i/2}(y)))\, U_1^*.$$

The last formula is valid for all $y \in \mathscr{D}(\sigma^{2,1}_{-i/2})$. Because U_1 implements α we may then conclude that $U_1 = \tilde{J}^*_{2,1}(\hat{J} \otimes J_{2,1})$.

Then we get

$$1 = U_1 U_1^* = \tilde{J}^*_{2,1}(\hat{J} \otimes J_{2,1})(\hat{J} \otimes J_1)\, \tilde{J}_1$$

and so $\tilde{J}_{2,1}\tilde{J}_1 = 1 \otimes J_{2,1}J_1$. Now $u = J_{2,1}J_1$ and $\tilde{J}_{2,1}\tilde{J}_1$ is the unitary intertwining the two standard representations of $M_\alpha \ltimes N$. This proves the first claim of the proposition. In particular we get

$$\begin{aligned}(1 \otimes u)\, U_1(1 \otimes u^*) &= (1 \otimes u)\, \tilde{J}_1(\hat{J} \otimes J_1)(1 \otimes u^*)\\ &= \tilde{J}_2(1 \otimes u)(\hat{J} \otimes J_1)(1 \otimes u^*) = \tilde{J}_1(\hat{J} \otimes J_2) = U_2.\end{aligned}$$

This proves the proposition. ∎

In the next proposition we will show how the unitary implementation of an action α changes when α is deformed with an α-cocycle.

Proposition 4.2. *Let α be an action of (M, Δ) on N and let $\mathscr{V} \in M \otimes N$ be an α-cocycle in the sense of Definition* 1.5. *Define the action β of (M, Δ) on N by $\beta(x) = \mathscr{V}\alpha(x)\mathscr{V}^*$ for all $x \in N$. If θ is a n.s.f. weight on N with GNS-construction $(K, \iota, \Lambda_\theta)$, the unitary implementations U_α and U_β of α and β obtained with θ satisfy*

$$U_\beta = \mathscr{V} U_\alpha(\hat{J} \otimes J_\theta)\, \mathscr{V}^*(\hat{J} \otimes J_\theta).$$

In particular U_β is a corepresentation if and only if U_α is a corepresentation.

Proof. Because $(\Delta \otimes \iota)(\mathscr{V}) = (1 \otimes \mathscr{V})(\iota \otimes \alpha)(\mathscr{V})$ we have

$$(1 \otimes \mathscr{V}^*)(W \otimes 1)(1 \otimes \mathscr{V}) = (W \otimes 1)(\iota \otimes \alpha)(\mathscr{V}^*).$$

So for every $\xi, \eta \in H$ and with $(e_i)_{i \in I}$ an orthonormal basis of H we have, by applying $\omega_{\xi, \eta} \otimes \iota \otimes \iota$

$$\mathscr{V}^*(\lambda(\omega_{\xi, \eta}) \otimes 1)\, \mathscr{V} = \sum_{i \in I} (\lambda(\omega_{e_i, \eta}) \otimes 1)\, \alpha((\omega_{\xi, e_i} \otimes \iota)(\mathscr{V}^*))$$

in the σ-strong* topology. From this it follows that $\mathscr{V}^*(a \otimes 1)\mathscr{V} \in M_\alpha \ltimes N$ for all $a \in \hat{M}$. But $\mathscr{V}^*\beta(x)\, \mathscr{V} = \alpha(x)$ for all $x \in N$. So

$$\rho\colon M_\beta \ltimes N \to M_\alpha \ltimes N\colon z \mapsto \mathscr{V}^* z \mathscr{V}$$

is a well-defined $*$-homomorphism. By symmetry ρ will be surjective and hence it is a $*$-isomorphism. Consider now the dual weights $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$ on $M_\alpha \ltimes N$ and $M_\beta \ltimes N$, with canonical GNS-constructions $(H \otimes K, \iota, \tilde{\Lambda}_\alpha)$ and $(H \otimes K, \iota, \tilde{\Lambda}_\beta)$. Take $\xi \in H$, $b \in \mathscr{T}_\varphi$ and $x \in \mathscr{N}_\theta$. Then

$$\mathscr{V}^*(\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \beta(x)\, \mathscr{V} = \sum_{i \in I} (\lambda(\omega_{e_i, \Lambda(b)}) \otimes 1)\, \alpha((\omega_{\xi, e_i} \otimes \iota)(\mathscr{V}^*)\, x)$$

in the σ-strong* topology. For every finite subset I_0 of I we define

$$z_{I_0} := \sum_{i \in I_0} (\lambda(\omega_{e_i, \Lambda(b)}) \otimes 1)\, \alpha((\omega_{\xi, e_i} \otimes \iota)(\mathscr{V}^*)\, x).$$

By Proposition 7.1 of the appendix we get that z_{I_0} belongs to $\mathscr{N}_{\tilde{\theta}_\alpha}$ and

$$\begin{aligned}\tilde{\Lambda}_\alpha(z_{I_0}) &= \sum_{i \in I_0} \hat{\Lambda}(\lambda(\omega_{e_i, \Lambda(b)})) \otimes (\omega_{\xi, e_i} \otimes \iota)(\mathscr{V}^*)\, \Lambda_\theta(x) \\ &= \sum_{i \in I_0} J\sigma_{i/2}(b)\, Je_i \otimes (\omega_{\xi, e_i} \otimes \iota)(\mathscr{V}^*)\, \Lambda_\theta(x) \\ &= (J\sigma_{i/2}(b)\, J \otimes 1)(P_{I_0} \otimes 1)\, \mathscr{V}^*(\xi \otimes \Lambda_\theta(x))\end{aligned}$$

where P_{I_0} denotes the projection onto $\operatorname{span}\{e_i \mid i \in I_0\}$. Now define $z := \mathscr{V}^*(\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \beta(x)\, \mathscr{V}$. Then we see that $z_{I_0} \to z$ σ-strong* and

$$\tilde{\Lambda}_\alpha(z_{I_0}) \to (J\sigma_{i/2}(b)\, J \otimes 1)\, \mathscr{V}^*(\xi \otimes \Lambda_\theta(x)) \qquad \text{in norm.}$$

So we get that $z \in \mathscr{N}_{\tilde{\theta}_\alpha}$ and

$$\begin{aligned}\tilde{\Lambda}_\alpha(z) &= (J\sigma_{i/2}(b)\, J \otimes 1)\, \mathscr{V}^*(\xi \otimes \Lambda_\theta(x)) \\ &= \mathscr{V}^*(J\sigma_{i/2}(b)\, J\xi \otimes \Lambda_\theta(x)) = \mathscr{V}^* \tilde{\Lambda}_\beta((\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \beta(x)).\end{aligned}$$

Because the elements $(\lambda(\omega_{\xi, \Lambda(b)}) \otimes 1)\, \beta(x)$ span a core for $\tilde{\Lambda}_\beta$ we have $\rho(y) \in \mathscr{N}_{\tilde{\theta}_\alpha}$ for every $y \in \mathscr{N}_{\tilde{\theta}_\beta}$ and $\tilde{\Lambda}_\alpha(\rho(y)) = \mathscr{V}^* \tilde{\Lambda}_\beta(y)$ in that case.

By symmetry $\rho(y) \in \mathcal{N}_{\tilde{\theta}_\alpha}$ if and only if $y \in \mathcal{N}_{\tilde{\theta}_\beta}$. But then it is clear that $\tilde{J}_\beta = \mathcal{V} \tilde{J}_\alpha \mathcal{V}^*$ and so

$$U_\beta = \tilde{J}_\beta(\hat{J} \otimes J_\theta) = \mathcal{V} \tilde{J}_\alpha \mathcal{V}^*(\hat{J} \otimes J_\theta) = \mathcal{V} U_\alpha(\hat{J} \otimes J_\theta) \mathcal{V}^*(\hat{J} \otimes J_\theta).$$

Now suppose that U_α is a corepresentation, meaning that $(\Delta \otimes \iota)(U_\alpha) = U_{\alpha 23} U_{\alpha 13}$. Then

$$(\Delta \otimes \iota)(U_\beta) = (\Delta \otimes \iota)(\mathcal{V}) U_{\alpha 23} U_{\alpha 13}(\Delta \otimes \iota)(R \otimes L_\theta)(\mathcal{V})$$

where L_θ is the $*$-anti-isomorphism from N to N' defined by $L_\theta(x) = J_\theta x^* J_\theta$ for all $x \in N$. Then we can compute

$$\begin{aligned}
(\Delta \otimes \iota)(U_\beta) &= \mathcal{V}_{23}(\iota \otimes \alpha)(\mathcal{V}) U_{\alpha 23} U_{\alpha 13}(R \otimes R \otimes L_\theta)(\Delta^{\text{op}} \otimes \iota)(\mathcal{V}) \\
&= \mathcal{V}_{23} U_{\alpha 23} \mathcal{V}_{13} U^*_{\alpha 23} U_{\alpha 23} U_{\alpha 13}(R \otimes R \otimes L_\theta)(\mathcal{V}_{13}(\iota \otimes \alpha)(\mathcal{V})_{213}) \\
&= \mathcal{V}_{23} U_{\alpha 23} \mathcal{V}_{13} U_{\alpha 13}(\hat{J} \otimes \hat{J} \otimes J_\theta) U_{\alpha 13} \mathcal{V}^*_{23} U^*_{\alpha 13} \mathcal{V}^*_{13}(\hat{J} \otimes \hat{J} \otimes J_\theta) \\
&= \mathcal{V}_{23} U_{\alpha 23} \mathcal{V}_{13}(\hat{J} \otimes \hat{J} \otimes J_\theta) \\
&\quad \times \mathcal{V}^*_{23}(\hat{J} \otimes \hat{J} \otimes J_\theta)(\hat{J} \otimes \hat{J} \otimes J_\theta) U^*_{\alpha 13} \mathcal{V}^*_{13}(\hat{J} \otimes \hat{J} \otimes J_\theta) \\
&= \mathcal{V}_{23} U_{\alpha 23}((\hat{J} \otimes J_\theta) \mathcal{V}^*(\hat{J} \otimes J_\theta))_{23} \\
&\quad \times \mathcal{V}_{13} U_{\alpha 13}((\hat{J} \otimes J_\theta) \mathcal{V}^*(\hat{J} \otimes J_\theta))_{13} \\
&= U_{\beta 23} U_{\beta 13}.
\end{aligned}$$

So, when U_α is a corepresentation then U_β is a corepresentation. By symmetry also the converse implication holds. ∎

In Proposition 2.4 we saw how to construct, with the methods of Enock and Schwartz, an implementation of an action α in the presence of a δ^{-1}-invariant weight. We will show now that this implementation coincides with the unitary implementation given in Definition 3.6.

PROPOSITION 4.3. *Let α be an action of (M, Δ) on N. Let θ be a n.s.f. and δ^{-1}-invariant weight on N with GNS-construction $(K, \iota, \Lambda_\theta)$. When V_θ is the unitary defined in Proposition* 2.4 *and when U is the unitary implementation of α defined in Definition* 3.6, *then $U = V_\theta$.*

Proof. Recall that

$$(\omega_{\xi,\eta} \otimes \iota)(V_\theta) \Lambda_\theta(x) = \Lambda_\theta((\omega_{\delta^{1/2}\xi,\eta} \otimes \iota) \alpha(x))$$

for all $\xi \in \mathscr{D}(\delta^{1/2})$ and $\eta \in H$. Because the positive operators δ and $\hat{\nabla}$ strongly commute, we can define the closure Q of the product $\delta \hat{\nabla}$. Denoting with $\chi_{\mathscr{U}}$ the characteristic function of a subset $\mathscr{U} \subset \mathbb{R}$ we consider the following subspace of H.

$$\mathscr{D}_0 = \bigcup_{n, m \in \mathbb{N}} \chi_{]1/n, n[}(\delta)\, \chi_{]1/m, m[}(\hat{\nabla})\, H.$$

Let now $\xi \in H$, $\eta \in \mathscr{D}_0$, $x \in \mathscr{T}_\theta$ and $y \in \mathscr{N}_\theta \cap \mathscr{N}_\theta^*$. Put again $\hat{T} = \hat{J}\hat{\nabla}^{1/2}$ and $\tilde{T} = \tilde{J}\tilde{\nabla}^{1/2}$. Then

$$\begin{aligned}
&(\theta_\xi^* \otimes 1)\, \tilde{T}\alpha(x^*)(\eta \otimes \Lambda_\theta(y)) \\
&\quad = (\theta_\xi^* \otimes 1)\, \alpha(y^*)(\hat{T}\eta \otimes \Lambda_\theta(x)) = \Lambda_\theta((\omega_{\hat{T}\eta, \xi} \otimes \iota)\, \alpha(y^*)\, x) \\
&\quad = J_\theta \sigma_{i/2}^\theta(x)^*\, J_\theta \Lambda_\theta((\omega_{\hat{T}\eta, \xi} \otimes \iota)\, \alpha(y^*)) \\
&\quad = J_\theta \sigma_{i/2}^\theta(x)^*\, J_\theta (\omega_{\delta^{-1/2}\hat{T}\eta, \xi} \otimes \iota)(V_\theta)\, \Lambda_\theta(y^*) \\
&\quad = (\theta_\xi^* \otimes 1)(1 \otimes J_\theta \sigma_{i/2}^\theta(x)^*\, J_\theta)\, V_\theta(\delta^{-1/2}\hat{T}\eta \otimes \Lambda_\theta(y^*)).
\end{aligned}$$

Now

$$\delta^{-1/2}\hat{T}\eta = \delta^{-1/2}\hat{J}\,\hat{\nabla}^{1/2}\eta = \hat{J}\delta^{1/2}\,\hat{\nabla}^{1/2}\eta = \hat{J}Q^{1/2}\eta.$$

So we may conclude that

$$\begin{aligned}
&\tilde{T}\alpha(x^*)(\eta \otimes \Lambda_\theta(y)) \\
&\quad = (1 \otimes J_\theta \sigma_{i/2}^\theta(x)^*\, J_\theta)\, V_\theta(\hat{J} \otimes J_\theta)(Q^{1/2} \otimes \nabla_\theta^{1/2})(\eta \otimes \Lambda_\theta(y))
\end{aligned}$$

for all $\eta \in \mathscr{D}_0$, $x \in \mathscr{T}_\theta$ and $y \in \mathscr{N}_\theta \cap \mathscr{N}_\theta^*$. Because $\tilde{T}$ is closed we can conclude that $\eta \otimes \Lambda_\theta(y) \in \mathscr{D}(\tilde{T})$ and

$$\tilde{T}(\eta \otimes \Lambda_\theta(y)) = V_\theta(\hat{J} \otimes J_\theta)(Q^{1/2} \otimes \nabla_\theta^{1/2})(\eta \otimes \Lambda_\theta(y))$$

for all $\eta \in \mathscr{D}_0$ and $y \in \mathscr{N}_\theta \cap \mathscr{N}_\theta^*$. Because $\mathscr{D}_0$ is a core for $Q^{1/2}$ and $\Lambda_\theta(\mathscr{N}_\theta \cap \mathscr{N}_\theta^*)$ for $\nabla_\theta^{1/2}$ we get

$$V_\theta(\hat{J} \otimes J_\theta)(Q^{1/2} \otimes \nabla_\theta^{1/2}) \subset \tilde{T}. \tag{4.1}$$

We now claim that $(Q^{it} \otimes \nabla_\theta^{it})\, \tilde{T} = \tilde{T}(Q^{it} \otimes \nabla_\theta^{it})$ for every $t \in \mathbb{R}$. Together with the fact that $\mathscr{D}(Q^{1/2} \otimes \nabla_\theta^{1/2}) \subset \mathscr{D}(\tilde{T})$ this leads to the conclusion that $\mathscr{D}(Q^{1/2} \otimes \nabla_\theta^{1/2})$ is a core for $\tilde{T}$. Then we get that the inclusion in Eq. 4.1 is in fact an equality. Uniqueness of the polar decomposition gives us $V_\theta(\hat{J} \otimes J_\theta) = \tilde{J}$ and so $V_\theta = U$.

So we only have to prove our claim. For this choose $x, y \in \mathcal{N}_\theta$ and $\xi \in \mathcal{D}(\hat{T})$. Then using Proposition 2.4 we get

$$(Q^{it} \otimes \nabla_\theta^{it})\, \tilde{T}\alpha(x^*)(\xi \otimes \Lambda_\theta(y)) = (Q^{it} \otimes \nabla_\theta^{it})\, \alpha(y^*)(\hat{T}\xi \otimes \Lambda_\theta(x))$$
$$= (Q^{it} \otimes \nabla_\theta^{it})\, V_\theta(1 \otimes y^*)\, V_\theta^*(\hat{T}\xi \otimes \Lambda_\theta(x))$$
$$= V_\theta(1 \otimes \sigma_t^\theta(y^*))\, V_\theta^*(Q^{it}\hat{T}\xi \otimes \nabla_\theta^{it}\Lambda_\theta(x))$$

because $Q^{it} \otimes \nabla_\theta^{it}$ and V_θ commute. Now observe that Q^{it} and $\hat{T}$ commute, so that $Q^{it}\xi \in \mathcal{D}(\hat{T})$ and

$$(Q^{it} \otimes \nabla_\theta^{it})\, \tilde{T}\alpha(x^*)(\xi \otimes \Lambda_\theta(y)) = \alpha(\sigma_t^\theta(y)^*)(\hat{T}Q^{it}\xi \otimes \Lambda_\theta(\sigma_t^\theta(x)))$$
$$= \tilde{T}\alpha(\sigma_t^\theta(x^*))(Q^{it}\xi \otimes \Lambda_\theta(\sigma_t^\theta(y)))$$
$$= \tilde{T}(Q^{it} \otimes \nabla_\theta^{it})\, \alpha(x^*)(\xi \otimes \Lambda_\theta(y)).$$

From this immediately follows our claim, and then the proof of the proposition is complete. ∎

With all these results at hand we can now prove the following theorem.

THEOREM 4.4. *The unitary implementation U of an action α of (M, Δ) on N is a corepresentation in the sense that $(\Delta \otimes \iota)(U) = U_{23} U_{13}$.*

Proof. Consider the bidual action $\hat{\hat{\alpha}}$ of $(M, \Delta)^{\wedge\text{op}\wedge\text{op}}$ on $\hat{M}_{\hat{\alpha}} \ltimes (M_\alpha \ltimes N)$. Let θ be a n.s.f. weight on N and denote with $\tilde{\tilde{\theta}}$ the bidual weight on $\hat{M}_{\hat{\alpha}} \ltimes (M_\alpha \ltimes N)$. It follows from Proposition 2.5 that $\tilde{\tilde{\theta}}$ is a $J\delta J$-invariant weight for the action $\hat{\hat{\alpha}}$. With the notation of Theorem 2.6 we define $\rho := \tilde{\tilde{\theta}} \circ \Phi$. Then ρ will be a n.s.f. and δ^{-1}-invariant weight on $B(H) \otimes N$ for the action γ of (M, Δ) on $B(H) \otimes N$. Combining Proposition 4.3 and Proposition 2.4 the unitary implementation of γ constructed with the weight ρ is a corepresentation. By Proposition 4.2 the unitary implementation of $\mu := (\sigma \otimes \iota)(\iota \otimes \alpha)$ constructed with ρ is a corepresentation as well. Then it follows from Proposition 4.1 that the unitary implementation U_μ of μ constructed with the n.s.f. weight $\text{Tr} \otimes \theta$ on $B(H) \otimes N$ will be a corepresentation. Here Tr denotes the usual trace on $B(H)$.

Represent N on the GNS-space of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction for θ. Let $(H_{\text{Tr}}, \pi_{\text{Tr}}, \Lambda_{\text{Tr}})$ be a GNS-construction for Tr. Then we have a canonical GNS-construction $(H_{\text{Tr}} \otimes K, \pi_{\text{Tr}} \otimes \iota, \Lambda_{\text{Tr}\otimes\theta})$ for $\text{Tr} \otimes \theta$. With this we construct the GNS-construction $(H \otimes H_{\text{Tr}} \otimes K,\ \iota \otimes \pi_{\text{Tr}} \otimes \iota,\ \tilde{\Lambda}_{\text{Tr}\otimes\theta})$ of the dual weight $(\text{Tr} \otimes \theta)\tilde{}$ on $M_\mu \ltimes (B(H) \otimes N)$. Denote with $\tilde{T}_{\text{Tr}\otimes\theta} = \tilde{J}_{\text{Tr}\otimes\theta} \tilde{\nabla}_{\text{Tr}\otimes\theta}^{1/2}$ the modular operator of this dual weight. As before we denote with $\tilde{T}=\tilde{J}\tilde{\nabla}^{1/2}$ the modular operator of the weight $\tilde{\theta}$ on $M_\alpha \ltimes N$ with GNS-construction $(H\otimes K, \iota, \tilde{\Lambda})$. It is an easy exercise to check that

$$\Sigma_{12}\tilde{T}_{\mathrm{Tr}\otimes\theta}\Sigma_{12}=J_{\mathrm{Tr}}\otimes\tilde{T},$$

where Σ_{12} flips the first two legs of $H\otimes H_{\mathrm{Tr}}\otimes K$. By uniqueness of the polar decomposition we get

$$\Sigma_{12}\tilde{J}_{\mathrm{Tr}\otimes\theta}\Sigma_{12}=J_{\mathrm{Tr}}\otimes\tilde{J}$$

and hence $\Sigma_{12}U_\mu\Sigma_{12}=1\otimes U$. Because U_μ is a corepresentation, also U will be a corepresentation. ∎

5. SUBFACTORS AND INCLUSIONS OF VON NEUMANN ALGEBRAS

It is well known that there is an important link between irreducible, depth 2 inclusions of factors and quantum groups. After a conjecture of Ocneanu the first result in this direction was proved by David in [3], Longo in [22] and Szymanski in [27]. They were able to prove that every irreducible, depth 2 inclusion of II_1-factors with finite index has the form $N^\alpha\subset N$, where N is a II_1-factor and α is an action of a finite Kac algebra (i.e. a finite dimensional locally compact quantum group, or a finite dimensional Hopf $*$-algebra with positive invariant integral). The restriction on type an index has been removed by Enock and Nest in [7] and [8]. There does not appear a finite quantum group but an arbitrary locally compact quantum group.

The theory of Enock and Nest is quite technical, but the results are deep and beautiful. They are important in themselves and serve as a motivation for the concept of a locally compact quantum group.

Before we describe their result we have to explain a little bit the basic theory of infinite index inclusions of factors or van Neumann algebras. So, let us look at an inclusion $N_0\subset N_1$ of von Neumann algebras. In this most general setting one can perform the well known basic construction of Jones. For this we have to choose a n.s.f. weight θ on N_1 and represent N_1 on the GNS-space of θ. Denote with J_θ the modular conjugation of θ. Then we define $N_2=J_\theta N_0' J_\theta$. Because $N_1'=J_\theta N_1 J_\theta$ we have $N_0\subset N_1\subset N_2$ and this inclusion of three von Neumann algebras is called the basic construction. One can continue in the same way and represent N_2 on the GNS-space of some n.s.f. weight. Then we obtain the von Neumann algebra N_3. Going on we get a tower of von Neumann algebras

$$N_0 \subset N_1 \subset N_2 \subset N_3 \subset \cdots$$

which is called the Jones tower.

But there is more. In the theory of inclusions of II_1-factors an important role is played by conditional expectations. In the more general theory being described now, this role will be taken over by operator valued weights. Before we can explain this, and also because we need it in the proof of Theorem 5.3, we have to explain Connes' spatial modular theory. For this we refer to e.g. [26, Section 7] and [28, Section III].

Suppose that N is a von Neumann algebra acting on a Hilbert space K. Let φ be a n.s.f. weight on N and ψ a n.s.f. weight on N'. Let $(K_\psi, \pi_\psi, \Lambda_\psi)$ be a GNS-construction for ψ. For every $\xi \in K$ we define the densely defined operator $R^\psi(\xi)$ with domain $\Lambda_\psi(\mathcal{N}_\psi) \subset K_\psi$ and range in K by $R^\psi(\xi)\,\Lambda_\psi(x) = x\xi$ for all $x \in \mathcal{N}_\psi$. When $\xi \in K$ we can define an operator $\Theta^\psi(\xi)$ in the extended positive part of $B(K)$ by

$$\langle \omega_\eta, \Theta^\psi(\xi) \rangle = \begin{cases} \|R^\psi(\xi)^* \eta\|^2 & \text{if} \quad \eta \in \mathcal{D}(R^\psi(\xi)^*) \\ +\infty & \text{else} \end{cases}.$$

In fact $\Theta^\psi(\xi) = R^\psi(\xi)\, R^\psi(\xi)^* + \infty P$ where P is the projection onto the orthogonal complement of $\mathcal{D}(R^\psi(\xi)^*)$. Then one can prove that $\Theta^\psi(\xi)$ belongs to N^+_{ext} and it is possible to define a strictly positive, self-adjoint operator $\frac{d\varphi}{d\psi}$ on K such that

$$\left\langle \omega_\xi, \frac{d\varphi}{d\psi} \right\rangle = \langle \varphi, \Theta^\psi(\xi) \rangle$$

for all $\xi \in K$. Here we used the extension of the weight φ to the extended positive part of N. The operator $\frac{d\varphi}{d\psi}$ is called the spatial derivative of φ with respect to ψ.

So, let $N_0 \subset N_1$ be an inclusion of von Neumann algebras and T_1 a n.s.f. operator valued weight from N_1 to N_0. Represent again N_1 on the GNS-space of a n.s.f. weight θ. Let $N_0 \subset N_1 \subset N_2$ be the basic construction. Then there exists a unique n.s.f. operator valued weight T_2 from N_2 to N_1 such that

$$\frac{d(\mu \circ T_2)}{d\nu'} = \frac{d\mu}{d((\nu \circ T_1)')}$$

for all n.s.f. weights μ on N_1 and ν on N_0. Here we denote with η' the n.s.f. weight on either $N_2' = J_\theta N_0 J_\theta$ or $N_1' = J_\theta N_1 J_\theta$, given by the formula

$\eta'(x) = \eta(J_\theta x J_\theta)$ for all positive x, whenever η is a n.s.f. weight on either N_0 or N_1. The existence of T_2 follows from [26, 12.11]. One can continue in the same way and construct n.s.f. operator valued weights T_i from N_i to N_{i-1} anywhere in the Jones tower.

Next recall that an inclusion of von Neumann algebras $N_0 \subset N_1$ is said to be

- irreducible, when $N_1 \cap N_0' = \mathbb{C}$.
- of depth 2, when $N_1 \cap N_0' \subset N_2 \cap N_0' \subset N_3 \cap N_0'$ is the basic construction.

Finally we describe the notion of regularity as it was introduced by Enock and Nest in [7, 11.12]. Let $N_0 \subset N_1$ be an inclusion of von Neumann algebras. Suppose that T_1 is a n.s.f. operator valued weight from N_1 to N_0. Let $N_0 \subset N_1 \subset N_2 \subset N_3 \subset \cdots$ be the Jones tower and construct as above the operator valued weights T_2 from N_2 to N_1 and T_3 from N_3 to N_2. Then T_1 is called regular when the restrictions of T_2 to $N_2 \cap N_0'$ and of T_3 to $N_3 \cap N_1'$ are both semifinite.

Then we can give the main result of Enock and Nest. Recall that for a locally compact quantum group (M, Δ) we denoted with $(M, \Delta)'$ the commutant locally compact quantum group, as described in the introduction.

THEOREM 5.1 (Enock and Nest). *Let $N_0 \subset N_1$ be an irreducible, depth 2 inclusion of factors and let T_1 be a regular n.s.f. operator valued weight from N_1 to N_0. Then the von Neumann algebra $M = N_3 \cap N_1'$ can be given the structure of a locally compact quantum group (M, Δ), such that there exists an outer action α of $(M, \Delta)'$ on N_1 satisfying $N_0 = N_1^\alpha$ and such that the inclusions $N_0 \subset N_1 \subset N_2$ and $\mathbb{C} \otimes N_1^\alpha \subset \alpha(N_1) \subset M'_\alpha \ltimes N_1$ are isomorphic.*

The definition of an outer action is given in Definition 5.5. Further we want to mention that in [8] it is proved that (M, Δ), together with invariant weights and antipode, is in fact a Woronowicz algebra. But it should be stressed that there is a small mistake in the proof that the Haar weight is invariant under the scaling group, so that in fact (M, Δ) is an arbitrary locally compact quantum group, possibly with scaling constant different from 1.

The main aim of this section is to clarify the conditions of Enock and Nest's theorem (in particular the regularity condition) and to prove a converse result: when α is an integrable and outer action of (M, Δ) on N, then the inclusion $N^\alpha \subset N$ is irreducible, of depth 2 and the operator valued weight $(\psi \otimes \iota)\alpha$ from N to N^α is regular. The same result is stated in [7, 11.14] for the special case of dual Kac algebra actions, but the proof is incomplete. The crucial point, our Proposition 5.7 identifying two operator valued weights, is not proved in [7]. We also remark that it will follow from Corollary 5.6 that the actions appearing in Enock and Nest's theorem are integrable. Further we refer to section 6 for the link between outer and minimal actions.

First of all we study the following problem. Let α be an action of (M, Δ) on N. Let $N^\alpha \subset N \subset N_2$ be the basic construction. When (M, Δ) is finite dimensional, it is known that N_2 is a quotient of the crossed product $M_\alpha \ltimes N$ (a proof can be found in [14, 4.1.3], but the result was undoubtedly known before). More precisely, there exists a surjective $*$-homomorphism ρ from $M_\alpha \ltimes N$ to N_2 sending $\alpha(x)$ to x for all $x \in N$. So, when $M_\alpha \ltimes N$ is a factor, the inclusion $\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N$ is the basic construction. More specifically, when N is a II_1-factor and α is an outer action (or equivalently a free action) of a finite group G on N it is well known that the crossed product $G_\alpha \ltimes N$ can be identified with $(N \cup \{u_t \mid t \in G\})''$, where N is represented standardly and $(u_t)_{t \in G}$ is the canonical implementation of α. This can be found in e.g. [16].

More generally we look at the following problem. Suppose that a locally compact group G acts on a von Neumann algebra N with action α. Then we can construct the crossed product $G_\alpha \ltimes N$ as follows. We represent N on a Hilbert space K and define operators on $L^2(G) \otimes K \cong L^2(G, K)$ by putting

$$(\alpha(x)\,\xi)(g) = \alpha_{g^{-1}}(x)\,\xi(g) \quad \text{for all } g \in G \quad \text{and } x \in N, \xi \in L^2(G, K)$$

$$(\lambda_g \xi)(h) = \xi(g^{-1}h) \qquad \text{for all } g, h \in G \text{ and } \xi \in L^2(G, K).$$

Then we define $G_\alpha \ltimes N = (\alpha(N) \cup \{\lambda_g \mid g \in G\})''$. But, when we represent N standardly on K and denote with $(u_g)_{g \in G}$ the canonical unitary implementation of α, we can also define

$$N_2 = (N \cup \{u_g \mid g \in G\})''.$$

Purely algebraicly one would expect to be able to define a $*$-homomorphism $\rho\colon G_\alpha \ltimes N \to N_2$ satisfying $\rho(\alpha(x)) = x$ for all $x \in N$ and $\rho(\lambda_g) = u_g$ for all $g \in G$. When the group G is finite, this can be done easily. In Theorem 5.3 we will prove that the construction of such a ρ is possible if and only if the action is integrable, and this will be proved for arbitrary locally compact quantum group actions. To see the link with the group case, recall that now the role of the regular representation (λ_g) is taken over by $\lambda(\omega) = (\omega \otimes \iota)(W)$ for all $\omega \in M_*$. So we work in fact with the regular representation of the L^1-functions.

Before we come to the proof of our main Theorem 5.3 we characterize the basic construction $N_2 = J_\theta (N^\alpha)' J_\theta$ in terms of the unitary implementation of α.

PROPOSITION 5.2. *Let α be an action of (M, Δ) on N. Fix a n.s.f. weight θ on N and let N act on the GNS-space of θ. Let U be the unitary implementation of α obtained with θ. Let $N_2 = J_\theta (N^\alpha)' J_\theta$ be the basic construction from $N^\alpha \subset N$. Then*

$$N_2 = (N \cup \{(\omega \otimes \iota)(U) \mid \omega \in M_*\})''.$$

Proof. Because $N_2 = J_\theta (N^\alpha)' J_\theta$ we get easily that

$$N_2' = N' \cap J_\theta \{(\omega \otimes \iota)(U^*) \mid \omega \in M_*\}' J_\theta.$$

But $(\hat{J} \otimes J_\theta)\, U^* (\hat{J} \otimes J_\theta) = U$, so that we have

$$N_2' = N' \cap \{(\omega \otimes \iota)(U) \mid \omega \in M_*\}'.$$

Because U is a corepresentation the σ-strong* closure of $\{(\omega \otimes \iota)(U) \mid \omega \in M_*\}$ is self-adjoint and then the result follows. ∎

Then we prove the following result.

THEOREM 5.3. *Let α be an action of (M, Δ) on N. Fix a n.s.f. weight θ on N and let N act on the GNS-space of θ. Let U be the unitary implementation of α obtained with θ. Let $N_2 = J_\theta (N^\alpha)' J_\theta$ be the basic construction from $N^\alpha \subset N$. Then the following statements are equivalent.*

- *There exists a normal and surjective $*$-homomorphism $\rho: M_\alpha \ltimes N \to N_2$ satisfying*

$$\rho(\alpha(x)) = x \qquad \textit{for all} \quad x \in N \qquad \textit{and}$$
$$\rho((\omega \otimes \iota)(W) \otimes 1) = (\omega \otimes \iota)(U^*) \qquad \textit{for all} \quad \omega \in M_*.$$

- *The action α is integrable.*

Proof of the first implication. Let us first suppose the first statement is valid. Because $\operatorname{Ker} \rho$ is a σ-strong* closed, two-sided ideal of $M_\alpha \ltimes N$ we can take a central projection $P \in M_\alpha \ltimes N$ such that

$$\operatorname{Ker} \rho = (M_\alpha \ltimes N)(1 - P).$$

Let ρ_P be the restriction of ρ to $(M_\alpha \ltimes N)\, P$. Then ρ_P is a $*$-isomorphism onto N_2. When η is a n.s.f. weight on $M_\alpha \ltimes N$ we have $\sigma_t^\eta(P) = P$ for all $t \in \mathbb{R}$, because P is central. So the restriction η_P of η to $(M_\alpha \ltimes N)\, P$ is a n.s.f. weight and $\sigma_t^{\eta_P}$ is the restriction of σ_t^η to $(M_\alpha \ltimes N)\, P$ for all $t \in \mathbb{R}$.

For every n.s.f. weight μ on N we define the n.s.f. weight $\check{\mu}$ on N_2 by $\check{\mu} = (\tilde{\mu})_P \circ \rho_P^{-1}$. Here $\tilde{\mu}$ denotes as before the dual weight on $M_\alpha \ltimes N$. For every $x \in N$ we have

$$\begin{aligned}\sigma_t^{\check{\mu}}(x) &= \rho_P(\sigma_t^{(\tilde{\mu})_P}(\rho_P^{-1}(x))) = \rho_P(\sigma_t^{(\tilde{\mu})_P}(\alpha(x)\, P)) \\ &= \rho_P(\sigma_t^{\tilde{\mu}}(\alpha(x))\, P) = \rho_P(\alpha(\sigma_t^\mu(x))\, P) = \sigma_t^\mu(x).\end{aligned}$$

When μ and ν are both n.s.f. weights on N we have

$$[D\check{\mu} : D\check{\nu}]_t = \rho_P([D(\tilde{\mu})_P : D(\tilde{\nu})_P]_t) = \rho_P([D\tilde{\mu} : D\tilde{\nu}]_t\, P)$$
$$= \rho_P(\alpha([D\mu : D\nu]_t)\, P) = [D\mu : D\nu]_t$$

for all $t \in \mathbb{R}$. So, by [26, 12.7], there exists a unique n.s.f. operator valued weight T_2 from N_2 to N such that $\check{\mu} = \mu \circ T_2$ for all n.s.f. weights μ on N. So $\mu \circ T_2 \circ \rho_P = (\tilde{\mu})_P$ for all n.s.f. weights μ on N.

When ν is a n.s.f. weight on either N^α or N we denote again with ν' the n.s.f. weight on either $J_\theta N^\alpha J_\theta = N_2'$ or $J_\theta N J_\theta = N'$ given by $\nu'(x) = \nu(J_\theta x J_\theta)$ for all positive x in either N^α or N. By [26, 12.11] there exists a unique n.s.f. operator valued weight T_1 from N to N^α such that

$$\frac{d(\mu \circ T_2)}{d\nu'} = \frac{d\mu}{d((\nu \circ T_1)')} \tag{5.1}$$

for all n.s.f. weights μ on N and ν on N^α.

Choose now a n.s.f. weight θ_0 on N^α. Put $\theta_1 = \theta_0 \circ T_1$ and $\theta_2 = \theta_1 \circ T_2$. When we change the weight θ which was chosen on N in the beginning of the story, the tower $N^\alpha \subset N \subset N_2$ will be transformed into a unitarily equivalent tower. The unitary implementing this transformation is the unique unitary intertwining the two standard representations of N. This unitary also intertwines the two implementations of α by Proposition 4.1. Hence also ρ can be transformed. So we may suppose that $\theta = \theta_1$.

From Eq. 5.1 follows that

$$\frac{d\theta_2}{d\theta_0'} = \frac{d\theta_1}{d\theta_1'}.$$

So we also have $\frac{d\theta_0'}{d\theta_2} = \frac{d\theta_1'}{d\theta_1}$. But $\frac{d\theta_1'}{d\theta_1} = \nabla_\theta^{-1}$ because K is the GNS-space of $\theta = \theta_1$. To compute $\frac{d\theta_0'}{d\theta_2}$ we need a GNS-construction for the weight θ_2. But $\theta_2 \circ \rho_P = \theta \circ T_2 \circ \rho_P = (\tilde{\theta})_P$. So we put $L = P(H \otimes K)$ and as before we denote with $(H \otimes K, \iota, \tilde{\Lambda})$ the GNS-construction of $\tilde{\theta}$. For every $x \in \mathcal{N}_{\theta_2}$ we define $\Lambda_{\theta_2}(x) = \tilde{\Lambda}(\rho_P^{-1}(x))$. Then $\Lambda_{\theta_2}(x) \in L$ and it is easy to check that $(L, \rho_P^{-1}, \Lambda_{\theta_2})$ is a GNS-construction for θ_2. Also observe that for all $a \in \mathcal{N}_{\hat{\varphi}}$ and $x \in \mathcal{N}_\theta$ we have $\rho(a \otimes 1)\, x \in \mathcal{N}_{\theta_2}$ and

$$\Lambda_{\theta_2}(\rho(a \otimes 1)\, x) = P(\hat{\Lambda}(a) \otimes \Lambda_\theta(x)).$$

Now choose $z \in \mathcal{T}_\theta$. Then

$$+\infty > \theta(z^*z) = \langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \nabla_\theta^{-1} \rangle = \left\langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \frac{d\theta_0'}{d\theta_2} \right\rangle$$
$$= \langle \theta_0', \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z))) \rangle. \tag{5.2}$$

Choose now a family $(\xi_i)_{i\in I}$ of vectors in K such that

$$\theta_0'(z) = \sum_{i\in I} \omega_{\xi_i}(z)$$

for all $z \in (J_\theta N^\alpha J_\theta)^+$. Fix $i \in I$. Then

$$\langle \omega_{\xi_i}, \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\rangle < +\infty$$

and so $\xi_i \in \mathscr{D}(R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^*)$. Further

$$\langle \omega_{\xi_i}, \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\rangle = \| R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i \|^2. \tag{5.3}$$

We will compute the final expression. For this we choose $\omega \in \mathscr{I}$ and $x \in \mathscr{N}_\theta$. Recall that the subset $\mathscr{I} \subset M_*$ was introduced in the introduction. Observe that

$$R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i \in L.$$

So we have

$$\begin{aligned}
&\langle R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i, \hat{\Lambda}((\omega\otimes\iota)(W)) \otimes \Lambda_\theta(x)\rangle \\
&\quad = \langle R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i, P(\hat{\Lambda}((\omega\otimes\iota)(W)) \otimes \Lambda_\theta(x))\rangle \\
&\quad = \langle R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i, \Lambda_{\theta_2}(\rho((\omega\otimes\iota)(W)\otimes 1)\, x)\rangle \\
&\quad = \langle \xi_i, R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\, \Lambda_{\theta_2}((\omega\otimes\iota)(U^*)\, x)\rangle \\
&\quad = \langle \xi_i, (\omega\otimes\iota)(U^*)\, x\Lambda_\theta(\sigma^\theta_{-i/2}(z))\rangle \\
&\quad = \langle \xi_i, (\omega\otimes\iota)(U^*)\, J_\theta z^* J_\theta \Lambda_\theta(x)\rangle \\
&\quad = \bar{\omega}((\iota\otimes\omega_{\xi_i,\Lambda_\theta(x)})((1\otimes J_\theta z J_\theta)\, U)).
\end{aligned}$$

By continuity we get that

$$\begin{aligned}
&\langle R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i, \hat{\Lambda}((\omega\otimes\iota)(W)) \otimes \eta\rangle \\
&\quad = \bar{\omega}((\iota\otimes\omega_{\xi_i,\eta})((1\otimes J_\theta z J_\theta)\, U))
\end{aligned}$$

for all $\omega \in \mathscr{I}$, $\eta \in K$ and $z \in \mathscr{T}_\theta$. By Proposition 7.3 of the appendix, it follows that

$$(\iota\otimes\omega_{\xi_i,\eta})((1\otimes J_\theta z J_\theta)\, U) \in \mathscr{N}_\varphi$$

and

$$\Lambda((\iota \otimes \omega_{\xi_i, \eta})((1 \otimes J_\theta z J_\theta) U)) = (1 \otimes \theta_\eta^*) R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i$$

for all $\eta \in K$ and $z \in \mathcal{T}_\theta$. Fix an orthonormal basis $(e_j)_{j \in J}$ for K. Then we may conclude that

$$\begin{aligned}
&\| R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i \|^2 \\
&\quad = \sum_{j \in J} \|(1 \otimes \theta_{e_j}^*) R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^* \xi_i\|^2 \\
&\quad = \sum_{j \in J} \varphi((\iota \otimes \omega_{\xi_i, e_j})((1 \otimes J_\theta z J_\theta) U)^* (\iota \otimes \omega_{\xi_i, e_j})((1 \otimes J_\theta z J_\theta) U)) \\
&\quad = \varphi((\iota \otimes \omega_{\xi_i})(U^*(1 \otimes J_\theta z^* z J_\theta) U)) \\
&\quad = \varphi(\hat{J}(\iota \otimes \omega_{J_\theta \xi_i}) \alpha(z^* z) \hat{J}) \\
&\quad = \psi((\iota \otimes \omega_{J_\theta \xi_i}) \alpha(z^* z)) \\
&\quad = \langle \omega_{J_\theta \xi_i}, (\psi \otimes \iota) \alpha(z^* z) \rangle.
\end{aligned}$$

Combining this with Eq. 5.3 we get that

$$\langle \omega_{\xi_i}, \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z))) \rangle = \langle \omega_{J_\theta \xi_i}, (\psi \otimes \iota) \alpha(z^* z) \rangle$$

for all $z \in \mathcal{T}_\theta$ and $i \in I$. Summing over i we get

$$\langle \theta'_0, \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z))) \rangle = \langle \theta_0, (\psi \otimes \iota) \alpha(z^* z) \rangle$$

for all $z \in \mathcal{T}_\theta$. Using Eq. 5.2 we get that

$$\theta(z^* z) = \langle \theta_0, (\psi \otimes \iota) \alpha(z^* z) \rangle$$

for all $z \in \mathcal{T}_\theta$. Hence the normal faithful weight $\theta_0 \circ (\psi \otimes \iota) \alpha$ is semifinite. From [26, 11.7] it follows that $(\psi \otimes \iota) \alpha$ is semifinite. So α is integrable.

Proof of the second implication. The second implication can be proved along the same lines as in the case of an Abelian group action, see [24]. So let us suppose that α is integrable. Choose a n.s.f. weight θ_0 on N^α and put $\theta = \theta_0 \circ (\psi \otimes \iota) \alpha$. Then θ is a n.s.f. weight on N. Represent N on the GNS-space K of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction for θ. Choose a family of vectors $(\xi_i)_{i \in I}$ in K such that

$$\theta_0(x) = \sum_{i \in I} \omega_{\xi_i}(x) \qquad \text{for all} \quad x \in (N^\alpha)^+.$$

Define $L = \bigoplus_{i \in I} H \otimes K$ and let π be the I-fold direct sum of the identical representation ι of $M_\alpha \ltimes N$ on $H \otimes K$. Recall that for any operator valued weight T we define $\mathcal{N}_T$ as the left ideal of elements x for which $T(x^*x)$ is bounded. Also recall that we introduced the canonical GNS-construction (H, ι, Γ) for ψ in the introduction. When $z \in \mathcal{N}_{\psi \otimes \iota}$ we define $(\Gamma \otimes \iota)(z) \in B(K, H \otimes K)$ by $(\Gamma \otimes \iota)(z)\, \Lambda_\theta(x) = (\Gamma \otimes \Lambda_\theta)(z(1 \otimes x))$ for all $x \in \mathcal{N}_\theta$, where $\Gamma \otimes \Lambda_\theta$ denotes the canonical GNS-map of $\psi \otimes \theta$. One can check easily that $(\Gamma \otimes \iota)(z)^*\, (\Gamma \otimes \iota)(z) = (\psi \otimes \iota)(z^*z)$. For this see e.g. [7, 10.6]. Put $T = (\psi \otimes \iota)\, \alpha$. For all $x \in \mathcal{N}_T \cap \mathcal{N}_\theta$ we define

$$\mathcal{V} \Lambda_\theta(x) = \bigoplus_{i \in I} (\Gamma \otimes \iota)\, \alpha(x)\, \xi_i.$$

Observe that $\mathcal{V}$ is well-defined:

$$\sum_{i \in I} \|(\Gamma \otimes \iota)\, \alpha(x)\, \xi_i\|^2 = \sum_{i \in I} \omega_{\xi_i}((\psi \otimes \iota)\, \alpha(x^*x))$$
$$= \langle \theta_0, (\psi \otimes \iota)\, \alpha(x^*x) \rangle = \theta(x^*x) < \infty.$$

Because $\mathcal{N}_T \cap \mathcal{N}_\theta$ is a σ-strong*-norm core for Λ_θ we get that $\Lambda_\theta(\mathcal{N}_T \cap \mathcal{N}_\theta)$ is dense in K. So $\mathcal{V}$ can be extended uniquely to an isometry from K to L.

We now want to prove that the range of $\mathcal{V}$ is invariant under $\pi(M_\alpha \ltimes N)$. So we first choose $y \in N$. Then for every $x \in \mathcal{N}_T \cap \mathcal{N}_\theta$ we have

$$\pi(\alpha(y))\, \mathcal{V} \Lambda_\theta(x) = \bigoplus_{i \in I} \alpha(y)(\Gamma \otimes \iota)\, \alpha(x)\, \xi_i$$
$$= \bigoplus_{i \in I} (\Gamma \otimes \iota)\, \alpha(yx)\, \xi_i = \mathcal{V} \Lambda_\theta(yx) = \mathcal{V} y \Lambda_\theta(x).$$

Next we will look at the invariance under $\pi(\hat{M} \otimes \mathbb{C})$. Analogously as in Proposition 7.2 of the appendix we have that for every $x \in \mathcal{N}_\psi$, $\xi \in \mathcal{D}(\delta^{1/2})$ and $\eta \in H$, $(\omega_{\delta^{1/2}\xi, \eta} \otimes \iota)\, \Delta(x) \in \mathcal{N}_\psi$ and

$$\Gamma((\omega_{\delta^{1/2}\xi, \eta} \otimes \iota)\, \Delta(x)) = (\omega_{\xi, \eta} \otimes \iota)(W^*)\, \Gamma(x).$$

Then it follows easily that for all $z \in \mathcal{N}_{\psi \otimes \iota}$ we have

$$x := (\omega_{\delta^{1/2}\xi, \eta} \otimes \iota \otimes \iota)(\Delta \otimes \iota)(z) \in \mathcal{N}_{\psi \otimes \iota} \qquad \text{and}$$
$$(\Gamma \otimes \iota)(x) = ((\omega_{\xi, \eta} \otimes \iota)(W^*) \otimes 1)(\Gamma \otimes \iota)(z).$$

So for all $\xi \in \mathscr{D}(\delta^{1/2})$, $\eta \in H$ and $x \in \mathcal{N}_T \cap \mathcal{N}_\theta$ we have

$$\begin{aligned}
&\pi((\omega_{\xi,\eta} \otimes \iota)(W^*) \otimes 1)\, \mathscr{V} \Lambda_\theta(x) \\
&\quad = \bigoplus_{i \in I} ((\omega_{\xi,\eta} \otimes \iota)(W^*) \otimes 1)(\Gamma \otimes \iota)\, \alpha(x)\, \xi_i \\
&\quad = \bigoplus_{i \in I} (\Gamma \otimes \iota)((\omega_{\delta^{1/2}\xi,\eta} \otimes \iota \otimes \iota)(\Delta \otimes \iota)\, \alpha(x))\, \xi_i \\
&\quad = \bigoplus_{i \in I} (\Gamma \otimes \iota)(\alpha((\omega_{\delta^{1/2}\xi,\eta} \otimes \iota)\, \alpha(x)))\, \xi_i \\
&\quad = \mathscr{V} \Lambda_\theta((\omega_{\delta^{1/2}\xi,\eta} \otimes \iota)\, \alpha(x)) = \mathscr{V} (\omega_{\xi,\eta} \otimes \iota)(U)\, \Lambda_\theta(x)
\end{aligned}$$

by Propositions 2.4 and 4.3. So the range of $\mathscr{V}$ is invariant under $\pi(M_\alpha \ltimes N)$. Then we can define a $*$-homomorphism

$$\rho\colon M_\alpha \ltimes N \to B(K)\colon \rho(z) = \mathscr{V}^* \pi(z)\, \mathscr{V}.$$

By the computations above we get that $\rho(\alpha(x)) = x$ for all $x \in N$ and $(\iota \otimes \rho)(W \otimes 1) = U^*$. Then it follows from Proposition 5.2 that $\rho(M_\alpha \ltimes N) = N_2$ and so the theorem is proved. ∎

One can also prove the following more general kind of result, where we do not specify what ρ should be.

Proposition 5.4. *Let α be an action of (M, Δ) on N. Fix a n.s.f. weight θ on N and let N act on the GNS-space K of θ. Consider the inclusions*

$$\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N \qquad \textit{and} \qquad N^\alpha \subset N \subset N_2 = J_\theta (N^\alpha)' J_\theta.$$

Then the following statements are equivalent.

- *There exists a surjective $*$-homomorphism ρ from $M_\alpha \ltimes N$ to N_2 such that ρ is an isomorphism of $\alpha(N)$ onto N and of $\mathbb{C} \otimes N^\alpha$ onto N^α.*
- *The action α is cocycle-equivalent with an integrable action β satisfying $N^\beta = N^\alpha$.*

Proof of the first implication. Suppose the first statement is true. Because N is represented on the GNS-space of θ, there exists a unitary u on K such that $\rho(\alpha(x)) = uxu^*$ for all $x \in N$ and $uJ_\theta = J_\theta u$. Define $\tilde{\rho}$ from $M_\alpha \ltimes N$ to $B(K)$ by $\tilde{\rho}(z) = u^* \rho(z)\, u$ for all $z \in M_\alpha \ltimes N$. Then $\tilde{\rho}(\alpha(x)) = x$ for all $x \in N$. Further

$$u^* N^\alpha u = u^* \rho(\mathbb{C} \otimes N^\alpha)\, u = u^* \rho(\alpha(N^\alpha))\, u = N^\alpha.$$

Because $uJ_\theta = J_\theta u$, we get $u^* N_2 u = N_2$, which leads to $\tilde{\rho}(M_\alpha \ltimes N) = N_2$.

So we may suppose from the beginning that $\rho(\alpha(x)) = x$ for all $x \in N$. Define the unitary $X \in M \otimes B(K)$ by

$$X = (\hat{J} \otimes J_\theta)(\iota \otimes \rho)(W \otimes 1)(\hat{J} \otimes J_\theta).$$

Put $\mathscr{V} = XU^*$. Clearly $\mathscr{V} \in M \otimes B(K)$ and

$$(\hat{J} \otimes J_\theta)\, \mathscr{V} (\hat{J} \otimes J_\theta) = (\iota \otimes \rho)(W \otimes 1)\, U.$$

For every $x \in N$ we have

$$\begin{aligned}(\iota \otimes \rho)(W \otimes 1)\, U(1 \otimes x) &= (\iota \otimes \rho)(W \otimes 1)\, \alpha(x)\, U \\ &= (\iota \otimes \rho)((W \otimes 1)(\iota \otimes \alpha)\, \alpha(x))\, U \\ &= (\iota \otimes \rho)((1 \otimes \alpha(x))(W \otimes 1))\, U \\ &= (1 \otimes x)(\iota \otimes \rho)(W \otimes 1)\, U.\end{aligned}$$

So we get $(\iota \otimes \rho)(W \otimes 1)\, U \in M \otimes N'$ and hence $\mathscr{V} \in M \otimes N$. In the next computation we denote again with L_θ the $*$-anti-automorphism of $B(K)$ given by $L_\theta(x) = J_\theta x^* J_\theta$ for all $x \in B(K)$. Then we have

$$\begin{aligned}(\Delta \otimes \iota)(\mathscr{V}) &= (\Delta \otimes \iota)(R \otimes L_\theta)(\iota \otimes \rho)(W^* \otimes 1)\, (\Delta \otimes \iota)(U^*) \\ &= (R \otimes R \otimes L_\theta)(\Delta^{\mathrm{op}} \otimes \rho)(W^* \otimes 1)\, U^*_{13} U^*_{23} \\ &= (R \otimes R \otimes L_\theta)(\iota \otimes \iota \otimes \rho)(W^*_{13} W^*_{23})\, U^*_{13} U^*_{23} \\ &= ((R \otimes L_\theta)(\iota \otimes \rho)(W^* \otimes 1))_{23}\, U^*_{23} U_{23} \\ &\quad \times ((R \otimes L_\theta)(\iota \otimes \rho)(W^* \otimes 1)\, U^*)_{13}\, U^*_{23} \\ &= \mathscr{V}_{23}(\iota \otimes \alpha)(\mathscr{V}).\end{aligned}$$

So $\mathscr{V}$ is a α-cocycle. Define the action β on (M, Δ) on N given by $\beta(x) = \mathscr{V} \alpha(x)\, \mathscr{V}^*$ for all $x \in N$. Then $\beta(x) = X(1 \otimes x)\, X^*$ for all $x \in N$. Because the σ-strong* closure of $\{(\omega \otimes \iota)(X) \mid \omega \in M_*\}$ equals $J_\theta \rho(\hat{M} \otimes \mathbb{C})\, J_\theta$ we get that $N^\beta = J_\theta \rho(M_\alpha \ltimes N)'\, J_\theta = N^\alpha$.

To conclude the proof of the first implication we have to show that β is integrable. For this we will use the previous theorem. From Proposition 4.2 it follows that the unitary implementation U_β of β is given by

$$U_\beta = \mathscr{V} U(\hat{J} \otimes J_\theta)\, \mathscr{V}^*(\hat{J} \otimes J_\theta) = \mathscr{V}(\iota \otimes \rho)(W^* \otimes 1).$$

From the proof of Proposition 4.2 we also get that $z \mapsto \mathscr{V}^* z \mathscr{V}$ gives an isomorphism from $M_\beta \ltimes N$ onto $M_\alpha \ltimes N$. So we can define

$$\tilde{\rho}\colon M_\beta \ltimes N \to N_2\colon \tilde{\rho}(z) = \rho(\mathscr{V}^* z \mathscr{V}).$$

Then $\tilde{\rho}$ is a surjective $*$-homomorphism onto $N_2 = J_\theta(N^\alpha)' J_\theta = J_\theta(N^\beta)' J_\theta$ and clearly $\tilde{\rho}(\beta(x)) = x$ for all $x \in N$. Because $\mathcal{V}$ is an α-cocycle we get that

$$(1 \otimes \mathcal{V}^*)(W^* \otimes 1)(1 \otimes \mathcal{V}) = (\iota \otimes \alpha)(\mathcal{V})(W^* \otimes 1).$$

From this it follows that

$$\begin{aligned}(\iota \otimes \tilde{\rho})(W^* \otimes 1) &= (\iota \otimes \rho)((1 \otimes \mathcal{V}^*)(W^* \otimes 1)(1 \otimes \mathcal{V})) \\ &= \mathcal{V}(\iota \otimes \rho)(W^* \otimes 1) = U_\beta.\end{aligned}$$

By the previous theorem we get that β is integrable.

Proof of the second implication. Conversely suppose that the second statement is valid and take such an action β. Let $\mathcal{V}$ be an α-cocycle such that $\beta(x) = \mathcal{V}\alpha(x)\mathcal{V}^*$ for all $x \in N$. It follows from the proof of Proposition 4.2 that

$$\Phi\colon M_\alpha \ltimes N \to M_\beta \ltimes N\colon \Phi(z) = \mathcal{V} z \mathcal{V}^*$$

is an isomorphism and $\Phi(\alpha(x)) = \beta(x)$ for all $x \in N$. By the previous theorem we can find a surjective $*$-homomorphism $\tilde{\rho}$ from $M_\beta \ltimes N$ onto $J_\theta(N^\beta)' J_\theta$ satisfying $\tilde{\rho}(\beta(x)) = x$ for all $x \in N$. Putting $\rho = \tilde{\rho} \circ \Phi$ and observing that $N^\alpha = N^\beta$ we get the first statement. ∎

We do not know an example of a non-integrable action α which is cocycle-equivalent with an integrable action β satisfying $N^\alpha = N^\beta$, but it seems to be natural that this kind of actions will exist. We will now specify a case in which it cannot exist. This should be compared with the example of a finite group acting outerly on a factor as described above.

Definition 5.5. An action α of a locally compact quantum group (M, Δ) on N is called outer when

$$M_\alpha \ltimes N \cap \alpha(N)' = \mathbb{C}.$$

Corollary 5.6. *Let α be an outer action of (M, Δ) on N. Choose again a n.s.f. weight θ on N and represent N on the GNS-space of θ. Let $N_2 = J_\theta(N^\alpha)' J_\theta$ be the basic construction. Then the inclusions*

$$\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N \qquad \textit{and} \qquad N^\alpha \subset N \subset N_2$$

are isomorphic if and only if α is integrable.

Proof. When α is integrable, one can use Theorem 5.3 and then observe that the $*$-homomorphism ρ is faithful because $M_\alpha \ltimes N$ is a factor.

Next suppose that the inclusions stated above are isomorphic. By Proposition 5.4 there exists an integrable action β which is cocycle equivalent with α and satisfies $N^\beta = N^\alpha$. Let $\mathscr{V} \in M \otimes N$ be an α-cocycle such that $\beta(x) = \mathscr{V} \alpha(x) \mathscr{V}^*$ for all $x \in N$. Then for all $x \in N^\alpha = N^\beta$ we have

$$1 \otimes x = \beta(x) = \mathscr{V} \alpha(x) \mathscr{V}^* = \mathscr{V}(1 \otimes x) \mathscr{V}^*.$$

Hence we get $\mathscr{V} \in M \otimes (N \cap (N^\alpha)')$. From our assumption and the fact that α is outer it follows that $N_2 \cap N' = \mathbb{C}$. But then also

$$(N^\alpha)' \cap N = J_\theta (N_2 \cap N') J_\theta = \mathbb{C}.$$

So we can take $u \in M$ such that $\mathscr{V} = u \otimes 1$. Because $\mathscr{V}$ is an α-cocycle we get that $\Delta(u) = u \otimes u$. By the unicity of right invariant weights on (M, Δ) there exists a number $\lambda > 0$ such that $\psi(u^* a u) = \lambda \psi(a)$ for all $a \in M^+$. Then we get that for all $x \in N^+$ we have $(\psi \otimes \iota)\, \alpha(x) = \lambda (\psi \otimes \iota)\, \beta(x)$. Because β is integrable it follows that α is integrable. ∎

There exist outer actions which are not integrable: see 6.3. Combining the previous result with Theorem 5.1 we get that all actions coming out of Enock and Nest's construction are integrable.

Next we turn towards the notion of a regular operator valued weight. Suppose α is an integrable action of (M, Δ) on N and suppose that the $*$-homomorphism ρ given by Theorem 5.3 is faithful. This will of course be the case whenever $M_\alpha \ltimes N$ is a factor, but also when α is a dual action or a semidual action. The latter follows from Proposition 5.12. Then we can prove that the operator valued weight $(\psi \otimes \iota)\, \alpha$ from N to N^α is regular. More precisely, we will do the following. By our assumption the basic construction $N^\alpha \subset N \subset N_2$ is isomorphic with $\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N$ through the isomorphism ρ. Let us denote with T_1 the operator valued weight $(\psi \otimes \iota)\, \alpha$ from N to N^α. Then we can construct the operator valued weight T_2 from N_2 to N by modular theory and the basic construction, as described above. Through the isomorphism ρ the operator valued weight T_2 is transformed to an operator valued weight from $M_\alpha \ltimes N$ to $\alpha(N)$. In the next proposition we prove that this operator valued weight is equal to the canonical operator valued weight $T = (\hat{\varphi} \otimes \iota \otimes \iota)\, \hat{\alpha}$ from $M_\alpha \ltimes N$ to $\alpha(N)$.

Proposition 5.7. *Let α be an integrable action of (M, Δ) on N. Suppose that the $*$-homomorphism ρ constructed in Theorem 5.3 is faithful. Denote with T_2 and T the operator valued weights defined above. Then $\rho \circ T = T_2 \circ \rho$.*

Proof. For clarity we stress that T_1 is the operator valued weight $(\psi \otimes \iota)\,\alpha$ from N to N^α, that T_2 is obtained out of T_1 by modular theory and the basic construction, and it goes from N_2 to N. Finally T is the canonical operator valued weight $(\hat{\varphi} \otimes \iota \otimes \iota)\,\hat{\alpha}$ from $M_\alpha \ltimes N$ to $\alpha(N)$, giving the dual weights by the formula $\tilde{\theta} = \theta \circ \alpha^{-1} \circ T$ for all n.s.f. weights θ on N.

Choose a n.s.f. weight θ_0 on N^α. Put $\theta = \theta_0 \circ T_1$ and let $\theta_2 = \tilde{\theta} \circ \rho^{-1}$. We will prove that $\theta_2 = \theta \circ T_2$. As in the proof of Theorem 5.3 we may suppose that N is represented on the GNS-space of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction for θ. Let $(H \otimes K, \iota, \tilde{\Lambda})$ be the canonical GNS-construction for $\tilde{\theta}$ and put $\Lambda_{\theta_2} = \tilde{\Lambda} \circ \rho^{-1}$. We now make a kind of converse reasoning of the proof of Theorem 5.3. Denote again with θ_0' the n.s.f. weight on $J_\theta N^\alpha J_\theta = N_2'$ given by $\theta_0'(x) = \theta_0(J_\theta x J_\theta)$ for all positive x. We claim that for all $z \in \mathcal{T}_\theta$

$$\left\langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \frac{d\theta'}{d\theta_2} \right\rangle = \langle \theta_0, (\psi \otimes \iota)\,\alpha(z^* z) \rangle. \tag{5.4}$$

So choose $z \in \mathcal{T}_\theta$. Take a family of vectors $(\xi_i)_{i \in I}$ in K such that

$$\theta_0(x) = \sum_{i \in I} \langle x J_\theta \xi_i, J_\theta \xi_i \rangle$$

for all $x \in (N^\alpha)^+$. Because $\langle \theta_0, (\psi \otimes \iota)\,\alpha(z^* z) \rangle = \theta(z^* z) < \infty$ we have

$$\langle \omega_{J_\theta \xi_i}, (\psi \otimes \iota)\,\alpha(z^* z) \rangle < \infty$$

for all $i \in I$. Fix $i \in I$. Then we conclude from the previous formula that

$$\varphi((\iota \otimes \omega_{\xi_i})(U^*(1 \otimes J_\theta z^* z J_\theta)\,U)) < \infty.$$

So, when $(e_j)_{j \in J}$ is an orthonormal basis for K we can define the element $\eta \in H \otimes K$ by

$$\eta := \sum_{j \in J} \Lambda((\iota \otimes \omega_{\xi_i, e_j})((1 \otimes J_\theta z J_\theta)\,U)) \otimes e_j.$$

It is easy to check that for all $\mu \in K$ we have $(\iota \otimes \omega_{\xi_i, \mu})((1 \otimes J_\theta z J_\theta)\,U) \in \mathcal{N}_\varphi$ and

$$\Lambda((\iota \otimes \omega_{\xi_i, \mu})((1 \otimes J_\theta z J_\theta)\,U)) = (1 \otimes \theta_\mu^*)\,\eta.$$

Using the notation $\mathcal{I} \subset M_*$ introduced in the introduction, we get for all $\omega \in \mathcal{I}$ and $x \in \mathcal{N}_\theta$ that

$$\begin{aligned}
&\langle \xi_i, R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\, \Lambda_{\theta_2}((\omega \otimes \iota)(U^*)\, x)\rangle \\
&\quad = \langle \xi_i, (\omega \otimes \iota)(U^*)\, J_\theta z^* J_\theta \Lambda_\theta(x)\rangle \\
&\quad = \bar{\omega}((\iota \otimes \omega_{\xi_i, \Lambda_\theta(x)})((1 \otimes J_\theta z J_\theta)\, U)) \\
&\quad = \langle \eta, \hat{\Lambda}((\omega \otimes \iota)(W)) \otimes \Lambda_\theta(x)\rangle \\
&\quad = \langle \eta, \Lambda_{\theta_2}((\omega \otimes \iota)(U^*)\, x)\rangle.
\end{aligned}$$

From this we get that

$$\langle \xi_i, R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\, \Lambda_{\theta_2}(y)\rangle = \langle \eta, \Lambda_{\theta_2}(y)\rangle$$

for all $y \in \mathcal{N}_{\theta_2}$. Hence $\xi_i \in \mathcal{D}(R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^*)$ and

$$\|R^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))^*\, \xi_i\|^2 = \|\eta\|^2 = \langle \omega_{J_\theta \xi_i}, (\psi \otimes \iota)\, \alpha(z^* z)\rangle.$$

This means that

$$\langle \omega_{\xi_i}, \Theta^{\theta_2}(\Lambda_\theta(\sigma^\theta_{-i/2}(z)))\rangle = \langle \omega_{J_\theta \xi_i}, (\psi \otimes \iota)\, \alpha(z^* z)\rangle.$$

Summing over i we get our claim stated in Eq. 5.4. But now

$$\langle \theta_0, (\psi \otimes \iota)\, \alpha(z^* z)\rangle = \theta(z^* z) = \langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \nabla_\theta^{-1}\rangle,$$

and so

$$\langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \nabla_\theta^{-1}\rangle = \left\langle \omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(z))}, \frac{d\theta'_0}{d\theta_2}\right\rangle$$

for all $z \in \mathcal{T}_\theta$. Next we claim that $\frac{d\theta'_0}{d\theta_2}$ and ∇_θ^{-1} commute strongly. Then we will be able to conclude that $\frac{d\theta'_0}{d\theta_2} = \nabla_\theta^{-1}$. But then $\frac{d\theta_2}{d\theta'_0} = \nabla_\theta$, and so we will get

$$\frac{d(\theta \circ T_2)}{d\theta'_0} = \frac{d\theta}{d((\theta_0 \circ T_1)')} = \frac{d\theta}{d\theta'} = \nabla_\theta = \frac{d\theta_2}{d\theta'_0}.$$

So we may conclude that $\theta_2 = \theta \circ T_2$. By definition of $\tilde{\theta}$ we have $\tilde{\theta} = \theta \circ \rho \circ T$ and then $\theta \circ \rho \circ T = \tilde{\theta} = \theta_2 \circ \rho = \theta \circ T_2 \circ \rho$. By [26, 11.13] we get that $\rho \circ T = T_2 \circ \rho$.

So we only have to prove our claim. Hence we want to prove that $\frac{d\theta_0'}{d\theta_2}$ and ∇_θ^{it} commute for every $t \in \mathbb{R}$. For this it is sufficient to prove that $\operatorname{Ad} \nabla_\theta^{it}$ leaves both $J_\theta N^\alpha J_\theta$ and N_2 invariant and

$$\theta_0' \circ \operatorname{Ad} \nabla_\theta^{it} \theta_0' \qquad \text{and} \qquad \theta_2 \circ \operatorname{Ad} \nabla_\theta^{it} = \theta_2$$

for all $t \in \mathbb{R}$. When $x \in N^\alpha$ we have

$$\nabla_\theta^{it} J_\theta x J_\theta \nabla_\theta^{-it} = J_\theta \sigma_t^\theta(x)\, J_\theta = J_\theta \sigma_t^{\theta_0}(x)\, J_\theta \in J_\theta N^\alpha J_\theta.$$

Then it is immediately clear that $\theta_0' \circ \operatorname{Ad} \nabla_\theta^{it} = \theta_0'$.

Because $N_2 = (J_\theta N^\alpha J_\theta)'$ we have that $\operatorname{Ad} \nabla_\theta^{it}$ leaves N_2 invariant. Recall that we denoted with $(\tilde{\sigma}_t)$ the modular group of $\tilde{\theta}$ on $M_\alpha \ltimes N$. Then we have, for all $x \in N$

$$\begin{aligned} \nabla_\theta^{it} \rho(\alpha(x))\, \nabla_\theta^{-it} &= \nabla_\theta^{it} x \nabla_\theta^{-it} = \sigma_t^\theta(x) \\ &= \rho(\alpha(\sigma_t^\theta(x))) = \rho(\tilde{\sigma}_t(\alpha(x))). \end{aligned} \qquad (5$$

Finally, for all $\omega \in B(H)_*$ we have by Proposition 4.3 and 2.4 that

$$\begin{aligned} \nabla_\theta^{it} P((\omega \otimes \iota)(W) \otimes 1)\, \nabla_\theta^{-it} &= \nabla_\theta^{it}(\omega \otimes \iota)(U^*)\, \nabla_\theta^{-it} = (Q^{it} \omega Q^{-it} \otimes \iota)(U^*) \\ &= \rho((Q^{it} \omega Q^{-it} \otimes \iota)(W) \otimes 1) \\ &= \rho((Q^{it} \otimes \nabla_\theta^{it})((\omega \otimes \iota)(W) \otimes 1)(Q^{-it} \otimes \nabla_\theta^{-it})) \end{aligned}$$

where we used that $W(Q^{it} \otimes Q^{it}) = (Q^{it} \otimes Q^{it})\, W$. From the proof of Proposition 4.3 it follows that $\tilde{\nabla}^{it} = Q^{it} \otimes \nabla_\theta^{it}$ and so we see that

$$\nabla_\theta^{it} \rho(a \otimes 1)\, \nabla_\theta^{-it} = \rho(\tilde{\sigma}_t(a \otimes 1))$$

for all $a \in \hat{M}$ and $t \in R$. Combining this with Eq. 5.5 we get that $\nabla_\theta^{it} \rho(z)\, \nabla_\theta^{-it} = \rho(\tilde{\sigma}_t(z))$ for all $z \in M_\alpha \ltimes N$ and $t \in \mathbb{R}$. Then we get immediately that $\theta_2 \circ \operatorname{Ad} \nabla_\theta^{it} = \theta_2$ for all $t \in \mathbb{R}$.

This proves our claim and ends the proof of the proposition. ∎

COROLLARY 5.8. *Under the same assumptions as in Proposition* 5.7, *the operator valued weight* $(\psi \otimes \iota)\, \alpha$ *from* N *to* N^α *is regular.*

Proof. Using the notations introduced above we will identify the inclusions $N^\alpha \subset N \subset N_2$ and $\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N$. Then we get that $T_2 = (\hat{\varphi} \otimes \iota \otimes \iota)\, \hat{\alpha}$. Now it is obvious that $\hat{M} \otimes \mathbb{C} \subset M_\alpha \ltimes N \cap (\mathbb{C} \otimes N^\alpha)'$ and $\mathcal{N}_{\hat{\varphi}} \otimes \mathbb{C} \subset \mathcal{N}_{T_2}$. So the restriction of T_2 to $N_2 \cap (N^\alpha)'$ is semifinite.

Next observe that $\alpha(N) = (M_\alpha \ltimes N)^{\hat{\alpha}}$. Applying the first part of the proof to the dual action $\hat{\alpha}$, which is integrable and for which the $*$-homomorphism ρ is faithful by Proposition 5.12, we get that the restriction of T_3 to $N_3 \cap N_1'$ is semifinite. ∎

As a final ingredient for the converse of Enock and Nest's theorem we look at depth 2 inclusions. The assumption of the following proposition may seem strange, but one can immediately look at the corollary for a more clear result.

PROPOSITION 5.9. *Let α be an action of (M, Δ) on N such that $\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N$ is the basic construction. Then the inclusion $N^\alpha \subset N$ has depth* 2.

Proof. Choose a n.s.f. weight θ on N and let $\tilde{\theta}$ be the dual weight on $M_\alpha \ltimes N$. Represent N on the GNS-space of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction for θ. Let $(H \otimes K, \iota, \tilde{\Lambda})$ be the canonical GNS-construction for $\tilde{\theta}$ and denote with $\tilde{J}$ the modular conjugation of $\tilde{\theta}$. Then it follows from Definition 3.6 that $U = \tilde{J}(\hat{J} \otimes J_\theta)$ is the unitary implementation of α. The basic construction from $\alpha(N) \subset M_\alpha \ltimes N$ is then given by

$$\tilde{J}\alpha(N)'\,\tilde{J} = \tilde{J}U(B(H) \otimes N')\,U^*\tilde{J} = B(H) \otimes N.$$

To prove that $N^\alpha \subset N$ has depth 2, we have to show that

$$\alpha(N \cap (N^\alpha)') \subset (M_\alpha \ltimes N) \cap (\mathbb{C} \otimes N^\alpha)' \subset B(H) \otimes (N \cap (N^\alpha)')$$

is the basic construction. But it is immediately clear that the restriction of α to $N \cap (N^\alpha)'$ is an action β of (M, Δ) on $N \cap (N^\alpha)'$. So by the first part of the proof it is sufficient to prove that

$$M_\beta \ltimes ((N \cap (N^\alpha)') = (M_\alpha \ltimes N) \cap (\mathbb{C} \otimes N^\alpha)'. \tag{5.6}$$

Now it follows from Theorem 2.6 and 2.7 that

$$M_\beta \ltimes ((N \cap (N^\alpha)') = \{z \in B(H) \otimes (N \cap (N^\alpha)') \mid (\iota \otimes \beta)(z) = V_{12} z_{13} V_{12}^*\}$$

and

$$M_\alpha \ltimes N = \{z \in B(H) \otimes N \mid (\iota \otimes \alpha)(z) = V_{12} z_{13} V_{12}^*\}.$$

From this we can immediately deduce Eq. 5.6, and that concludes the proof. ∎

Although the following result is an immediate corollary of the previous one, we include it for completeness. The first statement is clear and the next two statements follow from the first, using Proposition 5.12 for the last one.

COROLLARY 5.10. *Let α be an action of (M, Δ) on N.*

- *If α is integrable and the $*$-homomorphism in Theorem 5.3 is faithful, then the inclusion $N^\alpha \subset N$ has depth 2.*
- *If α is integrable and $M_\alpha \ltimes N$ is a factor, then the inclusion $N^\alpha \subset N$ has depth 2.*
- *The inclusion $\alpha(N) \subset M_\alpha \ltimes N$ has depth 2.*

We now prove the announced result giving a converse to the theorem of Enock and Nest.

PROPOSITION 5.11. *Let α be an integrable outer action of (M, Δ) on N. Then the operator valued weight $(\psi \otimes \iota)\,\alpha$ from N to N^α is regular. Further the inclusion $N^\alpha \subset N$ is irreducible and has depth 2.*

Proof. Because $M_\alpha \ltimes N$ is a factor the $*$-homomorphism ρ from Theorem 5.3 is faithful. Then we apply Corollary 5.8 to obtain the regularity of $(\psi \otimes \iota)\,\alpha$ and Corollary 5.10 to get that $N^\alpha \subset N$ has depth 2. It is clear that $N^\alpha \subset N$ is irreducible, because

$$N \cap (N^\alpha)' = J_\theta (N_2 \cap N')\, J_\theta = \mathbb{C}. \quad \blacksquare$$

As a complement to Theorem 5.3 we prove the following easy result. The terminology is taken from [23].

PROPOSITION 5.12. *Let α be an action of (M, Δ) on N. Then we call α semidual when there exists a unitary $v \in B(H) \otimes N$ satisfying $(\iota \otimes \alpha)(v) = v_{13} V_{12}^*$.*

- *Every dual action is semidual.*
- *Every semidual action is integrable and the $*$-homomorphism ρ from Theorem 5.3 is faithful.*

Proof. Let us first prove the first statement. Denote with $\hat{\alpha}$ the dual action, which is an action of $(\hat{M}, \hat{\Delta}^{\text{op}})$ on $M_\alpha \ltimes N$. Because $\hat{\varphi}$ is the right Haar weight of $(\hat{M}, \hat{\Delta}^{\text{op}})$, the role of V is played by $\Sigma \hat{W}^* \Sigma = W$. So we have to find a unitary $v \in B(H) \otimes (M_\alpha \ltimes N)$ satisfying $(\iota \otimes \hat{\alpha})(v) = v_{13} W_{12}^*$. Then it is clear that we can take $v = W^* \otimes 1$ and so $\hat{\alpha}$ is semidual.

To prove the second part suppose that $v \in B(H) \otimes N$ is unitary and $(\iota \otimes \alpha)(v) = v_{13} V_{12}^*$. Define the isomorphism Ψ: $B(H) \otimes N \to B(H) \otimes N$ by $\Psi(z) = vzv^*$. Using the notation of Theorem 2.6 we get that $\mu(\Psi(z)) = (\iota \otimes \Psi)\,\gamma(z)$ for all $z \in B(H) \otimes N$. So the action μ of (M, Δ) on $B(H) \otimes N$ is isomorphic with the action γ, which is integrable because it is isomorphic with the bidual action $\hat{\hat{\alpha}}$. Hence μ is integrable, and so α is integrable.

Fix now a n.s.f. weight θ on N and represent N on the GNS-space of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction. Let U be the canonical implementation of α. Let $N_2 = J_\theta (N^\alpha)' J_\theta$ be the basic construction from $N^\alpha \subset N$ and let $\rho: M_\alpha \ltimes N \to N_2$ be the $*$-homomorphism from Theorem 5.3. Then define $w = (\hat{J} \otimes J_\theta)\, v(\hat{J} \otimes J_\theta)$ and define

$$\eta: N_2 \to B(H \otimes K): \eta(z) = Uw^*(1 \otimes z)\, wU^* \qquad \text{for all} \quad z \in N_2.$$

Because $w \in B(H) \otimes N'$ we have

$$\eta(x) = U(1 \otimes x)\, U^* = \alpha(x)$$

for all $x \in N$. Further we have $(\iota \otimes \alpha)(v) = v_{13} V^*_{12}$ and so $U_{23} v_{13} U^*_{23} = v_{13} V^*_{12}$. Putting $\hat{J} \otimes \hat{J} \otimes J_\theta$ around this equation and using that $V = (\hat{J} \otimes \hat{J}) \Sigma W^* \Sigma (\hat{J} \otimes \hat{J})$ (see [21, 2.15]), we get

$$U^*_{23} w_{13} U_{23} = w_{13} (\Sigma W \Sigma)_{12}.$$

Flipping the first two legs of this equation and rewriting it we get

$$w^*_{23} U^*_{13} w_{23} = W_{12} U^*_{13}.$$

From this it follows that

$$\begin{aligned} U_{23} w^*_{23} U^*_{13} w_{23} U^*_{23} &= U_{23} W_{12} U^*_{13} U^*_{23} \\ &= U_{23} W_{12} (\Delta \otimes \iota)(U^*) = U_{23} U^*_{23} W_{12} = W_{12}. \end{aligned}$$

Then we get for all $\omega \in M_*$ that

$$\eta((\omega \otimes \iota)(U^*)) = (\omega \otimes \iota \otimes \iota)(U_{23} w^*_{23} U^*_{13} w_{23} U^*_{23}) = (\omega \otimes \iota)(W) \otimes 1.$$

Hence we may conclude that $\eta \circ \rho = \iota$ and so ρ is faithful. ∎

6. MINIMAL ACTIONS AND OUTER ACTIONS

In Definition 5.5 we already defined the notion of an outer action. In the literature one usually encounters the notion of outer action when dealing with discrete group actions and one encounters the notion of minimal action when dealing with compact group actions. In this section we will prove how both notions can be linked in a locally compact quantum group setting. We will also prove a generalization of the main theorem of Yamanouchi, [33]: when working on separable Hilbert spaces, we prove that every integrable outer action with infinite fixed point algebra is a dual action.

The following definition appears in [15, 4.3] when dealing with actions of compact Kac algebras.

DEFINITION 6.1. An action α of (M, Δ) on N is called minimal when

$$N \cap (N^\alpha)' = \mathbb{C} \qquad \text{and} \qquad \{(\iota \otimes \omega)\,\alpha(x) \mid \omega \in N_*, x \in N\}'' = M.$$

We will prove the following result.

PROPOSITION 6.2. *Let α be an action of (M, Δ) on N.*

- *If α is minimal, then α is outer.*
- *If α is outer and integrable, then α is minimal.*

Proof. Let α be minimal. Let $z \in (M_\alpha \ltimes N) \cap \alpha(N)'$. Then certainly $z \in (B(H) \otimes N) \cap (\mathbb{C} \otimes N^\alpha)'$ and hence $z \in B(H) \otimes \mathbb{C}$ by minimality. We now claim that for $x \in B(H)$ we have $x \otimes 1 \in M_\alpha \ltimes N$ if and only if $x \in \hat{M}$. Suppose $x \otimes 1 \in M_\alpha \ltimes N$. It is clear that for every $z \in M_\alpha \ltimes N$ we have $(\iota \otimes \alpha)(z) = V_{12} z_{13} V_{12}^*$. So we get $(x \otimes 1)\, V = V(x \otimes 1)$. From this it follows that $x \in \hat{M}$. So we may conclude that $z = x \otimes 1$, where $x \in \hat{M}$. Because $z \in \alpha(N)'$ we get that $(x \otimes 1)\,\alpha(y) = \alpha(y)(x \otimes 1)$ for all $y \in N$. By minimality we get $x \in M'$. But then $x \in M' \cap \hat{M} = \mathbb{C}$ and so $z \in \mathbb{C}$. Hence α is outer.

Let now α be outer and integrable. Choose a n.s.f. weight θ on N and represent N on the GNS-space of θ. Let J_θ denote the modular conjugation of θ and let $N_2 = J_\theta (N^\alpha)' J_\theta$ be the basic construction from $N^\alpha \subset N$. Let ρ be the $*$-homomorphism given in Theorem 5.3. Then ρ is faithful because $M_\alpha \ltimes N$ is a factor. Because ρ is an isomorphism we get $N_2 \cap N' = \mathbb{C}$ and so

$$N \cap (N^\alpha)' = J_\theta (N_2 \cap N')\, J_\theta = \mathbb{C}.$$

Next we claim that $(\alpha(N) \cup \mathbb{C} \otimes N')'' = M \otimes B(K)$. Because, by Theorem 2.6, $B(H) \otimes N = (M_\alpha \ltimes N \cup M \otimes \mathbb{C})''$, we get

$$\begin{aligned} B(H) \otimes (\alpha(N) \cup \mathbb{C} \otimes N')'' &= ((\iota \otimes \alpha)(B(H) \otimes N) \cup \mathbb{C} \otimes \mathbb{C} \otimes N')'' \\ &= ((\iota \otimes \alpha)(M_\alpha \ltimes N) \cup M \otimes \mathbb{C} \otimes N')'' \\ &= V_{12}((M_\alpha \ltimes N)_{13} \cup V^*(M \otimes \mathbb{C})\, V \otimes N')''\, V_{12}^*. \end{aligned}$$

When $\tilde{J}$ denotes the modular conjugation of the dual weight $\tilde{\theta}$, we already observed in the proof of Proposition 5.9 that $B(H) \otimes N = \tilde{J}\alpha(N)'\,\tilde{J}$. Then the outerness of α implies that $B(H) \otimes N \cap (M_\alpha \ltimes N)' = \mathbb{C}$ and so

$$(\mathbb{C} \otimes N' \cup M_\alpha \ltimes N)'' = B(H) \otimes B(K).$$

Then we may conclude from the previous computation that

$$\begin{aligned}
&B(H) \otimes (\alpha(N) \cup \mathbb{C} \otimes N')'' \\
&\quad = V_{12}(B(H) \otimes \mathbb{C} \otimes B(K) \cup V^*(M \otimes \mathbb{C})\, V \otimes \mathbb{C})''\, V_{12}^* \\
&\quad = (V(B(H) \otimes \mathbb{C})\, V^* \otimes B(K) \cup M \otimes \mathbb{C} \otimes \mathbb{C})'' \\
&\quad = (\varDelta(M) \otimes B(K) \cup (\hat{M} \cup M) \otimes \mathbb{C} \otimes \mathbb{C})'' \\
&\quad = B(H) \otimes M \otimes B(K),
\end{aligned}$$

where we have used that $V \in \hat{M}' \otimes M$, $(\hat{M} \cup M)'' = B(H)$ and $(\varDelta(M) \cup B(H) \otimes \mathbb{C})'' = B(H) \otimes M$. Then our claim follows and hence it is clear that

$$\{(\iota \otimes \omega)\, \alpha(x) \mid \omega \in N_*, x \in N\}'' = M.$$

So α is minimal. ∎

We will now give an example of an outer action which is not minimal.

COUNTEREXAMPLE 6.3. There exists an action α of $\mathbb{Z}$ on a II_1-factor N such that α is outer and $N^\alpha = \mathbb{C}$. Then α is clearly not minimal, and neither can $\mathbb{C} \otimes N^\alpha \subset \alpha(N) \subset M_\alpha \ltimes N$ be the basic construction.

Proof. Let G be the free group with a countably infinite number of generators $\{a_n \mid n \in \mathbb{Z}\}$. It is well known that the free group factor $N = \mathscr{L}(G)$ is a II_1-factor. Let β be the automorphism of G satisfying $\beta(a_n) = a_{n+1}$ for all $n \in \mathbb{Z}$. Let α be the automorphism of N satisfying $\alpha(\lambda_g) = \lambda_{\beta(g)}$ for all $g \in G$. Define the automorphism group $(\alpha_n)_{n \in \mathbb{Z}}$ in the usual way by $\alpha_n = \alpha^n$ for all $n \in \mathbb{Z}$. It is easy to verify that α is a free action and hence α is outer (see [16, Def. 1.4.2 and Prop. 1.4.4]). Further it is easy to check that $N^\alpha = \mathbb{C}$. ∎

We conclude this section with a generalization of the main theorem of Yamanouchi [33]. It is remarkable that the proof of our result is much more easy then Yamanouchi's proof. In [33] the following result is proved for minimal actions of compact Kac algebras, which are automatically integrable because the Haar weight is finite.

PROPOSITION 6.4. *Let α be an action of $(M, \varDelta)$ on N. Suppose that both M and N are σ-finite von Neumann algebras (i.e. with separable preduals). If the action α is minimal and integrable and if N^α is infinite, then α is a dual action.*

Proof. Consider the action β of (M, Δ) on $\tilde{N} = B(H) \otimes N \otimes M_2(\mathbb{C})$ given by

$$\beta \begin{pmatrix} x & y \\ x & r \end{pmatrix} = \begin{pmatrix} \mu(x) & \mu(g)\, \tilde{V}^* \\ \tilde{V}\mu(z) & \gamma(r) \end{pmatrix},$$

for $x, y, z, r \in B(H) \otimes N$. Here we used the notations of Theorem 2.6: $\mu(x) = (\sigma \otimes \iota)(\iota \otimes \alpha)(x)$, $\gamma(r) = \tilde{V}\mu(r)\, \tilde{V}^*$ and $\tilde{V} = \Sigma V^* \Sigma \otimes 1$. Let us define now

$$\mathscr{J} = \{ x \in B(H) \otimes N \mid (\iota \otimes \alpha)(x) = x_{13} V_{12}^* \}.$$

Using matrix notation and referring to Theorem 2.6 and 2.7, it is then clear that $x \in \tilde{N}^\beta$ if and only if $x_{11} \in B(H) \otimes N^\alpha$, $x_{22} \in M_\alpha \ltimes N$ and $x_{12}, x_{21}^* \in \mathscr{J}$.

Choose a n.s.f. weight θ on N and represent N on the GNS-space of θ such that $(K, \iota, \Lambda_\theta)$ is a GNS-construction. Then we fix $z \in \mathscr{N}_{(\psi \otimes \iota)\alpha}$ and $\xi \in H$ and we claim that the element $x \in B(H \otimes K)$ defined by

$$x := (\Gamma \otimes \iota)\, \alpha(z)(\theta_\xi^* \otimes 1)$$

belongs to $\mathscr{J}^*$. Here we used the notation $\Gamma \otimes \iota$ introduced in the second part of the proof of Theorem 5.3. To prove our claim we observe that for all $b \in \mathscr{N}_\psi$, $y \in \mathscr{N}_\theta$ and $\eta \in H$

$$\begin{aligned} (\theta_{\Gamma(b)}^* \otimes 1)\, x(\eta \otimes \Lambda_\theta(y)) &= \langle \eta, \xi \rangle (\theta_{\Gamma(b)}^* \otimes 1)(\Gamma \otimes \Lambda_\theta)(\alpha(z)(1 \otimes y)) \\ &= \langle \eta, \xi \rangle (\psi \otimes \iota)((b^* \otimes 1)\, \alpha(z))\, \Lambda_\theta(y). \end{aligned}$$

We can conclude that

$$(\omega_{\eta, \Gamma(b)} \otimes \iota)(x) = \langle \eta, \xi \rangle (\psi \otimes \iota)((b^* \otimes 1)\, \alpha(z)).$$

So $x \in B(H) \otimes N$ and for all $\eta \in H$, $b \in \mathscr{N}_\psi$ and $\omega \in N_*$ we have

$$\begin{aligned} (\omega_{\eta, \Gamma(b)} \otimes \iota \otimes \omega)(\iota \otimes \alpha)(x) &= (\iota \otimes \omega)\, \alpha(\langle \eta, \xi \rangle (\psi \otimes \iota)((b^* \otimes 1)\, \alpha(z))) \\ &= \langle \eta, \xi \rangle (\psi \otimes \iota)((b^* \otimes 1)\, \Delta((\iota \otimes \omega)\, \alpha(z))) \\ &= \langle \eta, \xi \rangle (\omega_{\Gamma((\iota \otimes \omega)\, \alpha(z)), \Gamma(b)} \otimes \iota)(V). \end{aligned}$$

Next we observe that for all $y \in \mathscr{N}_\psi$

$$\begin{aligned} \langle \langle \eta, \xi \rangle\, \Gamma((\iota \otimes \omega)\, \alpha(z)), \Gamma(y) \rangle &= \langle \eta, \xi \rangle\, \omega(\psi \otimes \iota)((y^* \otimes 1)\, \alpha(z)) \\ &= \omega((\omega_{\eta, \Gamma(y)} \otimes \iota)(x)) \\ &= \langle (\iota \otimes \omega)(x)\, \eta, \Gamma(y) \rangle. \end{aligned}$$

Inserting this in the computation above we get that

$$(\omega_{\eta,\Gamma(b)} \otimes \iota \otimes \omega)(\iota \otimes \alpha)(x) = (\omega_{(\iota \otimes \omega)(x)\eta,\Gamma(b)} \otimes \iota)(V)$$
$$= (\omega_{\eta,\Gamma(b)} \otimes \iota \otimes \omega)(V_{12} x_{13}).$$

Then it follows that $x \in \mathcal{J}^*$.

So we see that $\mathcal{J} \neq \{0\}$. Because α is minimal we also have that α is outer by Proposition 6.2. In particular $M_\alpha \ltimes N$ is a factor. Also N^α is a factor. Because $\mathcal{J} \neq \{0\}$ we then get immediately that $\tilde{N}^\beta$ is a factor. Because M is supposed to be σ-finite, the Hilbert space H is separable. So $\tilde{N}^\beta$ is σ-finite. Denoting with e_{ij} the matrix units in $M_2(\mathbb{C})$ we see that the projections $1 \otimes e_{11}$ and $1 \otimes e_{22}$ both belong to $\tilde{N}^\beta$. Because $\mathbb{C} \otimes N^\alpha \subset M_\alpha \ltimes N$ both projections are infinite. Hence they are equivalent in the σ-finite factor $\tilde{N}^\beta$. Take $w \in \tilde{N}^\beta$ such that $w^* w = 1 \otimes e_{22}$ and $ww^* = 1 \otimes e_{11}$. Then there exists a unitary $v \in \mathcal{J}$ such that $w = v \otimes e_{12}$.

Now we can consider the isomorphism

$$\Psi \colon B(H) \otimes N \to B(H) \otimes N \colon \Psi(z) = v^* z v.$$

It is easy to check that $(\iota \otimes \Psi)\,\mu(z) = \gamma(\Psi(z))$ for all $z \in B(H) \otimes N$. So the actions μ and γ are isomorphic. Because γ is isomorphic to the bidual action $\hat{\hat{\alpha}}$ by Theorem 2.6, we get that μ is a dual action. Because N^α is properly infinite and because H is a separable Hilbert space we get that the action α on N is isomorphic with the action μ on $B(H) \otimes N$. So α is a dual action. ∎

7. APPENDIX

In this appendix we collect four technical results which do not have anything to do with actions. The first three results are general results on locally compact quantum groups and the last one deals with n.s.f. weights on a von Neumann algebra. We will use freely the notations introduced in the introduction.

PROPOSITION 7.1. *Let (M, Δ) be a locally compact quantum group. For every $\xi \in H$ and $b \in \mathcal{T}_\varphi$ we have*

$$\lambda(\omega_{\xi,\Lambda(b)}) \in \mathcal{N}_{\hat{\varphi}} \qquad \text{and} \qquad \hat{\Lambda}(\lambda(\omega_{\xi,\Lambda(b)})) = J\sigma_{i/2}(b)\,J\xi.$$

Moreover $\operatorname{span}\{\lambda(\omega_{\xi,\Lambda(b)}) \mid \xi \in H, b \in \mathcal{T}_\varphi\}$ *is a σ-strong*-norm core for $\hat{\Lambda}$.*

Proof. The first statement follows easily from the definition of $\hat{\varphi}$. Let $x \in \mathcal{N}_\varphi$, then

$$\omega_{\xi, \Lambda(b)}(x^*) = \langle x^*\xi, \Lambda(b) \rangle = \langle J\sigma_{i/2}(b)\, J\xi, \Lambda(x) \rangle.$$

So we get the first statement. To prove the second one we define

$$\mathcal{L} = \{ a \in \mathcal{N}_\varphi \mid \text{there exists } \omega \in M_* \text{ such that } \omega(x) = \varphi(xa) \text{ for all } x \in \mathcal{N}_\varphi^* \}.$$

It is clear that for $a \in \mathcal{L}$ such a $\omega \in M_*$ is necessarily unique. We denote it with $a\varphi$. Then for every $a \in \mathcal{L}$ we have $\lambda(a\varphi) \in \mathcal{N}_{\hat{\varphi}}$ and $\hat{\Lambda}(\lambda(a\varphi)) = \Lambda(a)$. Define $\mathcal{D}_0 = \{ \lambda(a\varphi) \mid a \in \mathcal{L} \}$. We claim that $\mathcal{D}_0$ is a σ-strong*-norm core for $\hat{\Lambda}$.

Denote with $\mathcal{D}$ the domain of the σ-strong*-norm closure of the restriction of $\hat{\Lambda}$ to $\mathcal{D}_0$.

Let $a \in \mathcal{L}$ and $t \in \mathbb{R}$. Define $b = \tau_t(a)\, \delta^{-it}$. Then $b \in \mathcal{N}_\varphi$ and for all $x \in \mathcal{N}_\varphi^*$ we have

$$\begin{aligned}\varphi(xb) &= \varphi(x\tau_t(a)\, \delta^{-it}) = v^t \varphi(\delta^{-it} x \tau_t(a)) = \varphi(\delta^{-it}\tau_{-t}(x)\, a) \\ &= (a\varphi)(\delta^{-it}\tau_{-t}(x)) = (\rho_t(a\varphi))(x)\end{aligned}$$

where we used the notation of [19, 8.7]. So $b \in \mathcal{L}$ and $b\varphi = \rho_t(a\varphi)$. Hence

$$\hat{\sigma}_t(\lambda(a\varphi)) = \lambda(\rho_t(a\varphi)) = \lambda(b\varphi) \in \mathcal{D}_0.$$

So we get that $\mathcal{D}_0$ is invariant under $\hat{\sigma}_t$. Then it is easy to conclude that $\mathcal{D}$ is invariant under $\hat{\sigma}_t$ for all $t \in \mathbb{R}$.

Let now $\omega \in M_*$ and suppose that there exists a $\mu \in M_*$ such that $\mu(x) = \omega(S^{-1}(x))$ for all $x \in \mathcal{D}(S^{-1})$. Let $a \in \mathcal{L}$. Define $b = (\mu \otimes \iota)\, \Delta(a)$. Then $b \in \mathcal{N}_\varphi$ and for all $x \in \mathcal{N}_\varphi^*$ we have

$$\begin{aligned}\varphi(xb) &= \varphi(x(\mu \otimes \iota)\, \Delta(a)) = \mu((\iota \otimes \varphi)((1 \otimes x)\, \Delta(a))) \\ &= \omega((\iota \otimes \varphi)(\Delta(x)(1 \otimes a))) = \varphi((\omega \otimes \iota)\, \Delta(x)\, a) = (\omega \otimes a\varphi)\, \Delta(x).\end{aligned}$$

So we see that $b \in \mathcal{L}$ and $b\varphi = (\omega \otimes a\varphi)\, \Delta$. Then we may conclude that

$$\lambda(\omega)\, \lambda(a\varphi) = \lambda(b\varphi) \in \mathcal{D}_0.$$

Because such elements $\lambda(\omega)$ form a σ-strong* dense subset of $\hat{M}$ it is easy to conclude that $\mathcal{D}$ is a left ideal in $\hat{M}$.

Because $\mathcal{D}$ is a σ-strong* dense left ideal of $\hat{M}$, invariant under $\hat{\sigma}$ and because $\mathcal{D} \subset \mathcal{N}_{\hat{\varphi}}$, we may conclude that $\mathcal{D}$ is a σ-strong*-norm core for $\hat{\Lambda}$. But then $\mathcal{D} = \mathcal{N}_{\hat{\varphi}}$ and we have proven our claim.

Then it follows easily that also

$$\operatorname{span}\{\lambda(ab\varphi)\mid a\in\mathcal{L},\, b\in\mathcal{T}_\varphi\}$$

is a σ-strong*-norm core for $\hat{\Lambda}$. This last space equals

$$\operatorname{span}\{\lambda(\omega_{\Lambda(a),\,\Lambda(b)})\mid a\in\mathcal{L},\, b\in\mathcal{T}_\varphi\}$$

and so the proposition is proven. ∎

For completeness we also include the following easy result.

PROPOSITION 7.2. *Let (M, Δ) be a locally compact quantum group. For every $a\in\mathcal{N}_\varphi$, $\xi\in\mathcal{D}(\delta^{1/2})$ and $\eta\in H$ we have $(\iota\otimes\omega_{\xi,\eta})\,\Delta(a)\in\mathcal{N}_\varphi$ and*

$$\Lambda((\iota\otimes\omega_{\xi,\eta})\,\Delta(a))=(\iota\otimes\omega_{\delta^{1/2}\xi,\eta})(V)\,\Lambda(a).$$

Proof. Let (e_n) be the sequence of operators defined in the proof of [19, 7.6]. Because $\Delta(\delta)=\delta\otimes\delta$ it is clear that

$$((\iota\otimes\omega_{\xi,\eta})\,\Delta(ae_n))\,\delta^{-1/2}\subset(\iota\otimes\omega_{\delta^{1/2}\xi,\eta})\,\Delta(a(\delta^{-1/2}e_n)).$$

Because $a(\delta^{-1/2}e_n)\in\mathcal{N}_\psi$ we have $(\iota\otimes\omega_{\delta^{1/2}\xi,\eta})\,\Delta(a(\delta^{-1/2}e_n))\in\mathcal{N}_\psi$. We know that $\varphi=\psi_{\delta^{-1}}$, so that $(\iota\otimes\omega_{\xi,\eta})\,\Delta(ae_n)\in\mathcal{N}_\varphi$ and

$$\begin{aligned}\Lambda((\iota\otimes\omega_{\xi,\eta})\,\Delta(ae_n))&=\Gamma((\iota\otimes\omega_{\delta^{1/2}\xi,\eta})\,\Delta(a(\delta^{-1/2}e_n)))\\&=(\iota\otimes\omega_{\delta^{1/2}\xi,\eta})(V)\,\Gamma(a(\delta^{-1/2}e_n))\\&=(\iota\otimes\omega_{\delta^{1/2}\xi,\eta})(V)\,\Lambda(ae_n).\end{aligned}$$

Because Λ is σ-strong*-norm closed, the conclusion follows. ∎

We also need the following technical result.

PROPOSITION 7.3. *Let (M, Δ) be a locally compact quantum group and let $x\in M$. Suppose that there exists a vector $\eta\in H$ such that*

$$\omega(x^*)=\langle\xi(\omega),\eta\rangle$$

for all $\omega\in\mathcal{I}$. Then $x\in\mathcal{N}_\varphi$ and $\Lambda(x)=\eta$.

Proof. Let $\omega\in\mathcal{I}$ and $y\in\mathcal{T}_\varphi$. Then we have for all $a\in\mathcal{N}_\varphi$ that

$$(\omega y)(a^*)=\omega((ay^*)^*)=\langle\xi(\omega),\Lambda(ay^*)\rangle=\langle J\sigma_{i/2}(y^*)\,J\xi(\omega),\Lambda(a)\rangle.$$

So we get that $\omega y \in \mathcal{I}$ and $\xi(\omega y) = J\sigma_{i/2}(y^*) J\xi(\omega)$. Take now a net (e_α) in $\mathcal{T}_\varphi$ such that $\sigma_z(e_\alpha) \to 1$ in the σ-strong* topology for all $z \in \mathbb{C}$. Then we have for all $\omega \in \mathcal{I}$

$$\langle \xi(\omega), \Lambda(xe_\alpha) \rangle = \omega(e_\alpha^* x^*) = (\omega e_\alpha^*)(x^*) = \langle \xi(\omega e_\alpha^*), \eta \rangle$$

$$= \langle \xi(\omega), J\sigma_{i/2}(e_\alpha)^* J\eta \rangle.$$

Hence $\Lambda(xe_\alpha) = J\sigma_{i/2}(e_\alpha)^* J\eta$ for all α. Because Λ is σ-strong*-norm closed we get $x \in \mathcal{N}_\varphi$ and $\Lambda(x) = \eta$. ∎

The following result is probably well known, but we could not find it in the literature.

Proposition 7.4. *Let θ be a n.s.f. weight on a von Neumann algebra N with GNS-construction (H, π, Λ). Suppose that*

- *$\mathcal{D}$ is a weakly dense left ideal in N with $\mathcal{D} \subset \mathcal{N}_\theta$.*
- *K is a Hilbert space and $\Lambda_0 \colon \mathcal{D} \to K$ is a linear map such that $\Lambda_0(\mathcal{D})$ is dense in K.*
- *π_0 is normal representation of N on K such that $\pi_0(x) \Lambda_0(y) = \Lambda_0(xy)$ for all $x \in N$ and $y \in \mathcal{D}$.*
- *$\mathcal{V}$ is an isometry from K to H such that $\mathcal{V} \Lambda_0(x) = \Lambda(x)$ for all $x \in \mathcal{D}$.*
- *Λ_0 is σ-strong*-norm closed.*

Then there exists a unique n.s.f. weight μ on N such that $\mathcal{N}_\mu = \mathcal{D}$ and (K, π_0, Λ_0) is a GNS-construction for μ. In particular μ is a restriction of θ, which means that for every $x \in \mathcal{M}_\mu^+$ we have $x \in \mathcal{M}_\theta^+$ and $\mu(x) = \theta(x)$.

Proof. Because $\mathcal{V}$ is an isometry, Λ_0 is injective. Define $\mathcal{U} = \Lambda_0(\mathcal{D} \cap \mathcal{D}^*)$. Then $\mathcal{U}$ is a dense subspace of K. We make $\mathcal{U}$ into a $*$-algebra by using Λ_0 and the $*$-algebra structure on $\mathcal{D} \cap \mathcal{D}^*$. We claim that $\mathcal{U}$ is a left Hilbert algebra. The only non-trivial point is to prove that the map $\Lambda_0(x) \mapsto \Lambda_0(x^*)$ for $x \in \mathcal{D} \cap \mathcal{D}^*$ is closable. But, suppose that (x_n) is a sequence in $\mathcal{D} \cap \mathcal{D}^*$ such that $\Lambda_0(x_n) \to 0$ and $\Lambda_0(x_n^*) \to \xi \in K$. Applying $\mathcal{V}$ we get $\Lambda(x_n) \to 0$ and $\Lambda(x_n^*) \to \mathcal{V}\xi$. Because θ is a n.s.f. weight we get that $\mathcal{V}\xi = 0$ and so $\xi = 0$. This gives our claim.

It is clear that the von Neumann algebra generated by the left Hilbert algebra $\mathcal{U}$ is $\pi_0(N)$. Because Λ_0 is injective we have that π_0 is injective. So the n.s.f. weight on $\pi_0(N)$ which is canonically associated with $\mathcal{U}$ can be composed with π_0 to obtain a n.s.f. weight μ on N. Let (K, π_0, Λ_1) be the canonically associated GNS-construction.

Then, by definition of μ, every $x \in \mathscr{D} \cap \mathscr{D}^*$ will belong to $\mathscr{N}_\mu$ and $\Lambda_1(x) = \Lambda_0(x)$. Let now $x \in \mathscr{D}$. Take a net (e_α) in $\mathscr{D}$ such that $e_\alpha \to 1$ σ-strong*. Then we have $e_\alpha^* x \to x$ σ-strong*, $e_\alpha^* x \in \mathscr{D} \cap \mathscr{D}^*$ and

$$\Lambda_1(e_\alpha^* x) = \Lambda_0(e_\alpha^* x) = \pi_0(e_\alpha^*)\, \Lambda_0(x) \to \Lambda_0(x).$$

Because Λ_1 is σ-strong*-norm closed we get $x \in \mathscr{N}_\mu$ and $\Lambda_1(x) = \Lambda_0(x)$.

Conversely, suppose $x \in \mathscr{N}_\mu$. By the main theorem of [12] thee exists a net (x_α) in $\mathscr{D} \cap \mathscr{D}^*$ such that $\|x_\alpha\| \leqslant \|x\|$ for all α, $x_\alpha \to x$ σ-strong* and $\Lambda_1(x_\alpha) \to \Lambda_1(x)$ in norm. But $\Lambda_1(x_\alpha) = \Lambda_0(x_\alpha)$ for every α. Because Λ_0 is σ-strong*-norm closed we get that $x \in \mathscr{D}$. This concludes our proof. ∎

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