JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **198,** 227–236 (1996)
ARTICLE NO. 0078

C^*-Algebras That Are I-Rings

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Submitted by John Horváth

Received November 30, 1994

A ring $\mathfrak{R}$ is said to be an I-ring if and only if for every non-nilpotent element a in $\mathfrak{R}$, there exists $b \in \mathfrak{R}$, $b \neq 0$, such that $bab = b$. Equivalently, $\mathfrak{R}$ has this property if and only if each non-nil right ideal contains an idempotent. I-rings apparently first occurred in a paper of Köthe [14], who was then searching for a non-commutative substitute of the finiteness condition for rings which today usually is contained in the expression "noetherian." (It was later observed by Kaplansky [13] that an I-ring either contains an infinite number of mutually orthogonal idempotents or satisfies the descending chain condition modulo its radical.) In his paper, Köthe shows among other things that each minimal non-nil right ideal of an arbitrary ring contains an idempotent and that I-rings contain what today is occasionally called a Köthe-radical, i.e., a unique maximal bilateral ideal, which contains all left and right nil-ideals of the ring in question. I-rings appear next in the literature as a weak form of von Neumann's regular rings [11–13] and have then systematically been investigated by Levitzki [17, 18]. (For an investigation of I-rings with involution see [15, 16].)

The structure of regular rings within the category of Banach algebras is fairly well understood: All of them are finite dimensional [12]. The situa-

0022-247X/96 $18.00

tion for I-rings within this class seems to be more difficult to understand and only much more restrictive conditions have been investigated [19].

The idea behind the following is to connect I-rings to smoothness of the unit sphere in case that the ring in question is a C^*-algebra. More precisely, we are going to show that a C^*-algebra is an I-ring if and only if its norm is strongly subdifferentiable (see the definition below) on a dense set (Theorem 3). Part of this result extends an old characterization of commutative C^*-algebras that are I-rings due to Kaplansky [12, Theorem]. (Kaplansky's interest in such a characterization was to show that there is an abundance of Banach algebras which are I-rings but not regular. In fact, the proof for the commutative version is a matter of a few lines.) Another aspect of the present result is that there is no topological version of I-rings within the frame of C^*-algebras: A C^*-algebra which has the property that all *closed* left ideals contain an idempotent already is an I-ring. It is worth mentioning that the class of C^*-algebra thus characterized comprises the so-called spectral C^*-algebras [19], a fairly wide family of algebras including all von Neumann algebras (actually all Rickart C^*-algebras), AF-algebras, and C^*-algebras with real rank zero [19, 3]. Furthermore, each C^*-algebra is the factor of another C^*-algebra which is an I-ring and satisfies a finiteness condition for matrix units [19, Proposition 2.18].

For the sake of completeness we include a similar characterization for C^*-algebras with everywhere strongly subdifferentiable norm (Theorem 2): A C^*-algebra is sharing this property if and only if it is a modular annihilator algebra.

All of these results depend on a characterization of strong subdifferentiability of the norm of C^*-algebras in algebraic terms (Theorem 1). This latter result parallels similar characterizations of points of Fréchet-differentiability, which had been obtained in [21, 22]. Nevertheless, in the framework of C^*-algebras, strongly subdifferentiability of the norm happens to be much weaker than Fréchet-differentiability. Note that there are von Neumann algebras whose norm is Fréchet-differentiable at no point but strongly subdifferentiable on a dense set.

It is now time to recall the geometrical definition which will be crucial in the sequel. Following [5] we say that the norm of a Banach space X is strongly subdifferentiable at a point $x \in X$ when the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for y in the unit ball of X. Note that this condition is trivially satisfied for $x = 0$ and that it holds for ρx ($\rho > 0$) whenever it holds for x, so we shall mainly consider strong subdifferentiability of the norm at points in the unit sphere S_X. For $x \in S_X$ it is well known that

$$\lim_{t \to 0^+} \frac{\|x + ty\| - 1}{t} = \max\{\operatorname{Re} \varphi(y) \colon \varphi \in D(x)\},$$

where $D(x)$ (or $D(x, X)$ if it is necessary to be more precise) is the set of normalized support functionals for the unit ball at x, that is,

$$D(x) = \{\varphi \in S_{X^*} \colon \varphi(x) = 1\}.$$

It was shown by D. Gregory [9, Corollary 4.4] (see also [5, Theorem 1.2]) that the norm of X is strongly subdifferentiable at $x \in S_X$ if and only if the face $D(x)$ is strongly exposed by x, that is, the distance $d(\varphi_n, D(x))$ tends to zero for any sequence $\{\varphi_n\}$ in the dual unit ball such that $\varphi_n(x) \to 1$. Norm-to-norm upper semicontinuity of the set-valued mapping D in the sense of Giles, Gregory, and Sims [6] is another equivalent condition. We should also mention that a continuous convex function defined on an open subset of X is differentiable on a dense G_δ if it is strongly subdifferentiable on another dense G_δ (D. Preiss, unpublished) and that G. Godefroy ([7], see also [4]) has shown that Banach spaces with everywhere strongly subdifferentiable norm are Asplund spaces.

Our first goal will be the announced algebraic characterization of strong subdifferentiability points for the norm of a C^*-algebra. Let us fix some notation: In what follows, $\mathfrak{A}$ will always denote a C^*-algebra. Recall that the (topological) bidual $\mathfrak{A}^{**}$ of a C^*-algebra is itself a C^*-algebra and that this makes it possible to equip $\mathfrak{A}^*$ in a natural way with an $\mathfrak{A}^{**}$-bimodule structure via

$$fu(a) := ua(f) \qquad \text{and} \qquad uf(a) := au(f),$$

where $f \in \mathfrak{A}^*$ and $u \in \mathfrak{A}^{**}$. Note that

$$\|fu\| = \sup_{a \in B_{\mathfrak{A}}} ua(f) \le \|u\| \, \|f\|$$

and similarly $\|uf\| \le \|u\| \, \|f\|$. For unexplained notation and standard results on C^*-algebras which are used without comment we refer to the book of Pedersen [20]. We will need the following lemma.

LEMMA 1 [21, Lemma 2.7]. *Let $\{\varphi_n\}$ be a sequence in $\mathfrak{A}^*$, with $\|\varphi_n\| \le 1$, for all n. Suppose p, q are projections in $\mathfrak{A}^{**}$ such that $\|p\varphi_n q\| \to 1$. Then $\|p\varphi_n q - \varphi_n\| \to 0$.*

Consider the trivial fact that strong subdifferentiability points are preserved under isometric linear bijections. This applies, in particular, to multiplication by unitary elements in a unitary C^*-algebra. Our next lemma shows that multiplication by a partial isometry still preserves some points of strong subdifferentiability.

LEMMA 2. *Strong subdifferentiability of the norm passes from* $a \in S_{\mathfrak{A}}$ *to* va *for any partial isometry* $v \in \mathfrak{A}^{**}$ *such that* $va \in \mathfrak{A}$ *and* $v^*va = a$.

Proof. Given a sequence $\{\varphi_n\}$ in $S_{\mathfrak{A}^*}$ with $\varphi_n(va) \to 1$, write $\varphi_n' = \varphi_n v$. Since $\varphi_n'(a) = \varphi_n(va) \to 1$ and $\|\varphi_n'\| \le 1$, the assumption yields a sequence $\{\psi_n'\}$ in $D(a)$ such that $\|\varphi_n' - \psi_n'\| \to 0$ and we write $\psi_n = \psi_n' v^*$. Note that $\|\psi_n\| \le 1$ and $\psi_n(va) = \psi_n'(v^*va) = \psi_n'(a) = 1$, so $\psi_n \in D(va)$ and we are left with showing that $\|\varphi_n - \psi_n\| \to 0$. Actually, we have

$$\|\varphi_n vv^* - \psi_n\| = \|\varphi_n' v^* - \psi_n' v^*\| \le \|\varphi_n' - \psi_n'\| \to 0,$$

and also $\|\varphi_n vv^* - \varphi_n\| \to 0$, for

$$(\varphi_n vv^*)(va) = \varphi_n(va) \to 1,$$

and Lemma 1 applies. ∎

Strong subdifferentiability at $x \in B_X$ implies this property for (the canonical image of) $x \in B_{X^{**}}$ [6, Corollary 2.1], and one might consequently dispense with the condition that $va \in \mathfrak{A}$ in the above lemma. (For our present purposes, the above form suffices.) It is on the other hand easy to see that the assumption $v^*va = a$ cannot be dropped.

LEMMA 3. *Let* p *be a projection in the* C^**-algebra* $\mathfrak{A}$. *Then the norm of* $\mathfrak{A}$ *is strongly subdifferentiable at* p.

Proof. The norm of a unital Banach algebra is always strongly subdifferentiable at the unit (see [1, Theorem 5.5], for example). Thus, the norm of the C^*-algebra $p\mathfrak{A}p$ is strongly subdifferentiable at p, but we want something better. Given a sequence $\{\varphi_n\}$ in $S_{\mathfrak{A}^*}$ with $\varphi_n(p) \to 1$ the above observation and the Hahn–Banach Theorem provide a sequence $\{\psi_n\}$ in $D(p, \mathfrak{A})$ such that $\varphi_n - \psi_n \to 0$ uniformly on the unit ball of $p\mathfrak{A}p$, equivalently, $\|p(\varphi_n - \psi_n)p\| \to 0$. Lemma 1 now gives $\|\varphi_n - p\psi_n p\| \to 0$, but $p\psi_n p \in D(p, \mathfrak{A})$ for all n. ∎

Therefore the norm of a C^*-algebra containing non-zero projections is strongly subdifferentiable at some non-zero points. The converse is also true as a by-product of our next result.

THEOREM 1. *Let* $\mathfrak{A}$ *be a* C^**-algebra and* $a \in S_{\mathfrak{A}}$. *The following assertions are equivalent.*

(i) *The norm of* $\mathfrak{A}$ *is strongly subdifferentiable at* a.

(ii) 1 *is an isolated point in the spectrum of* $|a|$.

(iii) *There exists a partial isometry* $v \in \mathfrak{A}$ *such that*

$$av^* = vv^* \qquad \text{and} \qquad \|a - v\| < 1.$$

Proof. (i) $\Rightarrow$ (ii). By using the polar decomposition and Lemma 2 we get from (i) that the norm of $\mathfrak{A}$ is strongly subdifferentiable at $|a|$. Let us denote by $\mathfrak{B}$ the C^*-subalgebra generated by $|a|$ and consider the elements of $\mathfrak{B}$ as continuous functions on the locally compact space $L = \mathrm{sp}(|a|) \setminus \{0\}$ vanishing at infinity, the function $|a|$ being the identity function on L. If we denote by δ_t the evaluation at the point $t \in L$, we clearly have $D(|a|, \mathfrak{B}) = \{\delta_1\}$. Arguing by contradiction, we may suppose that there is a sequence $\{t_n\}$ in $\mathrm{sp}(|a|)$ such that $\{t_n\}$ converges to 1 and $t_n < 1$ for all n. Since $\delta_{t_n}(|a|) \to 1$, and the sequence $\{\delta_{t_n}\}$ does not converge to δ_1 (even in the weak topology) we have that the norm of $\mathfrak{B}$ is not strongly subdifferentiable at $|a|$, the desired contradiction.

(ii) $\Rightarrow$ (iii). Use the continuous functional calculus to find a projection $p \in \mathfrak{A}$ satisfying

$$|a|p = p \qquad \text{and} \qquad \|\,|a| - p\| < 1. \tag{$*$}$$

If $a = u|a|$ is the polar decomposition of a, we claim that (iii) holds with $v = up$. In fact, since

$$v = u|a|p = ap,$$

we have $v \in \mathfrak{A}$ and

$$v^*v = pu^*ap = p|a|p = p,$$

hence v is a partial isometry. Moreover, $vv^* = upa^* = va^*$, that is, $av^* = vv^*$. Since $v^*a = p\,|a| = p$ we finally have

$$\begin{aligned} \|a - v\|^2 &= \|a^*a - v^*a - a^*v + v^*v\| \\ &= \|\,|a|^2 - p\| \\ &= \|\,|a|(|a| - p)\| < 1. \end{aligned}$$

(iii) $\Rightarrow$ (i). We first prove that $p = v^*v$ satisfies $(*)$. In fact,

$$p|a|^2p = v^*(av^*)^*(av^*)v = p,$$

that is, $p(1 - |a|^2)p = 0$ and it follows that $p|a| = |a|p = p$. On the other hand, since $\|a - v\| < 1$ and

$$p = p|a|^2 = (v^*vv^*)a = v^*a,$$

we get $\|\,|a| - p\|^2 = \|a^*a - p\| = \|a^*a - v^*a\| < 1$.

To prove (i), we show that the norm of $\mathfrak{A}$ is strongly subdifferentiable at $|a|$ and the result will follow from Lemma 2. By Lemma 3 and the dual characterization of strong subdifferentiability (see [5, Proposition 3.1]), it is enough to show that $D(|a|) = D(p)$. Since the C^*-subalgebra generated by $|a|$ and p is commutative, an obvious application of the Hahn–Banach Theorem reduces the problem to the commutative case, that is, we may assume that $\mathfrak{A} = C_0(L)$, for some locally compact space L. Then p is the characteristic function of a clopen set Ω, the relation between p and $|a|$ given by $(*)$ clearly implies that $\Omega = \{t \in L: |a|(t) = 1\}$, and the equality $D(p, C_0(L)) = D(|a|, C_0(L))$ follows. ▮

If the norm of $\mathfrak{A}$ is Fréchet-differentiable at a point a and $\mathfrak{B}$ is a larger C^*-algebra, then the norm of $\mathfrak{B}$ need not be Fréchet-differentiable at a but it is still strongly subdifferentiable, as shown by the following immediate consequence of the above theorem.

COROLLARY 1. *Let $\mathfrak{B}$ be a C^*-algebra, $\mathfrak{A}$ a C^*-subalgebra of $\mathfrak{B}$, and $a \in \mathfrak{A}$. If the norm of $\mathfrak{A}$ is strongly subdifferentiable at a, then the norm of $\mathfrak{B}$ is also strongly subdifferentiable at a.*

If K is a compact space, the norm of $C(K)$ is strongly subdifferentiable at $a \in S_{C(K)}$ if and only if $\sup\{|a(t)|: t \in K, |a(t)| < t\} < 1$. This elementary fact, covered by Theorem 1, also admits the following noncommutative version in terms of irreducible representations.

COROLLARY 2. *The norm of $\mathfrak{A}$ is strongly subdifferentiable at $a \in S_{\mathfrak{A}}$ if, and only if, there is a family F of irreducible representations of $\mathfrak{A}$, satisfying the following conditions.*

(i) $\|\pi(a)\| = 1$ *for all* $\pi \in F$ *and there exists* $\rho > 0$ *such that* $\|\pi(a)\| \le 1 - \rho$ *for any irreducible representation* π *not belonging to* F.

(ii) *For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $d(\varphi, D(\pi(a))) < \varepsilon$ *whenever* $\pi \in F$ *and* $\varphi \in B_{L(H_\pi)^*}$ *satisfy* $\mathrm{Re}(\varphi(\pi(a))) > 1 - \delta$.

Proof. $\mathfrak{A}$ is isometric to a C^*-subalgebra of the l_∞-sum $\mathfrak{B} = \oplus_\infty L(H_\pi)$, where π runs through the family of all irreducible representations of $\mathfrak{A}$. In view of the above corollary we can dispose of $\mathfrak{A}$ and consider a as an element in $\mathfrak{B}$. Then the result follows from the behavior of points of strong subdifferentiability in l_∞-sums of Banach spaces [5, Theorem 2.5]. ▮

We come now to a description of C^*-algebras with everywhere strongly subdifferentiable norm. Recall the following definition: A ring $\mathfrak{R}$ is a modular annihilator ring, if and only if

$$L(\mathfrak{M}) = \{r \in \mathfrak{R}: r\mathfrak{M} = 0\} \neq 0$$

for every maximal modular right ideal $\mathfrak{M}$ of $\mathfrak{R}$. (Note that a proper exchange of "left" and "right" in this definition doesn't lead to a new concept [23, Theorem 3.4].)

THEOREM 2. *For a C^*-algebra $\mathfrak{A}$, the following assertions are equivalent.*

(i) *The relative weak and weak-$*$ topologies agree on $S_{\mathfrak{A}^*}$.*

(ii) *The norm of $\mathfrak{A}$ is strongly subdifferentiable at every point.*

(iii) *For every normal element $a \in \mathfrak{A}$, zero is the only possible accumulation point in* $\operatorname{sp}(a)$.

(iv) *$\mathfrak{A}$ is a modular annihilator algebra.*

(v) *$\mathfrak{A}$ is a c_0-sum of algebras of compact operators on Hilbert space.*

Proof. (i) ⇒ (ii). Given any element a in the unit sphere of a Banach space, it is easy to show that for any weak-* neighbourhood of zero V in the dual space there is a $\delta > 0$ such that

$$\left.\begin{array}{c}\operatorname{Re}\varphi(a) > 1-\delta\\ \|\varphi\| = 1\end{array}\right\} \Rightarrow \varphi \in D(a) + V.$$

We can take for a any positive element in $S_{\mathfrak{A}}$ and our assumption (i) allows replacing "weak-*" with "weak" in the above statement. Then, arguing like in the proof of the assertion (i) ⇒ (ii) in Theorem 1 we get that 1 is an isolated point of $\operatorname{sp}(a)$ and (ii) follows.

(ii) ⇒ (iii). Let a be a normal element in $\mathfrak{A}$, and $\alpha \neq 0$ an accumulation point of $\operatorname{sp}(a)$. Consider the continuous function $f\colon \operatorname{sp}(a) \to \mathbb{C}$ given by $f(t) = |t|/(|t| + |t-\alpha|)$. Since 1 is an accumulation point of $\operatorname{sp}(f(a))$, the norm of $\mathfrak{A}$ is not strongly subdifferentiable at $f(a)$.

(iii) ⇒ (iv). This is a special case of [2, Théorème 3, p. 84].

(iv) ⇒ (v). The conclusion follows from results of B. Yood (every C^*-algebra which is a modular annihilator algebra is dual, i.e., every closed ideal is an annihilator ideal [23, Theorem 4.1]) and I. Kaplansky (every C^*-algebra is of the form announced in (v) [11, Theorem 8.3]).

(v) ⇒ (i). By [10, Proposition III.2.9] $\mathfrak{A}$ is an M-ideal of $\mathfrak{A}^{**}$ and (i) follows from the fact that for this class of spaces condition (i) always is satisfied [10, Corollary III.2.15]. ∎

Let us finally prove the main result of this paper, a geometric characterization of C^*-algebras that are I-rings. Recall that a projection $Q \in \mathfrak{A}^{**}$ is called open, if and only if there is an increasing net $\{a_\lambda\}$ of positive elements in $\mathfrak{A}$ that converges to Q in the weak*-topology. We will also make use of the fact that $Q \to \mathfrak{A}^{**}Q \cap \mathfrak{A}$ establishes a one-to-one correspondence of open projections and (closed) left ideals of $\mathfrak{A}$ (see [20, 3.10.7, 3.11.10] for details).

THEOREM 3. *For a C^*-algebra, the following assertions are equivalent.*

(i) *The norm of $\mathfrak{A}$ is strongly subdifferentiable on a dense subset of $\mathfrak{A}$.*

(ii) *For every $a \in S_{\mathfrak{A}}$ there exists $x \in S_{\mathfrak{A}}$ such that $\|x - a\| < 1$ and the norm of $\mathfrak{A}$ is strongly subdifferentiable at x.*

(iii) *$\mathfrak{A}$ is an I-ring.*

(iv) *Every non-zero closed left ideal of $\mathfrak{A}$ contains a non-zero projection.*

(v) *For every open projection Q in $\mathfrak{A}^{**}$, $Q \neq 0$, there is a projection $q \in \mathfrak{A}$, $q \neq 0$, such that $q \leq Q$.*

Proof. (i) ⇒ (ii). This is evident.

(ii) ⇒ (iii). For $a \in S_{\mathfrak{A}}$, let $x \in S_{\mathfrak{A}}$ be given by (ii). By Theorem 1, there is a partial isometry $v \in \mathfrak{A}$ such that

$$xv^* = vv^* \qquad \text{and} \qquad \|x - v\| < 1.$$

Since

$$\|av^* - vv^*\| = \|av^* - xv^*\| < 1,$$

the element $vv^*av^* = vv^*av^*vv^*$ is invertible in the unital C^*-algebra $vv^*\mathfrak{A}vv^*$ so there is an element $c \in \mathfrak{A}$, $c \neq 0$, such that $vv^*av^*cvv^* = vv^*$. Condition (iii) now follows with $b = v^*cvv^*$, which clearly is different from zero.

(iii) ⇒ (iv). A non-zero left ideal of a C^*-algebra cannot be a nil ideal. Thus, if (iii) holds, every non-zero (closed or not) left ideal of A contains a non-zero idempotent. The conclusion then follows from the fact that

$$p = [1 + (e - e^*)(e^* - e)]^{-1} e^*e$$

is a projection whenever e is idempotent (see, e.g., the proof of [8, Proposition 19.1] for details).

(iv) ⇒ (v). Let Q be a non-zero open projection in $\mathfrak{A}^{**}$. Then $\mathfrak{A}^{**}Q \cap \mathfrak{A}$ is a non-zero, closed left ideal of $\mathfrak{A}$ which will contain a non-zero projection p, and it is easy to see that $p \leq Q$.

(v) ⇒ (i). An appeal to Lemma 2 shows that it is sufficient to prove that any positive element $a \in \mathfrak{A}$ is the limit of a sequence of points of strong subdifferentiability $\{a_n\}$ such that $s(a)a_n = a_n$ for all natural numbers n, where $s(a)$ denotes the support of a. (This by definition is the smallest projection $p \in \mathfrak{A}^{**}$ with $ap = pa = a$. Equivalently, $a = pa$ is the polar decomposition of a.) Hence let $a \in S_{\mathfrak{A}}$, $a \geq 0$, be given and fix $\varepsilon > 0$. Denote by $\mathfrak{B}$ the C^*-subalgebra generated by a, and select a positive element $\hat{a} \in S_{\mathfrak{B}}$ as well as a projection $Q \in \mathfrak{B}^{\infty\infty}$, open for $\mathfrak{B}$ (and hence for $\mathfrak{A}$), so that

$$\|a - \hat{a}\| < \frac{\varepsilon}{2} \qquad \text{and} \qquad Q\hat{a} = Q.$$

Pick $q \in \mathfrak{A} \setminus \{0\}$ with $q \leq Q$ and put $a_\varepsilon := \varepsilon/2\ q + (1 - \varepsilon/2)\hat{a}$. Since $s(a)\hat{a} = \hat{a}$ and $q = q\hat{a} = \hat{a}q$ it follows that $s(a)a_\varepsilon = a_\varepsilon$. Furthermore, $qa_\varepsilon = q$ and $\|(1 - q)a_\varepsilon\| \leq 1 - \varepsilon/2$. Also $\|a - a_\varepsilon\| \leq \|a - \hat{a}\| + \|\hat{a} - a_\varepsilon\| < \varepsilon$, and (i) follows from Theorem 1. ∎

ACKNOWLEDGMENTS

The authors express their gratitude to Angel Rodríguez Palacios for helpful discussions on the subject. Part of this work has been carried out while the third named author was visiting the University of Granada. He expresses his gratitude to the members of the Departamento de Análisis Matemático for their hospitality.

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