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On the connectivity of close to regular multipartite tournaments

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D)=\max\{d^+(x), d^-(x)\}-\min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x=y$) and the local irregularity of a digraph D is $i_l(D)=\max|d^+(x)-d^-(x)|$ over all vertices x of D. Clearly, $i_l(D)\leqslant i_g(D)$. If $i_g(D)=0$, then D is regular and if $i_g(D)\leqslant 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament such that $|V_1|\leqslant|V_2|\leqslant\cdots\leqslant|V_c|$. In 1998, Yeo proved

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|-2i_l(D)}{3}\right\rceil$$

for each c-partite tournament D, where $\kappa(D)$ is the connectivity of D. Using Yeo's proof, we will present the structure of those multipartite tournaments, which fulfill the last inequality with equality. These investigations yield the better bound

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|-2i_l(D)+1}{3}\right\rceil$$

in the case that $|V_c|$ is odd. Especially, we obtain a 1980 result by Thomassen for tournaments of arbitrary (global) irregularity. Furthermore, we will give a shorter proof of the recent result of Volkmann that

$$\kappa(D)\geqslant\left\lceil\frac{|V(D)|-|V_c|+1}{3}\right\rceil$$

for all regular multipartite tournaments with exception of a well-determined family of regular $(3q+1)$-partite tournaments. Finally we will characterize all almost regular tournaments with this property.

Keywords: Multipartite tournaments; Almost regular multipartite tournaments; Connectivity

1. Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D is denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x\to y$ and say that x *dominates* y, and if

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X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$. Furthermore, let $E(X, Y) = d(X, Y) + d(Y, X)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. In [11], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}$$

and the *local irregularity* by $i_l(D) = \max\{|d^+(x) - d^-(x)| \,|\, x \in V(D)\}$. Clearly $i_l(D) \leqslant i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leqslant 1$, then D is called *almost regular*.

A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v, there are a directed path from u to v, and a directed path from v to u in D. A digraph D with at least $k + 1$ vertices is *k-connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If S is a set of vertices of D such that the subdigraph $D - S$ is not strongly connected, then S is called a *separating set*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. A *semicomplete multipartite digraph* is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$. Note that especially for tournaments, the global and the local irregularity have the same value. Hence, in this case we shortly speak of the *irregularity* $i(T)$ of a tournament T.

In 1998, Yeo [10] proved the following useful bound.

Theorem 1.1 (*Yeo [10]*). *Let D be a c-partite tournament. Then*

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil. \tag{1}$$

In general, this bound cannot be improved as the following example demonstrates (see also [6]).

Example 1.2 (*Volkmann [6]*). Let $q \geqslant 1$ be an integer, and let $c = 3q + 1$. We define the families $\mathscr{F}_q$ of c-partite tournaments with the partite sets $W_1, W_2, \ldots, W_q$ and

$$W_{q+1} = A_{q+1} \cup B_{q+1},\ W_{q+2} = A_{q+2} \cup B_{q+2}, \ldots, W_c = A_c \cup B_c$$

with $2|A_i| = 2|B_i| = |W_j| = 2t$ for $i = q+1, q+2, \ldots, c$ and $j = 1, 2, \ldots, q$ as follows. The partite sets $W_1, W_2, \ldots, W_q$ induce a $t(q-1)$-regular q-partite tournament H, the sets $A_{q+1}, A_{q+2}, \ldots, A_c$ induce a tq-regular $(2q+1)$-partite tournament A, and the sets $B_{q+1}, B_{q+2}, \ldots, B_c$ induce a tq-regular $(2q+1)$-partite tournament B. In addition, let $H \to A \leadsto B \to H$. Obviously, if $D \in \mathscr{F}_q$, then D is a $3qt$-regular c-partite tournament with the separating set $V(H)$ and thus $\kappa(D) = 2qt = q\alpha(D)$.

Since Yeo's result is often used to solve problems depending on the global irregularity, it would be interesting to solve the following general problem.

Problem 1.3. For each integer $i \geqslant 0$ find all multipartite tournaments D with $i_g(D) = i$ and the property that

$$\kappa(D) = \left\lceil \frac{|V(D)| - |V_c| - 2i}{3} \right\rceil.$$

In Section 2, we will analyze the proof of Theorem 1.1. With this method we will extend this result by working out—for each given integer $j \geqslant 0$—the structure of those multipartite tournaments D with $i_l(D) = j$ for which the bound (1) is tight. This structure implies a well-known bound of Thomassen [4] on the connectivity of tournaments of given irregularity. Furthermore, the results of Section 2 will be useful for Section 3 and to prove a result in [9] about Hamiltonian paths through a given arc.

In Section 3, we will study Problem 1.3 for $i = 0$ and $i = 1$. For the case that D is a regular tournament, Volkmann [6] proved the following bound, which solves Problem 1.3 for $i = 0$.

Theorem 1.4 (*Volkmann [6]*). *Let D be a regular c-partite tournament with* $c \geqslant 2$. *Then*,

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) + 1}{3} \right\rceil,$$

with exception of the case that D is a member of the families $\mathscr{F}_q$.

Using the structure of the multipartite tournaments, which fulfill (1) with equality, in the beginning of Section 3, we will present a shorter proof of Theorem 1.4. Note that Theorem 1.4 generalizes Theorem 2.10 in [7], which is needed to prove a theorem about complementary cycles. Furthermore, we will extend Theorem 1.4 to almost regular multipartite tournaments, which means that we will present a solution of Problem 1.3 for $i = 1$.

For more information on multipartite tournaments we refer the reader to Bang-Jensen and Gutin [1], Gutin [2], and Volkmann [5].

2. An analysis of Yeo's result

The following results were given in [10] and [11]. The information about the cases of equality can implicitly be found in the proofs of the lemmas.

Lemma 2.1 (*Yeo [11]*). *Let* $\vec{v} = (v_1, v_2, \ldots, v_c)$ *be c integers with* $\sum_{i=1}^{c} v_i = B$ *and* $v_i \geqslant 1$ *for all* $i = 1, 2, \ldots, c$. *For any set of c reals* $\vec{x} = (x_1, x_2, \ldots, x_c)$ *with* $0 \leqslant x_i \leqslant v_i$ $(i = 1, 2, \ldots, c)$ *and* $0 < \sum_{i=1}^{c} x_i = A \leqslant B/2$ *we have the following*:

$$\frac{e(\vec{x}, \vec{v})}{A} + \frac{e(\vec{x}, \vec{v})}{B - A} \geqslant B - \max\{v_i | i = 1, 2, \ldots, c\}, \tag{2}$$

where $e(\vec{x}, \vec{v}) = A(B - A) - \sum_{i=1}^{c} x_i(v_i - x_i)$.

Furthermore, if equality holds above, then $v_i - 2x_i = v_j - 2x_j$ *and* $v_j - x_j = v_i - x_i$ *for all* $1 \leqslant i, j \leqslant c$.

Lemma 2.2 (*Yeo [11]*). *Let D be a semicomplete multipartite digraph with the partite sets* $V_1, V_2, \ldots, V_c$. *Let* $X \subseteq Y \subseteq V(D)$ *be arbitrary. Let* $x_i = |V_i \cap X|$ *and* $v_i = |Y \cap V_i|$ *for all* $i = 1, 2, \ldots, c$. *This implies the following*:

$$\frac{E(X, Y - X)}{|X|} + \frac{E(X, Y - X)}{|Y - X|} \geqslant |Y| - \max\{v_i | i = 1, 2, \ldots, c\}. \tag{3}$$

In the case of equality in (3) *we have also equality in* (2) *with* x_i *and* v_i *defined here*.

Lemma 2.3 (*Yeo [10]*). *If D is a digraph and* $X \subset V(D)$ *is non-empty*, *then*

$$i_l(D) \geqslant \frac{|d(X, V(D) - X) - d(V(D) - X, X)|}{|X|}. \tag{4}$$

If equality holds above, *then it follows that* $d^+(x) = d^-(x) + i_l(D)$ *for all* $x \in X$ *or* $d^-(x) = d^+(x) + i_l(D)$ *for all* $x \in X$.

Theorem 2.4 (*Yeo [10]*). *Let D be a semicomplete multipartite digraph with the partite sets* $V_1, V_2, \ldots, V_c$, *and let S be a separating set in D. Let* Q_1 *and* Q_2 *be a partition of* $V(D) - S$, *such that* $Q_1 \rightsquigarrow Q_2$, *and let*

$v' = \max\{|V_i \cap (V(D) - S)| \mid i = 1, 2, \ldots, c\}$. *Then the following holds*:

$$i_l(D) \geqslant \frac{|V(D)| - 3|S| - v'}{2}. \tag{5}$$

In the case of equality in (5) *we have also equality in* (3) *with* $X = Q_1$ *and* $Y = V(D) - S$. *Furthermore, it follows that* $|Q_1| = |Q_2|$, $S \to Q_1$ *and* $d(Q_1, V(D) - Q_1) \geqslant |Q_1||S|$, *and we have equality in* (4) *with* $X = Q_1$.

This immediately leads to Yeo's main result.

Theorem 2.5 (*Yeo [10]*). *If D is a semicomplete multipartite digraph, then* (1) *holds.*

Furthermore, if equality holds in (1), *then we observe that* (5) *is fulfilled with equality and there is a partite set* V_i *such that* $|V_i| = \alpha(D)$ *and* $V_i \subseteq V(D) - S$.

The following slight extension of a result of the authors [8] is useful to structure the multipartite tournaments that fulfill (1) with equality.

Lemma 2.6 (*Volkmann and Winzen [8]*). *If D is a multipartite tournament with* $i_l(D) \leqslant l$ *and* $x \in V(D)$ *such that* $|V(x)| = p$, *then*

$$\frac{|V(D)| - p - l}{2} \leqslant d^+(x), d^-(x) \leqslant \frac{|V(D)| - p + l}{2}.$$

All results above yield the following corollary.

Corollary 2.7. *Let D be a multipartite tournament with* $\kappa(D) = (|V(D)| - 2i_l(D) - \alpha(D))/3$ *and let S be a separating set with* $|S| = \kappa(D)$. *Then the following holds*:

- (i) $(|V(D)| - 2i_l(D) - \alpha(D))/3 \in \mathbb{N}_0$.
- (ii) *There is no partite set* V_i *of D such that* $V_i \cap (V(D) - S) \neq \emptyset$ *and* $V_i \cap S \neq \emptyset$.
- (iii) *For all partite sets* V_i *of D with* $V_i \subseteq V(D) - S$ *it follows that* $|V_i| = \alpha(D)$.
- (iv) $V(D) - S$ *can be partitioned in the sets* Q_1 *and* Q_2 *with* $Q_1 \leadsto Q_2$ *such that* $|Q_1| = |Q_2|$, $Q_2 \to S \to Q_1$ *and* $D[Q_1]$ *and* $D[Q_2]$ *are strong.*
- (v) $d^+(q_1) = d^-(q_1) + i_l(D) = (|V(D)| - \alpha(D) + i_l(D))/2$ *for all* $q_1 \in Q_1$ *and* $d^-(q_2) = d^+(q_2) + i_l(D) = (|V(D)| - \alpha(D) + i_l(D))/2$ *for all* $q_2 \in Q_2$.
- (vi) $\alpha(D)$ *is even.*
- (vii) *Every partite set* V_i *of D with* $V_i \subseteq V(D) - S$ *can be partitioned in two disjoined sets of vertices* V_i' *and* V_i'' *such that* $|V_i'| = |V_i''|$, $V_i' \subseteq Q_1$ *and* $V_i'' \subseteq Q_2$.
- (viii) $D[Q_1]$ *and* $D[Q_2]$ *are regular multipartite tournaments.*

Proof. Since $\kappa(D)$ is a non-negative integer, (i) follows immediately. Let Q_1 and Q_2 be a partition of $V(D) - S$ such that $Q_1 \leadsto Q_2$. According to Theorem 2.5, there is a partite set V_i of D such that $V_i \subseteq V(D) - S$ and $|V_i| = \alpha(D)$. Now Lemma 2.1 with $x_i = |Q_1 \cap V_i|$ and $v_i = |V_i \cap (V(D) - S)|$ yields that

$$|V_i \cap (V(D) - S)| - 2|V_i \cap Q_1| = |V_j \cap (V(D) - S)| - 2|V_j \cap Q_1|$$

and

$$|V_i \cap (V(D) - S)| - |V_i \cap Q_1| = |V_j \cap (V(D) - S)| - |V_j \cap Q_1|$$

for all indices j with $V_j \cap (V(D) - S) \neq \emptyset$. This is possible only if $|V_i \cap Q_1| = |V_j \cap Q_1|$ and $|V_i \cap (V(D) - S)| = |V_i| = \alpha(D) = |V_j \cap (V(D) - S)|$ for all these indices j. This implies (ii) and (iii).

According to Theorem 2.4, we have $|Q_1| = |Q_2|$. If $D - S$ does not consist of two strong components of the same cardinality, then we can choose a partition Q_1 and Q_2 of $V(D) - S$ such that $Q_1 \leadsto Q_2$ and $|Q_1| \neq |Q_2|$, a contradiction. Furthermore, Theorem 2.4 leads to $S \to Q_1$. Observing the converse D^{-1} of D, we arrive at $Q_2 \to S$. Altogether we have shown (iv).

Since, according to Theorem 2.4, $d(Q_1, V(D) - Q_1) \geqslant |Q_1||S| = d(V(D) - Q_1, Q_1)$, Lemma 2.3 yields $d^+(q_1) = d^-(q_1) + i_l(D)$ for all $q_1 \in Q_1$ and, caused by symmetry, $d^-(q_2) = d^+(q_2) + i_l(D)$ for all $q_2 \in Q_2$. Using Lemma 2.6 with $p = \alpha(D)$, we arrive at (v).

As seen above, Lemma 2.1 implies $|V_i \cap Q_1| = |V_j \cap Q_1|$ for all indices i and j with $V_i, V_j \subseteq V(D) - S$. Because of $|Q_1| = |Q_2|$, this exactly means (vii) and thus with (iii) we deduce that (vi) is valid.

According to (vii), we have $d(x, Q_2) = d(y, Q_2)$ for all $x, y \in Q_1$. Because of (v), $D[Q_1]$ has to be a regular multipartite tournament. Caused by symmetry, $D[Q_2]$ is also a regular multipartite tournament, which means that (viii) is valid.

This completes the proof of this corollary. □

This result yields a simple method to check, whether the inequality (1) can be improved.

Corollary 2.8. *Let D be a multipartite tournament. If* $\alpha(D)$ *is odd, then it follows that*:

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D) + 1}{3} \right\rceil.$$

In the case of a tournament T we observe that $\alpha(T) = 1$ is odd and $i_g(T) = i_l(T) = i(T)$. Hence, Corollary 2.8 implies the following result of Thomassen [4].

Theorem 2.9 (*Thomassen [4]*). *If T is a tournament with* $i(T) \leqslant k$, *then*

$$\kappa(D) \geqslant \left\lceil \frac{|V(T)| - 2k}{3} \right\rceil.$$

Another consequence of Corollary 2.8 is the following result.

Corollary 2.10. *Let D be a c-partite tournament with* $c \geqslant 2$, $i_g(D) = 2k + 1$ *for an integer* $k \geqslant 0$ *and* $\alpha(D) = \gamma(D)$. *Then the following holds*:

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D) + 1}{3} \right\rceil = \left\lceil \frac{|V(D)| - \alpha(D) - 4k - 1}{3} \right\rceil.$$

3. Connectivity in almost regular multipartite tournaments

With the results of the last section we are able to present a shorter proof of Theorem 1.4.

Theorem 3.1 (*Volkmann [6]*). *Let D be a regular c-partite tournament with* $c \geqslant 2$. *Then*,

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) + 1}{3} \right\rceil,$$

with exception of the case that D is a member of the families $\mathscr{F}_q$.

Proof. If $V_1, V_2, \ldots, V_c$ are the partite sets of D, then $|V_1| = |V_2| = \cdots = |V_c| = r$, $\alpha(D) = r$, and $i_l(D) = 0$. Suppose that $\kappa(D) = (|V(D)| - \alpha(D))/3 = (c-1)r/3$. It follows that (i)–(viii) of Corollary 2.7 holds. Especially (ii) yields that $|S| = sr$ for an integer s. On the other hand, we see that $|S| = \kappa(D) = (c-1)r/3$ and thus $s = (c-1)/3 \in \mathbb{N}$, which means that $c = 3q + 1$ for an integer q and $|S| = qr = q\alpha(D)$. Since, according to (iv), $Q_2 \to S \to Q_1$ with $|Q_1| = |Q_2|$ and D is regular, $D[S]$ has also to be regular. With Corollary 2.7 (vii) and (viii) we conclude that D belongs to the families $\mathscr{F}_q$. □

Now we will examine almost regular multipartite tournaments. At first we want to derive a result that will help us in the following. According to Tewes, Volkmann and Yeo [3], the following lemma holds.

Lemma 3.2 (*Tewes, Volkmann and Yeo [3]*). *If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D such that* $|V_1| \leqslant |V_2| \leqslant$
$\cdots \leqslant |V_c|$, *then* $|V_c| \leqslant |V_1| + 2i_g(D)$.

The following lemma presents a lower bound for the degree of a vertex.

Lemma 3.3. *Let D be a multipartite tournament with $i_g(D) \leqslant l$ and $\gamma(D) = r$. Then we have*

$$\frac{|V(D)| - \gamma(D) - 2l}{2} \leqslant d^+(x), d^-(x)$$

for all $x \in V(D)$. If furthermore $|V(x)| = r + 2l$, then it follows that:

$$d^+(x), d^-(x) = \frac{|V(D)| - r - 2l}{2}.$$

Proof. Let $x \in V(D)$ be arbitrary. If $|V(x)| \leqslant r+l$, then the first assertion holds by Lemma 2.6. Hence, let $|V(x_1)| \geqslant r+l+1$. Suppose that $d^+(x_1) \leqslant (|V(D)|-r-2l-1)/2$. Because of $i_g(D) \leqslant l$, we conclude that $d^+(y), d^-(y) \leqslant (|V(D)|-r-1)/2$ for all $y \in V(D)$. If we take a vertex $x_2 \in V(D)$ with $|V(x_2)| = r$, then we arrive at the contradiction

$$|V(D)| = d^+(x_2) + d^-(x_2) + r \leqslant |V(D)| - r - 1 + r = |V(D)| - 1.$$

Hence, it has to be $d^+(x) \geqslant (|V(D)| - \gamma(D) - 2l)/2$ for all vertices $x \in V(D)$. Since the proof for $d^-(x)$ follows the same lines, the first assertion of this lemma is completed.

Now, let $x \in V(D)$ with $|V(x)| = r + 2l$. Suppose that $d^+(x) \geqslant (|V(D)| - r - 2l + 1)/2$. The fact that $|V(D)| = d^+(x) + d^-(x) + r + 2l$ yields that $d^-(x) \leqslant (|V(D)| - r - 2l - 1)/2$, a contradiction to the first assertion of this lemma. This completes the proof of the lemma. □

Together with Corollary 2.7, this yields the following result.

Corollary 3.4. *Let D be a multipartite tournament such that $\kappa(D) = (|V(D)| - 2i_l(D) - \alpha(D))/3$ and $i_g(D) = i_l(D) \geqslant 1$. Then it follows that $\alpha(D) < \gamma(D) + 2i_g(D)$.*

Proof. According to (iii) and (v) in Corollary 2.7, we observe that

$$d^+(q_1) = d^-(q_1) + i_g(D) = \frac{|V(D)| - \alpha(D) + i_g(D)}{2}$$

and $|V(q_1)| = \alpha(D)$ for all $q_1 \in Q_1$. Assume that $\alpha(D) \geqslant \gamma(D) + 2i_g(D)$. Lemma 3.2 yields that $\alpha(D) = \gamma(D) + 2i_g(D)$. Now Lemma 3.3 leads to the contradiction

$$d^+(q_1) = \frac{|V(D)| - \gamma(D) - 2i_g(D)}{2} = \frac{|V(D)| - \alpha(D)}{2}. \qquad \square$$

The following examples will present the families of the multipartite tournaments with $i_g(D) = 1$, which realize (1).

Example 3.5. Let the integers k, m, r, p, l, v, q, c and k_1 fulfill one of the following properties:

(1) $r = 2p + 1 \geqslant 1$, $k = 3m \geqslant 3$, $l = 2v$, $0 \leqslant v \leqslant (m-1)/(4p+2)$, $k_1 = m - 1 - 2v(2p+1)$, $q = 2v + 2vp + m$ and $c = 3q + 1$.
(2) $r = 4p + 3 \geqslant 3$, $k = 3m \geqslant 3$, $0 \leqslant l \leqslant (m-1)/(4p+3)$, $k_1 = m - 1 - l(4p+3)$, $q = 2l + 2lp + m$ and $c = 3q + 1$.
(3) $r = 12p + 3 \geqslant 3$, $k = 3m$, $m \geqslant 8p + 3$, $1 \leqslant l \leqslant (4p+m)/(12p+3)$, $k_1 = 4p + m - l(12p+3)$, $q = m - 2p - 1 + l(6p+2)$ and $c = 3q + 2$.
(4) $r = 6p + 3 \geqslant 3$, $k = 3m$, $m \geqslant 4p + 3$, $l = 2v + 1 \geqslant 1$, $0 \leqslant v \leqslant (m - 4p - 3)/(12p+6)$, $k_1 = 2p + m - (2v+1)(6p+3)$, $q = m - p - 1 + (2v+1)(3p+2)$ and $c = 3q + 2$.
(5) $r = 12p + 11 \geqslant 11$, $k = 3m + 1$, $m \geqslant 8p + 8$, $1 \leqslant l \leqslant (m + 4p + 3)/(12p+11)$, $k_1 = 4p + 3 + m - l(12p+11)$, $q = m - 2p - 2 + l(6p+6)$ and $c = 3q + 2$.

(6) $r=6p+5\geqslant 5, k=3m+1, m\geqslant 4p+4, l=2v+1, 0\leqslant v\leqslant (m-4p-4)/(12p+10), k_1=2p+1+m-(2v+1)(6p+5)$, $q=m-p-1+(2v+1)(3p+3)$ and $c=3q+2$.
(7) $r=12p+7\geqslant 7, k=3m+2, m\geqslant 8p+5, 1\leqslant l\leqslant (m+4p+2)/(12p+7), k_1=4p+2+m-l(12p+7)$, $q=m-2p-1+l(6p+4)$ and $c=3q+2$.
(8) $r=6p+1\geqslant 1, k=3m+2, m\geqslant 4p+1, l=2v+1, 0\leqslant v\leqslant (m-4p-1)/(12p+2), k_1=2p+m-(2v+1)(6p+1)$, $q=m-p+(2v+1)(3p+1)$ and $c=3q+2$.
(9) $r=12p+3\geqslant 3, k=3m, m\geqslant 4p+2, 0\leqslant l\leqslant (m-4p-2)/(12p+3), k_1=m-4p-2-l(12p+3)$, $q=m+2p+1+l(6p+2)$ and $c=3q$.
(10) $r=6p+3\geqslant 3, k=3m, m\geqslant 8p+5, l=2v+1, 0\leqslant v\leqslant (m-8p-5)/(12p+6), k_1=m-2p-2-(2v+1)(6p+3)$, $q=m+p+1+(2v+1)(3p+2)$ and $c=3q$.
(11) $r=6p+1\geqslant 1, k=3m+1, m\geqslant 8p+2, l=2v+1, 0\leqslant v\leqslant (m-8p-2)/(12p+2), k_1=m-2p-1-(2v+1)(6p+1)$, $q=m+p+1+(2v+1)(3p+1)$ and $c=3q$.
(12) $r=12p+7\geqslant 7, k=3m+1, m\geqslant 4p+3, 0\leqslant l\leqslant (m-4p-3)/(12p+7), k_1=m-4p-3-l(12p+7)$, $q=m+2p+2+l(6p+4)$ and $c=3q$.
(13) $r=6p+5\geqslant 5, k=3m+2, m\geqslant 8p+7, l=2v+1, 0\leqslant v\leqslant (m-8p-7)/(12p+10), k_1=m-2p-2-(2v+1)(6p+5)$, $q=m+2+p+(2v+1)(3p+3)$ and $c=3q$.
(14) $r=12p+11\geqslant 11, k=3m+2, m\geqslant 4p+4, 0\leqslant l\leqslant (m-4p-4)/(12p+11), k_1=m-4p-4-l(12p+11)$, $q=m+3+2p+l(6p+6)$ and $c=3q$.

If the properties in (i) are valid ($i=1,2,\ldots,14$) for the indices k,m,r,p,l,v,q,c and k_1, then we define the families $\mathscr{G}_q^i$ of c-partite tournaments with the partite sets $W_1=A_1\cup B_1, W_2=A_2\cup B_2,\ldots,W_{k-k_1}=A_{k-k_1}\cup B_{k-k_1}$ and $W_{k-k_1+1}, W_{k-k_1+2},\ldots,W_c$ with $2|A_i|=2|B_i|=|W_j|=r+1$ for $i=1,2,\ldots,k-k_1$ and $j=k-k_1+1,k-k_1+2,\ldots,k$ and $|W_{k+1}|=|W_{k+2}|=\cdots=|W_c|=r$ as follows.

The partite sets $W_{k-k_1+1}, W_{k-k_1+2},\ldots,W_c$ induce a $(q+l)$-partite tournament H such that $d_H^+(x)=d_H^-(x)$ for all $x\in W_{k+1}\cup W_{k+2}\cup\cdots\cup W_c$ and $|d_H^+(x)-d_H^-(x)|=1$ for all $x\in W_{k-k_1+1}\cup W_{k-k_1+2}\cup\cdots\cup W_k$; the sets $A_1, A_2,\ldots, A_{k-k_1}$ induce a $[(c-q-l-1)(r+1)/4]$-regular $(c-q-l)$-partite tournament A; and analogously the sets $B_1, B_2,\ldots,B_{k-k_1}$ induce a $[(c-q-l-1)(r+1)/4]$-regular $(c-q-l)$-partite tournament B. In addition, let $H\rightarrow A\rightsquigarrow B\rightarrow H$. If $D\in\mathscr{G}_q^i$ for $i=1,2,\ldots,14$, then it is straightforward to show that D is a c-partite tournament with $i_g(D)=i_l(D)=1$ containing the separating set $V(H)$ such that $|V(H)|=\kappa(D)=(c-k+k_1)r+k_1=(|V(D)|-\alpha(D)-2)/3$.

Example 3.6. Let the integers k,m,r,p,l,v,q,c and k_1 fulfill one of the following properties:

(1) $r=2+4p\geqslant 2, k=3m+1, m\geqslant 1+2p, 1\leqslant l\leqslant m/(1+2p), k_1=m-l(1+2p), q=m+l(1+p)$ and $c=3q+1$.
(2) $r=4+4p\geqslant 4, k=3m+1, m\geqslant 4+4p, l=2v+2, 0\leqslant v\leqslant (m-4-4p)/(4+4p), k_1=m-(2v+2)(2p+2)$, $q=m+(v+1)(2p+3)$ and $c=3q+1$.
(3) $r=6+12p\geqslant 6, k=3m+1, m\geqslant 5+10p, l=2v, 1\leqslant v\leqslant \frac{1}{6}(1+(m/(1+2p))), k_1=1+2p+m-6v-12pv$, $q=4v+6pv+m-p-1$ and $c=3q+2$.
(4) $r=10+12p\geqslant 10, k=3m+2, m\geqslant 10p+8, l=2v, 1\leqslant v\leqslant (m+2p+2)/(12p+10), k_1=2p+2+m-12pv-10v$, $q=6v+6pv+m-p-1$ and $c=3q+2$.
(5) $r=2+12p\geqslant 2, k=3m, m\geqslant 10p+2, l=2v, 1\leqslant v\leqslant (m+2p)/(12p+2), k_1=2p+m-2v-12pv$, $q=2v+6pv+m-p-1$ and $c=3q+2$.
(6) $r=6p+2\geqslant 2, k=3m, m\geqslant 2p+1, l=2v+1, 0\leqslant v\leqslant (m-2p-1)/(6p+2), k_1=m-2p-2v-6pv-1$, $q=2v+3pv+m+p$ and $c=3q+2$.
(7) $r=6+6p\geqslant 6, k=3m+1, m\geqslant 2p+2, l=2v+1, 0\leqslant v\leqslant \frac{1}{6}((m/(p+1))-2), k_1=m-2p-6pv-6v-2$, $q=4v+3vp+m+p+1$ and $c=3q+2$.
(8) $r=4+6p\geqslant 4, k=3m+2, m\geqslant 2p+1, l=2v+1, 0\leqslant v\leqslant (m-2p-1)/(4+6p), k_1=m-2p-1-6pv-4v$, $q=3v+3pv+m+p+1$ and $c=3q+2$.
(9) $r=10+12p\geqslant 10, k=3m, m\geqslant 2p+2, l=2v, 0\leqslant v\leqslant (m-2p-2)/(10+12p), k_1=m-2-2p-10v-12pv$, $q=6v+6pv+m+1+p$ and $c=3q$.
(10) $r=6+12p\geqslant 6, k=3m+1, m\geqslant 2p+1, l=2v, 0\leqslant v\leqslant (m-2p-1)/(6+12p), k_1=m-2p-1-12pv-6v$, $q=4v+6pv+m+p+1$ and $c=3q$.

(11) $r = 2 + 12p \geqslant 2$, $k = 3m + 2$, $m \geqslant 2p$, $l = 2v$, $0 \leqslant v \leqslant (m - 2p)/(2 + 12p)$, $k_1 = m - 2p - 2v - 12pv$, $q = 2v + 6pv + m + 1 + p$ and $c = 3q$.
(12) $r = 6p + 4 \geqslant 4$, $k = 3m$, $m \geqslant 3 + 4p$, $l = 2v + 1$, $0 \leqslant v \leqslant (m - 3 - 4p)/(4 + 6p)$, $k_1 = m - 3 - 4p - 4v - 6pv$, $q = 3v + 2 + 3pv + m + 2p$ and $c = 3q$.
(13) $r = 6 + 6p \geqslant 6$, $k = 3m + 1$, $m \geqslant 4 + 4p$, $l = 2v + 1$, $0 \leqslant v \leqslant (m - 4 - 4p)/(6 + 6p)$, $k_1 = m - 4 - 4p - 6v - 6pv$, $q = 4v + 3pv + 2p + m + 3$ and $c = 3q$.
(14) $r = 2 + 6p \geqslant 2$, $k = 3m + 2$, $m \geqslant 4p + 1$, $l = 2v + 1$, $0 \leqslant v \leqslant (m - 4p - 1)/(2 + 6p)$, $k_1 = m - 4p - 2v - 6pv - 1$, $q = 2v + 2 + 3pv + m + 2p$ and $c = 3q$.

If the properties in (i) are valid ($i = 1, 2, \ldots, 14$) for the indices k, m, r, p, l, v, q, c and k_1, then we define the families $\mathscr{H}_q^i$ of c-partite tournaments with the partite sets $W_1 = A_1 \cup B_1$, $W_2 = A_2 \cup B_2, \ldots, W_{k-k_1} = A_{k-k_1} \cup B_{k-k_1}$ and $W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ with $2|A_i| = 2|B_i| = |W_j| = r + 2$ for $i = 1, 2, \ldots, k - k_1$ and $j = k - k_1 + 1, k - k_1 + 2, \ldots, k$ and $|W_{k+1}| = |W_{k+2}| = \cdots = |W_c| = r$ as follows.

The partite sets $W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ induce a local regular $(q + l)$-partite tournament H; the sets $A_1, A_2, \ldots, A_{k-k_1}$ induce a $[(c - q - l - 1)(r + 2)/4]$-regular $(c - q - l)$-partite tournament A; and the sets $B_1, B_2, \ldots, B_{k-k_1}$ induce a $[(c - q - l - 1)(r + 2)/4]$-regular $(c - q - l)$-partite tournament B. In addition, let $H \to A \rightsquigarrow B \to H$. If $D \in \mathscr{H}_q^i$ for $i = 1, 2, \ldots, 14$, then it is left to the reader to show that D is a c-partite tournament with $i_g(D) = 1$ and $i_l(D) = 0$ containing the separating set $V(H)$ such that $|V(H)| = \kappa(D) = (c - k + k_1)r + 2k_1 = (|V(D)| - \alpha(D))/3$.

There are no other c-partite tournaments with $i_g(D) = 1$ and $\kappa(D) = (|V(D)| - \alpha(D) - 2i_l(D))/3$ as we can see in the following theorem.

Theorem 3.7. *Let D be an almost regular c-partite tournament with $c \geqslant 2$. Then,*

$$\kappa(D) \geqslant \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D) + 1}{3} \right\rceil,$$

with exception of the case that D is a member of one of the families $\mathscr{F}_q$, $\mathscr{G}_q^i$ or $\mathscr{H}_q^i$ with $i \in \{1, 2, \ldots, 14\}$.

Proof. If $i_g(D) = 0$, then the assertion follows from Theorem 3.1. Hence, let D be a c-partite tournament with $i_g(D) = 1$ and the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| = \alpha(D)$. According to Lemma 3.2, we have $r \leqslant \alpha(D) \leqslant r + 2$. Suppose that $\kappa(D) = (|V(D)| - 2i_l(D) - \alpha(D))/3$. Let S, Q_1 and Q_2 be defined as in Corollary 2.7 and observe that (i)–(viii) of this corollary holds. Now we distinguish different cases.

Case 1. Let $\alpha(D) = r$. In this case, Corollary 2.10 yields a contradiction.

Case 2. Assume that $\alpha(D) = r + 1$. This implies that $i_l(D) = i_g(D) = 1$ and according to Corollary 2.7 (vi), r is odd. Hence, we may suppose that $r = 2p + 1$ for an integer $p \geqslant 0$. Let $|V(D)| = cr + k$ with $0 < k < c$. Because of Corollary 2.7 (v), we deduce that the number of partite sets with the cardinality r has to be odd, which means that $c - k$ is odd. Again with Corollary 2.7 we see that S consists of all the $c - k$ partite sets of cardinality r and of $k_1 \geqslant 0$ partite sets of cardinality $r + 1$. This yields that $|S| = (c - k + k_1)r + k_1$. Since $|Q_1| = |Q_2|$ and $Q_2 \to S \to Q_1$, it follows that $d^+_{D[S]}(x) = d^-_{D[S]}(x)$ for all vertices x belonging to a partite set of cardinality r and $|d^+_{D[S]}(x) - d^-_{D[S]}| = 1$ for all vertices $x \in S$ belonging to a partite set of cardinality $r + 1$.

Subcase 2.1. Let $c = 3q + 1$. If $D[S]$ is $(q + l)$-partite, then we arrive at

$$\begin{aligned} |S| &= (c - k + k_1)r + k_1 = (q + l)r + k_1 = \frac{cr + k - 2 - (r + 1)}{3} \\ &= qr + \frac{k}{3} - 1 = (q + l)r + \frac{k}{3} - 1 - lr. \end{aligned}$$

This implies that $k = 3m$ for an integer $m \geqslant 1$, $c - k + k_1 = q + l$ and $k_1 = m - 1 - l(2p + 1) \geqslant 0$. It follows that $l \leqslant (m - 1)/(2p + 1)$ and

$$\begin{aligned} & c - k + k_1 = 3q + 1 - 3m + m - 1 - l(2p + 1) = q + l \\ & \Leftrightarrow 2q = 2l + 2lp + 2m \Rightarrow q = l + lp + m. \end{aligned}$$

This leads to

$$c = 3q + 1 = 3l + 3lp + 3m + 1 = k + 1 + 3l + 3lp < c + 1 + 3l + 3lp$$
$$\Rightarrow 1 + 3l + 3lp > 0 \Rightarrow l > -\frac{1}{3 + 3p} \Rightarrow l \geqslant 0.$$

Since $c - k = 1 + 3l + 3lp = 1 + 3l(p + 1)$ has to be odd, it follows that l is even or p is odd.

If $l = 2v$, then we deduce that $0 \leqslant v \leqslant (m - 1)/(4p + 2)$, $m \geqslant 1$, $k_1 = m - 1 - 2v(2p + 1)$ and $q = 2v + 2vp + m$. Corollary 2.7 yields that D belongs to the families $\mathscr{G}_q^1$.

If $p = 2s + 1 \geqslant 1$, then it follows that $r = 4s + 3 \geqslant 3$, $m \geqslant 1$, $0 \leqslant l \leqslant (m - 1)/(4s + 3)$ and $q = 2l + 2ls + m$. Corollary 2.7 implies that D is an element of the families $\mathscr{G}_q^2$.

Subcase 2.2. Suppose that $c = 3q + 2$ for an integer $q \geqslant 0$. If $D[S]$ is $(q + l)$-partite, then we observe that

$$|S| = (c - k + k_1)r + k_1 = (q + l)r + k_1 = \frac{cr + k - r - 3}{3}$$
$$= qr + \frac{r + k}{3} - 1 = (q + l)r + \frac{r + k}{3} - 1 - lr,$$

and thus $c - k + k_1 = q + l$ and $k_1 = ((r + k)/3) - 1 - lr$. This leads to

$$3q + 2 - k + \frac{r + k}{3} - 1 - lr = 3q + 1 + \frac{r - 2k}{3} - lr = q + l$$
$$\Rightarrow 2q = \frac{2k - r}{3} - 1 + l(r + 1) \Rightarrow q = \frac{2k - r - 3}{6} + l\frac{r + 1}{2}.$$

Since $r = 2p + 1$, we have

$$q = \frac{2k - 2p - 4}{6} + l(p + 1) = \frac{k - p - 2}{3} + l(p + 1). \tag{6}$$

Subcase 2.2.1. Let $k = 3m$ for an integer $m \geqslant 1$. With (6) we arrive at $q = m - ((p + 2)/3) + l(p + 1)$ and thus $p = 3s + 1$, $r = 6s + 3$ and $q = m - s - 1 + l(3s + 2)$ for an $s \in \mathbb{N}_0$. Furthermore we see that

$$c = 3q + 2 = 3m - 3s - 3 + 3l(3s + 2) + 2 = k - 3s - 1 + 3l(3s + 2) < c - 3s - 1 + 3l(3s + 2)$$
$$\Rightarrow -3s - 1 + 3l(3s + 2) > 0 \Rightarrow l > \frac{3s + 1}{9s + 6} \Rightarrow l \geqslant 1$$

and

$$k_1 = \frac{r + k}{3} - 1 - lr = 2s + m - l(6s + 3) \geqslant 0 \Rightarrow l \leqslant \frac{2s + m}{6s + 3}.$$

Since $c - k = -3s - 1 + 3l(3s + 2) = 3(3ls + 2l - s) - 1$ is odd, we conclude that $3ls - s = s(3l - 1)$ is even and thus s is even or l is odd.

If $s = 2n$ with $n \in \mathbb{N}_0$, then we arrive at $r = 12n + 3$, $q = m - 2n - 1 + l(6n + 2)$, $k_1 = 4n + m - l(12n + 3)$, $1 \leqslant l \leqslant (4n + m)/(12n + 3)$ and thus $m \geqslant 8n + 3$. According to Corollary 2.7, D is a member of the families $\mathscr{G}_q^3$.

If $l = 2v + 1$ for an integer v, then it follows that $0 \leqslant v \leqslant (m - 4s - 3)/(12s + 6)$, $m \geqslant 4s + 3$, $q = m - s - 1 + (2v + 1)(3s + 2)$ and $k_1 = 2s + m - (2v + 1)(6s + 3)$. Again with Corollary 2.7 we deduce that D is an element of the families $\mathscr{G}_q^4$.

Subcase 2.2.2. Assume that $k = 3m + 1$ with $m \in \mathbb{N}_0$. According to (6), we have $q = m - ((p + 1)/3) + l(p + 1)$ and thus $p = 3s + 2$ for an integer $s \geqslant 0$, $r = 6s + 5$ and $q = m - s - 1 + l(3s + 3)$. Furthermore we conclude that

$$c = 3q + 2 = 3m - 3s - 3 + 3l(3s + 3) + 2 = k - 2 - 3s + 3l(3s + 3)$$
$$\Rightarrow -2 - 3s + 3l(3s + 3) > 0 \Rightarrow l > \frac{3s + 2}{3(3s + 3)} \Rightarrow l \geqslant 1$$

and

$$k_1 = \frac{r + k}{3} - 1 - lr = 2s + 1 + m - l(6s + 5) \geqslant 0 \Rightarrow l \leqslant \frac{m + 2s + 1}{6s + 5}.$$

Since $c-k=-2-3s+3l(3s+3)=-2+3(3ls+3l-s)$ is odd, we observe that $3ls+3l-s$ is odd. This is possible only if s is odd or l is odd.

If $s=2n+1$ for an integer $n\geqslant 0$, then it follows that $r=12n+11$, $1\leqslant l\leqslant (m+4n+3)/(12n+11)$, $m\geqslant 8n+8$, $k_1=4n+3+m-l(12n+11)$ and $q=m-2n-2+l(6n+6)$. According to Corollary 2.7, we deduce that D belongs to the families $\mathscr{G}_q^5$.

If $l=2v+1$ for an integer v, then it follows that $0\leqslant v\leqslant (m-4s-4)/(12s+10)$, $m\geqslant 4s+4$, $k_1=2s+1+m-(2v+1)(6s+5)$ and $q=m-s-1+(2v+1)(3s+3)$. Hence, using Corollary 2.7 we conclude that D is a member of the families $\mathscr{G}_q^6$.

Subcase 2.2.3. Suppose that $k=3m+2$ with $m\in\mathbb{N}_0$. According to (6), we observe that $q=m-(p/3)+l(p+1)$, and thus $p=3s$, $r=6s+1$ and $q=m-s+l(3s+1)$ with $s\geqslant 0$. Furthermore we see that

$$c=3q+2=3m-3s+3l(3s+1)+2=k-3s+3l(3s+1)$$
$$\Rightarrow -3s+3l(3s+1)>0\Rightarrow l>\frac{3s}{9s+3}\Rightarrow l\geqslant 1$$

and

$$k_1=\frac{r+k}{3}-1-lr=2s+m-l(6s+1)\geqslant 0\Rightarrow l\leqslant\frac{m+2s}{6s+1}.$$

Since $c-k=-3s+3l(3s+1)=3(3ls+l-s)$ is odd, we see that $3ls+l-s$ is odd. This is possible only if s is odd or l is odd.

If $s=2n+1$ with $n\in\mathbb{N}_0$, then it follows that $r=12n+7$, $1\leqslant l\leqslant (m+4n+2)/(12n+7)$, $m\geqslant 8n+5$, $q=m-2n-1+l(6n+4)$ and $k_1=4n+2+m-l(12n+7)$. Using Corollary 2.7 we deduce that D is an element of the families $\mathscr{G}_q^7$.

If $l=2v+1$ for an integer v, then we have $0\leqslant v\leqslant (m-4s-1)/(12s+2)$, $m\geqslant 4s+1$, $q=m-s+(2v+1)(3s+1)$ and $k_1=2s+m-(2v+1)(6s+1)$. Again with Corollary 2.7 we conclude that D is a member of the families $\mathscr{G}_q^8$.

Subcase 2.3. Let $c=3q$ for an integer $q\geqslant 1$. If S is $(q+l)$-partite, then it follows that:

$$|S|=(c-k+k_1)r+k_1=(q+l)r+k_1=\frac{cr+k-r-3}{3}$$
$$=qr+\frac{k-r}{3}-1=(q+l)r+\frac{k-r}{3}-1-lr,$$

and thus $c-k+k_1=q+l$ and $k_1=((k-r)/3)-1-lr$. This implies that

$$3q-k+\frac{k-r}{3}-1-lr=q+l\Rightarrow 2q=\frac{r+2k}{3}+1+l(r+1)$$
$$\Rightarrow q=\frac{r+2k+3}{6}+l\frac{r+1}{2}.$$

Because of $r=2p+1$ this means that

$$q=\frac{2k+2p+4}{6}+l(p+1)=\frac{k+p+2}{3}+l(p+1). \tag{7}$$

Subcase 2.3.1. Assume that $k=3m$ with $m\in\mathbb{N}$. According to (7), this leads to $q=m+((p+2)/3)+l(p+1)$, and thus $p=3s+1$, $r=6s+3$ and $q=m+s+1+l(3s+2)$ for an integer $s\geqslant 0$. Furthermore we observe that

$$c=3q=3m+3s+3+3l(3s+2)=k+3s+3+3l(3s+2)$$
$$\Rightarrow 3s+3+3l(3s+2)>0\Rightarrow l>\frac{-s-1}{3s+2}\Rightarrow l\geqslant 0$$

and

$$k_1=\frac{k-r}{3}-1-lr=m-2s-2-l(6s+3)\geqslant 0\Rightarrow l\leqslant\frac{m-2s-2}{6s+3}.$$

Since $c-k=3s+3+3l(3s+2)=3+3(s+3ls+2l)$ is odd, we deduce that $s+3ls=s(1+3l)$ is even, which means that s is even or l is odd.

If $s=2n$ with $s\in\mathbb{N}_0$, then it follows that $r=12n+3$, $q=m+2n+1+l(6n+2)$, $0\leqslant l\leqslant (m-4n-2)/(12n+3)$, $m\geqslant 4n+2$ and $k_1=m-4n-2-l(12n+3)$. According to Corollary 2.7, we see that D is member of the families $\mathcal{G}_q^9$.

If $l=2v+1$ for an integer v, then we arrive at $0\leqslant v\leqslant (m-8s-5)/(12s+6)$, $m\geqslant 8s+5$, $q=m+s+1+(2v+1)(3s+2)$ and $k_1=m-2s-2-(2v+1)(6s+3)$. Again with Corollary 2.7 we observe that D belongs to the families $\mathcal{G}_q^{10}$.

Subcase 2.3.2. Suppose that $k=3m+1$ for an integer $m\geqslant 0$. With (7) this yields $q=m+(p/3)+1+l(p+1)$ and thus $p=3s$, $r=6s+1$ and $q=m+s+1+l(3s+1)$ with $s\in\mathbb{N}_0$. Furthermore, we conclude that

$$c=3q=3m+3s+3+3l(3s+1)=k+3s+2+3l(3s+1)$$
$$\Rightarrow 3s+2+3l(3s+1)>0\Rightarrow l>-\frac{3s+2}{9s+3}\Rightarrow l\geqslant 0$$

and

$$k_1=\frac{k-r}{3}-1-lr=m-2s-1-l(6s+1)\geqslant 0\Rightarrow l\leqslant\frac{m-2s-1}{6s+1}.$$

Since $c-k=3s+2+3l(3s+1)=2+3(3ls+l+s)$ is odd, it follows that $3ls+l+s$ is odd. This is possible only if s is odd or l is odd.

If $l=2v+1$ for an integer v, then we arrive at $0\leqslant v\leqslant (m-8s-2)/(12s+2)$, $m\geqslant 8s+2$, $q=m+s+1+(2v+1)(3s+1)$ and $k_1=m-2s-1-(2v+1)(6s+1)$. According to Corollary 2.7, D is an element of the families $\mathcal{G}_q^{11}$.

If $s=2n+1$ for an integer $n\geqslant 0$, then it follows that $r=12n+7$, $0\leqslant l\leqslant (m-4n-3)/(12n+7)$, $m\geqslant 4n+3$, $q=m+2n+2+l(6n+4)$ and $k_1=m-4n-3-l(12n+7)$. Again with Corollary 2.7 we see that D belongs to the families $\mathcal{G}_q^{12}$.

Subcase 2.3.3. Let $k=3m+2$ with $m\in\mathbb{N}_0$. Using (7), we observe that $q=m+1+((p+1)/3)+l(p+1)$, and thus $p=3s+2$, $r=6s+5$ and $q=m+2+s+l(3s+3)$. Furthermore we have

$$c=3q=3m+6+3s+3l(3s+3)=k+4+3s+3l(3s+3)$$
$$\Rightarrow 4+3s+3l(3s+3)>0\Rightarrow l>-\frac{4+3s}{9s+9}\Rightarrow l\geqslant 0$$

and

$$k_1=\frac{k-r}{3}-1-lr=m-2s-2-l(6s+5)\geqslant 0\Rightarrow l\leqslant\frac{m-2s-2}{6s+5}.$$

The fact that $c-k=4+3(3ls+3l+s)$ is odd implies that $3ls+3l+s=s(3l+1)+3l$ is odd. This is possible only if l is odd or if s is odd.

If $l=2v+1$ for an integer v, then we deduce that $0\leqslant v\leqslant (m-8s-7)/(12s+10)$, $m\geqslant 8s+7$, $q=m+2+s+(2v+1)(3s+3)$ and $k_1=m-2s-2-(2v+1)(6s+5)$. According to Corollary 2.7, we have that D is a member of the families $\mathcal{G}_q^{13}$.

If $s=2n+1$ for an integer $n\geqslant 0$, then we observe that $r=12n+11$, $0\leqslant l\leqslant (m-4n-4)/(12n+11)$, $m\geqslant 4n+4$, $q=m+3+2n+l(6n+6)$ and $k_1=m-4n-4-l(12n+11)$. Using Corollary 2.7 it follows that D belongs to the families $\mathcal{G}_q^{14}$.

Case 3. Let $\alpha(D)=r+2$. According to Corollary 3.4 we have $i_l(D)<i_g(D)$ and hence $i_l(D)=0$. Because of $Q_2\to S\to Q_1$, $D[S]$ has to be local regular. Since $V_i\subseteq D$ for all partite sets V_i with $|V_i|\leqslant r+1$, this implies that D does not contain any partite set of order $r+1$. Hence, let $|V(D)|=cr+2k$ such that $0<k<c$. Using Corollary 2.7 we observe that S contains all the $c-k$ partite sets of order r. If S contains additionally k_1 partite sets of order $r+2$, then it follows that $|S|=(c-k+k_1)r+2k_1$.

Subcase 3.1. Suppose that $c=3q+1$ with $q\in\mathbb{N}$. If S is $(q+l)$-partite, then it follows that:

$$|S|=(c-k+k_1)r+2k_1=(q+l)r+2k_1=\frac{cr+2k-(r+2)}{3}$$
$$=qr+\frac{2k-2}{3}=(q+l)r+\frac{2k-2}{3}-lr.$$

Since $|S| \in \mathbb{N}_0$, we observe that $k = 3m + 1$ for an integer $m \geqslant 0$, and thus $c - k + k_1 = q + l$ and $k_1 = m - l(r/2)$. This yields that

$$3q + 1 - (3m + 1) + m - l\frac{r}{2} = 3q - 2m - l\frac{r}{2} = q + l$$
$$\Rightarrow 2q = 2m + l\left(1 + \frac{r}{2}\right) \Rightarrow q = m + l\frac{r+2}{4}.$$

If $l \leqslant 0$, then we arrive at $q \leqslant m$ and thus $c \leqslant k$, a contradiction. Hence, let $l \geqslant 1$. Because of $q \in \mathbb{N}$, we conclude that $l(r + 2)/4 \in \mathbb{N}$. Since r is even this implies that $r = 2 + 4p$ or $r = 4 + 4p$ and $l = 2v + 2$ for integers $p, v \geqslant 0$.

If $r = 2+4p$, then we see that $q = m + l(1+p)$ and $k_1 = m - l(1+2p)$. The fact that $k_1 \geqslant 0$ yields that $1 \leqslant l \leqslant m/(1+2p)$ and thus $m \geqslant 1 + 2p$. Using Corollary 2.7, it is obvious that D belongs to the families $\mathscr{H}_q^1$.

If $r = 4 + 4p$ and $l = 2v + 2$, then it follows that $q = m + (v + 1)(2p + 3)$ and $k_1 = m - (2v + 2)(2p + 2)$. Because of $k_1 \geqslant 0$ we have $0 \leqslant v \leqslant (m - 4 - 4p)/(4 + 4p)$ and thus $m \geqslant 4 + 4p$. Again with Corollary 2.7 we deduce that D is a member of the families $\mathscr{H}_q^2$.

Subcase 3.2. Assume that $c = 3q + 2$ for an integer $q \geqslant 0$. Let $r = 2 + 2p$ with $p \in \mathbb{N}_0$. If S is $(q + l)$-partite, then we observe that

$$|S| = (c - k + k_1)r + 2k_1 = (q + l)r + 2k_1 = \frac{cr + 2k - r - 2}{3}$$
$$= qr + \frac{r + 2k - 2}{3} = (q + l)r + \frac{r + 2k - 2}{3} - lr,$$

and thus $c - k + k_1 = q + l$ and $k_1 = ((r + 2k - 2)/6) - l(r/2)$. This implies

$$3q + 2 - k + \frac{r + 2k - 2}{6} - l\frac{r}{2} = 3q + 2 + \frac{r - 4k - 2}{6} - l\frac{r}{2} = q + l$$
$$\Rightarrow 2q = l\left(1 + \frac{r}{2}\right) - 2 + \frac{4k + 2 - r}{6}$$
$$\Rightarrow q = \frac{r+2}{4}l + \frac{4k - r + 2}{12} - 1 = \frac{3lr + 6l + 4k - r + 2}{12} - 1.$$

This leads to

$$c = 3q + 2 = k + \frac{3lr + 6l - r + 2}{4} - 1 < c + \frac{3lr + 6l - r + 2}{4} - 1$$
$$\Rightarrow \frac{3lr + 6l - r + 2}{4} - 1 > 0 \Rightarrow 3lr + 6l - r + 2 > 4$$
$$\Rightarrow 3l(r + 2) > r + 2 \Rightarrow l > \frac{1}{3} \Rightarrow l \geqslant 1,$$

which means that $v \geqslant 1$, if $l = 2v$, and $v \geqslant 0$, if $l = 2v + 1$. Since $r = 2 + 2p$, we observe that

$$q = l - 1 + \frac{3lp + 2k - p}{6}. \tag{8}$$

Subcase 3.2.1. Let $l = 2v$ for an integer v. Then (8) leads to

$$q = 2v + vp + \frac{2k - p}{6} - 1. \tag{9}$$

Subcase 3.2.1.1. Assume that $k = 3m + 1$ with $m \in \mathbb{N}_0$. Now (9) yields

$$q = 2v + vp + m + \frac{2 - p}{6} - 1,$$

and thus $p=6s+2$, $r=6+12s$ and $q=4v+6vs+m-s-1$ for an integer $s\geqslant 0$. Furthermore, we conclude that

$$k_1=\frac{r+2k-2}{6}-l\frac{r}{2}=1+2s+m-6v-12vs\geqslant 0$$
$$\Rightarrow 1\leqslant v\leqslant \frac{1+2s+m}{6+12s}=\frac{1}{6}\left(1+\frac{m}{1+2s}\right),$$

and thus $m\geqslant 5+10s$. Using Corollary 2.7 we see that D is an element of the families $\mathscr{H}_q^3$.

Subcase 3.2.1.2. Let $k=3m+2$ with $m\in\mathbb{N}_0$. Using (9) we arrive at

$$q=2v+vp+m+\frac{4-p}{6}-1,$$

and thus $p=6s+4$, $r=12s+10$ and $q=6v+6vs+m-s-1$ for an integer $s\geqslant 0$. Furthermore we observe that

$$k_1=\frac{r+2k-2}{6}-l\frac{r}{2}=2s+2+m-12vs-10v\geqslant 0\Rightarrow 1\leqslant v\leqslant \frac{2s+2+m}{10+12s},$$

which implies that $m\geqslant 10s+8$. According to Corollary 2.7, D belongs to the families $\mathscr{H}_q^4$.

Subcase 3.2.1.3. Suppose that $k=3m$ with $m\in\mathbb{N}$. Then (9) leads to

$$q=2v+vp+m-\frac{p}{6}-1,$$

and thus $p=6s$, $r=2+12s$ and $q=2v+6vs+m-s-1$ for an integer $s\geqslant 0$. Furthermore we see that

$$k_1=\frac{r+2k-2}{6}-l\frac{r}{2}=2s+m-2v-12vs\geqslant 0\Rightarrow 1\leqslant v\leqslant \frac{2s+m}{12s+2},$$

which yields that $m\geqslant 10s+2$. Using Corollary 2.7 it follows that D is a member of the families $\mathscr{H}_q^5$.

Subcase 3.2.2. Assume that $l=2v+1$ for an integer v. In this case (8) yields that

$$q=2v+vp+\frac{p+k}{3}. \tag{10}$$

Subcase 3.2.2.1. Let $k=3m$ with $m\in\mathbb{N}$. Using (10) we deduce that

$$q=2v+vp+m+\frac{p}{3},$$

which leads to $p=3s$, $r=2+6s$ and $q=2v+3vs+m+s$ for an integer $s\geqslant 0$. Furthermore it follows that:

$$k_1=\frac{r+2k-2}{6}-l\frac{r}{2}=m-2s-2v-6vs-1\geqslant 0\Rightarrow 0\leqslant v\leqslant \frac{m-2s-1}{2+6s},$$

and thus $m\geqslant 2s+1$. According to Corollary 2.7, D is an element of the families $\mathscr{H}_q^6$.

Subcase 3.2.2.2. Suppose that $k=3m+1$ with $m\in\mathbb{N}_0$. With (10) we arrive at

$$q=2v+vp+m+\frac{p+1}{3},$$

and thus $p=3s+2$, $r=6s+6$ and $q=4v+3vs+m+s+1$ for an integer $s\geqslant 0$. Furthermore we see that

$$k_1=\frac{r+2k-2}{6}-l\frac{r}{2}=m-2s-6vs-6v-2\geqslant 0\Rightarrow 0\leqslant v\leqslant \frac{1}{6}\left(\frac{m}{s+1}-2\right),$$

which implies that $m\geqslant 2s+2$. Hence, again with Corollary 2.7 we observe that D belongs to the families $\mathscr{H}_q^7$.

Subcase 3.2.2.3. Let $k = 3m + 2$ with $m \in \mathbb{N}_0$. According to (10), we have

$$q = 2v + vp + m + \frac{p+2}{3},$$

and thus $p = 3s + 1$, $r = 6s + 4$ and $q = 3v + 3vs + m + s + 1$ for an integer $s \geqslant 0$. Furthermore, we observe that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = m - 2s - 1 - 6vs - 4v \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m - 2s - 1}{4 + 6s},$$

which means that $m \geqslant 2s + 1$. Using Corollary 2.7 we conclude that D is a member of the families $\mathscr{H}_q^8$.

Subcase 3.3. Assume that $c = 3q$ with $q \in \mathbb{N}$. Let $r = 2 + 2p$ for an integer $p \geqslant 0$. If S is $(q + l)$-partite, then we conclude that

$$\begin{aligned} |S| &= (c - k + k_1)r + 2k_1 = (q + l)r + 2k_1 = \frac{cr + 2k - (r + 2)}{3} \\ &= qr + \frac{2k - 2 - r}{3} = (q + l)r + \frac{2k - 2 - r}{3} - lr. \end{aligned}$$

This implies that $c - k + k_1 = q + l$ and $k_1 = ((2k - 2 - r)/6) - l(r/2)$, and thus

$$\begin{aligned} &3q - k + \frac{2k - 2 - r}{6} - l\frac{r}{2} = 3q - \frac{4k + 2 + r}{6} - l\frac{r}{2} = q + l \\ &\Rightarrow 2q = \frac{r + 2 + 4k}{6} + l\frac{r + 2}{2} \Rightarrow q = \frac{r + 2 + 4k + 3lr + 6l}{12}. \end{aligned}$$

This leads to

$$\begin{aligned} &c = 3q = k + \frac{r + 2 + 3lr + 6l}{4} < c + \frac{r + 2 + 3lr + 6l}{4} \\ &\Rightarrow \frac{r + 2 + 3lr + 6l}{4} > 0 \Rightarrow l > -\frac{1}{3} \Rightarrow l \geqslant 0, \end{aligned}$$

which means that $v \geqslant 0$, if $l = 2v$ or $l = 2v + 1$ for an integer v. Furthermore, since $r = 2 + 2p$, we deduce that

$$q = l + \frac{2 + p + 2k + 3lp}{6}. \tag{11}$$

Subcase 3.3.1. Let $l = 2v$ for an integer v. Using (11) we see that

$$q = 2v + vp + \frac{2 + p + 2k}{6}. \tag{12}$$

Subcase 3.3.1.1. Assume that $k = 3m$ with $m \in \mathbb{N}$. According to (12), we have

$$q = 2v + vp + m + \frac{2 + p}{6},$$

and thus $p = 6s + 4$, $r = 10 + 12s$ and $q = 6v + 6vs + m + 1 + s$ for an integer $s \geqslant 0$. Furthermore it follows that:

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 2 - 2s - 10v - 12vs \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-2-2s}{10+12s},$$

which yields that $m \geqslant 2s + 2$. Using Corollary 2.7 we conclude that D belongs to the families $\mathscr{H}_q^9$.

Subcase 3.3.1.2. Suppose that $k = 3m + 1$ with $m \in \mathbb{N}_0$. Using (12) we see that

$$q = 2v + vp + m + \frac{p+4}{6},$$

and thus $p = 6s + 2$, $r = 12s + 6$ and $q = 4v + 6vs + m + s + 1$ for an integer $s \geqslant 0$. Furthermore we observe that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 2s - 1 - 12vs - 6v \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-2s-1}{6+12s},$$

which means that $m \geqslant 2s + 1$. According to Corollary 2.7, we deduce that D is an element of the families $\mathscr{H}_q^{10}$.

Subcase 3.3.1.3. Let $k = 3m + 2$ with $m \in \mathbb{N}_0$. According to (12), we arrive at

$$q = 2v + vp + m + 1 + \frac{p}{6},$$

and thus $p = 6s$, $r = 2 + 12s$ and $q = 2v + 6vs + m + 1 + s$ for an integer $s \geqslant 0$. Furthermore we conclude that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 2s - 2v - 12vs \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-2s}{2+12s},$$

which leads to $m \geqslant 2s$. Using Corollary 2.7 we observe that D is a member of the families $\mathscr{H}_q^{11}$.

Subcase 3.3.2. Assume that $l = 2v + 1$ for an integer v. According to (11), this yields

$$q = 2v + 1 + vp + \frac{1+2p+k}{3}. \tag{13}$$

Subcase 3.3.2.1. Suppose that $k = 3m$ with $m \in \mathbb{N}$. Using (13) we observe that

$$q = 2v + 1 + vp + m + \frac{1+2p}{3},$$

and thus $p = 3s + 1$, $r = 4 + 6s$ and $q = 3v + 2 + 3vs + m + 2s$. Furthermore we see that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 3 - 4s - 4v - 6vs \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-3-4s}{4+6s},$$

which leads to $m \geqslant 3 + 4s$. According to Corollary 2.7, D belongs to the families $\mathscr{H}_q^{12}$.

Subcase 3.3.2.2. Let $k = 3m + 1$ with $m \in \mathbb{N}_0$. Using (13) we have

$$q = 2v + 1 + vp + m + \frac{2p+2}{3},$$

and thus $p = 3s + 2$, $r = 6s + 6$ and $q = 4v + 3vs + 2s + m + 3$ for an integer $s \geqslant 0$. Furthermore we see that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 4 - 4s - 6v - 6vs \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-4-4s}{6+6s},$$

which yields that $m \geqslant 4 + 4s$, According to Corollary 2.7, it follows that D is an element of the families $\mathscr{H}_q^{13}$.

Subcase 3.3.2.3. Assume that $k = 3m + 2$ with $m \in \mathbb{N}_0$. Using (13) we observe that

$$q = 2v + 2 + vp + m + \frac{2p}{3},$$

and thus $p = 3s$, $r = 6s + 2$ and $q = 2v + 2 + 3vs + m + 2s$. Furthermore it follows that:

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m - 4s - 2v - 6vs - 1 \geqslant 0 \Rightarrow 0 \leqslant v \leqslant \frac{m-4s-1}{2+6s},$$

which means that $m \geqslant 4s + 1$. According to Corollary 2.7, we conclude that D belongs to the families $\mathscr{H}_q^{14}$. This completes the proof of the theorem. □

References

[1] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, London, 2000.
[2] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems, and algorithms: a survey, J. Graph Theory 19 (1995) 481–505.
[3] M. Tewes, L. Volkmann, A. Yeo, Almost all almost regular c-partite tournaments with $c \geqslant 5$ are vertex pancyclic, Discrete Math. 242 (2002) 201–228.
[4] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory B 28 (1980) 142–163.
[5] L. Volkmann, Cycles in multipartite tournaments: results and problems, Discrete Math. 245 (2002) 19–53.
[6] L. Volkmann, On the connectivity of regular multipartite tournaments, 2002 (unpublished manuscript).
[7] L. Volkmann, Complementary cycles in regular multipartite tournaments, where one cycle has length 4, Kyungpook Math. J. 44 (2) (2004) 219–247.
[8] L. Volkmann, S. Winzen, Cycles through a given arc in almost regular multipartite tournaments, Austral. J. Combin. 27 (2003) 223–245.
[9] S. Winzen, Close to regular multipartite tournaments, Ph. D. Thesis, RWTH Aachen University, 2004.
[10] A. Yeo, Semicomplete multipartite digraphs, Ph. D. Thesis, Odense University, 1998.
[11] A. Yeo, How close to regular must a semicomplete multipartite digraph be to secure Hamiltonicity?, Graphs Combin. 15 (1999) 481–493.