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The Stability of Difference Schemes of Second-Order of Accuracy for Hyperbolic-Parabolic Equations

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Abstract—A nonlocal boundary value problem for hyperbolic-parabolic equations in a Hilbert space H is considered. Difference schemes of second order of accuracy difference schemes for approximate solution of this problem are presented. Stability estimates for the solution of these difference schemes are established.

Keywords—Abstract hyperbolic-parabolic problem, Cosine operator-function, Hilbert spaces, Hyperbolic-parabolic equation, Difference schemes, Stability inequality, Order of accuracy.

1. INTRODUCTION

It is known (see, for example, [1–4]) that various boundary value problems for hyperbolic-parabolic equations can be reduced to a nonlocal boundary value problem for differential equations of mixed type,

$$
\begin{aligned}
&\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \qquad (0 \le t \le 1), \quad u(-1) = \alpha u(1) + \varphi, \\
&\frac{du(t)}{dt} + Au(t) = g(t) \qquad (-1 \le t \le 0), \quad 0 \le \alpha \le 1,
\end{aligned}
\tag{1.1}
$$

in a Hilbert space H with self-adjoint positive definite operator A.

THEOREM 1.1. *(See [5].) Suppose that $\varphi \in D(A)$, that $f(t)$ is a continuously differentiable on $[0,1]$ function and that $g(t)$ is a continuously differentiable on $[-1,0]$ function. Then, there is a unique solution $u(t)$ of the problem (1.1) and the stability inequalities,*

$$
\max_{-1\le t\le 1} \|u(t)\|_H \le M \left[\|\varphi\|_H + \max_{-1\le t\le 0} \|g(t)\|_H + \max_{0\le t\le 1} \left\| (\sqrt{A})^{-1} f(t) \right\|_H \right],
$$

$$
\max_{-1\le t\le 1} \left\| \sqrt{A} u(t) \right\|_H \le M \left[\left\| \sqrt{A}\varphi \right\|_H + \|g(0)\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H + \max_{0\le t\le 1} \|f(t)\|_H \right],
$$

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$$\max_{-1\le t\le 0}\left\|\frac{du(t)}{dt}\right\|_H+\max_{0\le t\le 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H+\max_{-1\le t\le 1}\|Au(t)\|_H$$
$$\le M\left[\|A\varphi\|_H+\left\|\sqrt{A}g(0)\right\|_H+\max_{-1\le t\le 0}\|g'(t)\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H\right]$$

hold, where M does not depend on $f(t)$, $g(t)$, or φ.

Methods for the numerical solution of boundary value problems for evolution differential equations have been studied extensively by many researchers (see [5–16] and the references therein). In the present paper, difference schemes of second order of accuracy for the approximate solution of boundary value problem (1.1) are presented. Stability estimates for the solution of these difference schemes are established.

2. THE DIFFERENCE SCHEMES— THE MAIN THEOREM

Applying difference schemes of second order of accuracy for hyperbolic equations (see [16]) and modified Crank-Nicholson difference schemes for parabolic equations (see [17–19]), we obtain the following difference schemes of second order of accuracy

$$\begin{gathered}
\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k,\\
f_k=f(t_k),\qquad t_k=k\tau,\qquad 1\le k\le N-1,\qquad f_0=f(0),\\
\tau^{-1}\left(I+\frac{\tau^2}{4}A\right)(u_1-u_0)=\frac{\tau}{2}(f_0-Au_0)+(g_0-Au_0),\qquad g_0=g(0),\\
\tau^{-1}(u_k-u_{k-1})+\frac{1}{2}(Au_k+Au_{k-1})=g_{k-1},\\
g_{k-1}=g\left(t_k-\frac{\tau}{2}\right),\qquad t_k=k\tau,\quad -(N-r-1)\le k\le 0,\\
\tau^{-1}(u_k-u_{k-1})+Au_k=g_{k-1},\qquad g_{k-1}=g\left(t_k-\frac{\tau}{2}\right),\\
t_k=k\tau,\qquad -N+1\le k\le -(N-r),\qquad r=2,\dots,u_{-N}=\alpha u_N+\varphi,
\end{gathered}\tag{2.1}$$

for the approximate solution of the boundary value problem (1.1).

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A\ge\delta I$, where $\delta>\delta_0>0$. Throughout this paper, $\{c(t),t\ge0\}$ is a strongly continuous cosine operator-function defined by the formula,

$$c(t)=\frac{e^{it\sqrt{A}}+e^{-it\sqrt{A}}}{2}.$$

Then, from the definition of the sine operator-function $s(t)$

$$s(t)u=\int_0^t c(s)u\,ds,$$

it follows that

$$s(t)=(\sqrt{A})^{-1}\frac{e^{it\sqrt{A}}-e^{-it\sqrt{A}}}{2i}.$$

For the theory of cosine operator-function, we refer to [20,21]. First of all, let us prove some lemmas that will be needed below.

LEMMA 2.1. *The following estimate holds:*

$$\left\|\left[B^N\left(\pm\tau\sqrt{A}\right)-\exp\left\{\mp i\sqrt{A}\right\}\right]A^{-1}\right\|\le\frac{\tau}{4}.\tag{2.2}$$

Here and below,

$$B\left(\pm\tau\sqrt{A}\right) = \left(I \mp \frac{1}{2}i\tau\sqrt{A}\right)\left(I \pm \frac{1}{2}i\tau\sqrt{A}\right)^{-1}$$

PROOF. We use the identity

$$B^N\left(\pm\tau\sqrt{A}\right) - \exp\left\{\mp i\sqrt{A}\right\} = \int_0^1 \Psi'\left(s\tau\sqrt{A}\right)\,ds,$$

where

$$\Psi\left(s\tau\sqrt{A}\right) = B^N\left(\pm s\tau\sqrt{A}\right)\exp\left\{\mp i\,(1-s)\sqrt{A}\right\}.$$

The derivative $\Psi'(s\tau\sqrt{A})$ is given by

$$\Psi'\left(s\tau\sqrt{A}\right) = B^{N-1}\left(\pm s\tau\sqrt{A}\right)\left(\mp i\sqrt{A}\right)$$
$$\times\left(-\frac{1}{4}\tau^2 s^2 A\right)\left(I \pm \frac{1}{2}is\tau\sqrt{A}\right)^{-2}\exp\left\{\mp i(1-s)\sqrt{A}\right\}.$$

Thus,

$$B^N\left(\pm\tau\sqrt{A}\right) - \exp\left\{\mp i\sqrt{A}\right\}$$
$$= \pm\int_0^1 B^{N-1}\left(\pm s\tau\sqrt{A}\right) i\left(\sqrt{A}\right)^3 \frac{1}{4}\tau^2 s^2\left(I \pm \frac{1}{2}is\tau\sqrt{A}\right)^{-2}\exp\left\{\mp i\,(1-s)\sqrt{A}\right\}\,ds.$$

Using the last identity and the estimates,

$$\left\|B\left(\pm\tau s\sqrt{A}\right)\right\| \le 1, \qquad \left\|\exp\left\{\mp i(1-s)\sqrt{A}\right\}\right\| \le 1, \tag{2.3}$$
$$\left\|is\sqrt{A}\frac{1}{2}\tau\left(I \pm \frac{1}{2}is\tau\sqrt{A}\right)^{-2}\right\| \le \frac{1}{2},$$

we obtain

$$\left\|\left[B^N\left(\pm\tau\sqrt{A}\right) - \exp\left\{\mp i\sqrt{A}\right\}\right]A^{-1}\right\| \le \frac{\tau}{2}\int_0^1 \left\|B^{N-1}\left(\pm s\tau\sqrt{A}\right)\right\|$$
$$\times\left\|is\sqrt{A}\frac{1}{2}\tau\left(I \pm \frac{1}{2}is\tau\sqrt{A}\right)^{-2}\right\|\left\|\exp\left\{\mp i(1-s)\sqrt{A}\right\}\right\| s\,ds \le \frac{1}{2}\tau\int_0^1 s\,ds = \frac{\tau}{4}. \blacksquare$$

LEMMA 2.2. *For any $r = 2, 3, \dots$, the following estimate holds:*

$$\left\|\sqrt{A}\left[B^{N-r}R^r - \exp\{-A\}\right]\right\| \le M\sqrt{\tau}, \tag{2.4}$$

where M does not depend on τ. Here and below,

$$R = (I + \tau A)^{-1}, \qquad B = \left(I - \frac{\tau A}{2}\right)C, \qquad C = \left(I + \frac{\tau A}{2}\right)^{-1}$$

PROOF. Since

$$\sqrt{A}\left[B^{N-r}R^r - \exp\{-A\}\right] = \sqrt{A}\left[B^{N-r} - \exp\{-(N-r)\tau A\}\right]R^r$$
$$+\sqrt{A}\exp\{-(N-r)\tau A\}\left[R^r - \exp\{-r\tau A\}\right],$$

to prove (2.4) it suffices to establish the estimates,

$$\left\|\sqrt{A}\exp\{-(N-r)\tau A\}\left[R^r-\exp\{-r\tau A\}\right]\right\|\le M\tau, \tag{2.5}$$

$$\left\|\sqrt{A}\left[B^{N-r}-\exp\{-(N-r)\tau A\}\right]R^r\right\|\le M\sqrt{\tau}. \tag{2.6}$$

Let us establish the estimate (2.5). Using the identity

$$R^r-\exp\{-r\tau A\}=\int_0^1 r\tau^2 sA^2(I+s\tau A)^{-(r+1)}\exp\{-r(1-s)\tau A\}\,ds$$

and the estimates

$$\left\|\sqrt{A}(I+s\tau A)^{-(r+1)}\right\|\le\frac{1}{\sqrt{rs\tau}},\qquad \left\|A^\beta\exp\{-tA\}\right\|\le\frac{1}{t^\beta},\quad 0\le\beta\le 1,\quad t>0, \tag{2.7}$$

we obtain

$$\left\|\sqrt{A}\exp\{-(N-r)\tau A\}\left[R^r-\exp\{-r\tau A\}\right]\right\|\le\left\|\left(\sqrt{A}\right)^3\exp\{-(N-r)\tau A\}\right\|$$
$$\times\int_0^1 r\tau^2 s\left\|\sqrt{A}(I+s\tau A)^{-(r+1)}\right\|\left\|\sqrt{A}\exp\{-r(1-s)\tau A\}\right\|\,ds\le M(r)\tau.$$

Now, let us obtain the estimate (2.6). We have that

$$B^{N-r}-\exp\{-(N-r)\tau A\}=-\int_0^1(N-r)\tau^3\frac{s}{4}A^3\left\{\left(I-\frac{s\tau}{2}A\right)\left(I+\frac{s\tau}{2}A\right)^{-1}\right\}^{N-r-1}$$
$$\times\left(I+\frac{s\tau}{2}A\right)^{-2}\exp\{-(N-r)(1-s)\tau A\}\,ds.$$

Therefore,

$$\sqrt{A}\left[B^{N-r}-\exp\{-(N-r)\tau A\}\right]R^r=-\left\{\int_0^{1/2}+\int_{1/2}^1\right\}A^3\sqrt{A}\left\{\left(I-\frac{s\tau}{2}A\right)\left(I+\frac{s\tau}{2}A\right)^{-1}\right\}^{N-r-1}$$
$$\times(N-r)\tau^3\frac{s}{4}\left(I+\frac{s\tau}{2}A\right)^{-2}\exp\{-(N-r)(1-s)\tau A\}\,dsR^r=J_1+J_2.$$

The proof of the estimate (2.6) is based on the estimates,

$$\|J_1\|\le M\tau^2,\qquad \|J_2\|\le M\sqrt{\tau}.$$

The proof of these estimates is based on the estimate (2.7) and

$$\left\|A^\beta B^k C\right\|\le 1,\qquad 0\le\beta\le\frac{1}{2},\quad 1\le k\le N. \tag{2.8}$$

∎

LEMMA 2.3. *For any $0\le\alpha\le 1$ and $r=2,3,\dots$, one has the estimate*

$$\|T\|\le M, \tag{2.9}$$

where M does not depend on τ and α. Here and below,

$$T=\left\{I-\alpha\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)\right.\right.$$
$$\left.\left.+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right]B^{N-r}R^r\right\}^{-1}$$

PROOF. Since

$$T-\{I-\alpha\left[c(1)-As(1)\right]\exp\{-A\}\}^{-1}$$
$$=\alpha\left\{\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right]\right.$$
$$\left.\times B^{N-r}R^r-\left[c(1)-As(1)\right]\exp\{-A\}\right\}T\{I-\alpha\left[c(1)-As(1)\right]\exp\{-A\}\}^{-1}$$

and

$$\left\|\{I-\alpha\left[c(1)-As(1)\right]\exp\{-A\}\}^{-1}\right\|\leq M, \tag{2.10}$$

to prove (2.9), it suffices to establish the estimate

$$\left\|\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right.$$
$$\left.\times B^{N-r}R^r-\left[c\left(1\right)-As\left(1\right)\right]\exp\left\{-A\right\}\right\|\leq M\sqrt{\tau}. \tag{2.11}$$

The estimate (2.10) was proved in [4]. Finally, using the identity,

$$\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right]$$
$$\times B^{N-r}R^r-\left[c\left(1\right)-As\left(1\right)\right]\exp\left\{-A\right\}=\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)\right.$$
$$\left.+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right]\left(\sqrt{A}\right)^{-1}\sqrt{A}\exp\left\{-A\right\}$$
$$+\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right)+B^N\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right)-B^N\left(\tau\sqrt{A}\right)\right)\right]$$
$$\times\left(\sqrt{A}\right)^{-1}\sqrt{A}\left[B^{N-r}R^r-\exp\left\{-A\right\}\right],$$

and the estimates (2.2), (2.3), and (2.4), we obtain the estimate (2.11). ∎

THEOREM 2.1. *Let $\varphi\in D(A)$, $g_0\in D(\sqrt{A})$. Then, there is a unique solution $u_k, -N\leq k\leq N$ of the difference schemes (2.1) and the stability inequalities,*

$$\max_{-N\leq k\leq N}\|u_k\|_H\leq M\left[\max_{0\leq k\leq N-1}\left\|(\sqrt{A})^{-1}f_k\right\|_H+\max_{-N\leq k\leq 0}\|g_k\|_H+\|\varphi\|_H\right], \tag{2.12}$$

$$\max_{-N\leq k\leq N}\left\|\sqrt{A}u_k\right\|_H$$
$$\leq M\left[\max_{0\leq k\leq N-1}\|f_k\|_H+\left\|\sqrt{A}\varphi\right\|_H+\|g_0\|_H+\max_{-N+1\leq k\leq 0}\|g_k-g_{k-1}\|_H\right], \tag{2.13}$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\|_H+\max_{-(N-1)\le k\le 0}\left\|\tau^{-1}(u_k-u_{k-1})\right\|_H$$

$$\le M\left[\max_{1\le k\le N-1}\|f_k-f_{k-1}\|_H+\|f_0\|_H\right. \tag{2.14}$$

$$\left.+\left\|\sqrt{A}g_0\right\|_H+\|A\varphi\|_H+\max_{-N+1\le k\le 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right],$$

hold, where M does not depend on f_k, $0\le k\le N-1$, g_k, $-N\le k\le 0$ or φ.

PROOF. We obtain the formula for the solution of the difference schemes (2.1). First, we consider two auxiliary problems

$$\begin{gathered}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k,\\ f_k=f(t_k),\qquad t_k=k\tau,\qquad 1\le k\le N-1,\\ \tau^{-1}\left(I+\frac{\tau^2}{4}A\right)(u_1-u_0)=Z_1,\qquad u_0\text{ is given},\\ Z_1=\frac{\tau}{2}\left(f_0-Au_0\right)+\left(g_0-Au_0\right),\qquad f_0=f(0),\quad g_0=g(0),\end{gathered} \tag{2.15}$$

$$\begin{gathered}\tau^{-1}\left(u_k-u_{k-1}\right)+\frac{1}{2}\left(Au_k+Au_{k-1}\right)=g_{k-1},\\ g_{k-1}=g\left(t_k-\frac{\tau}{2}\right),\qquad t_k=k\tau,\quad -(N-r-1)\le k\le 0,\\ \tau^{-1}\left(u_k-u_{k-1}\right)+Au_k=g_{k-1},\qquad u_{-N}\text{ is given},\\ g_{k-1}=g\left(t_k-\frac{\tau}{2}\right),\qquad t_k=k\tau,\quad -N+1\le k\le -(N-r),\quad r=2,\dots.\end{gathered} \tag{2.16}$$

There are unique solutions of these problems and the following formulas hold:

$$\begin{aligned}u_1&=\left(I+\frac{\tau^2}{4}A\right)^{-1}\left[\left(I-\frac{\tau^2}{4}A-\tau A\right)u_0+\tau g_0+\frac{\tau^2}{2}f_0\right],\\ u_k&=\left[\frac{1}{2}\left(B^k\left(\tau\sqrt{A}\right)+B^k\left(-\tau\sqrt{A}\right)\right)\right.\\ &\quad\left.+i\sqrt{A}\frac{1}{2}\left(B^k\left(-\tau\sqrt{A}\right)-B^k\left(\tau\sqrt{A}\right)\right)\right]u_0\\ &\quad-\sum_{s=1}^{k-1}\frac{\tau}{2i}\left(\sqrt{A}\right)^{-1}\left[B^{k-s}\left(\tau\sqrt{A}\right)-B^{k-s}\left(-\tau\sqrt{A}\right)\right]f_s,\qquad 2\le k\le N,\end{aligned} \tag{2.17}$$

$$\begin{aligned}u_k&=R^{N+k}u_{-N}+\tau\sum_{s=-N}^{k-1}R^{k-s}g_s, && -N+1\le k\le -N+r,\\ u_k&=B^{N-r+k}\left[R^ru_{-N}+\tau\sum_{s=-N}^{-N+r-1}R^{-N+r-s}g_s\right]\\ &\quad+\tau\sum_{s=-N+r}^{k-1}B^{k-1-s}Cg_s, && -N+r+1\le k\le 0.\end{aligned} \tag{2.18}$$

Second, using the formulas (2.17) and (2.18), we obtain the following formula,

$$u_1 = \left(I + \frac{\tau^2}{4}A\right)^{-1}\left[\left(I - \frac{\tau^2}{4}A - \tau A\right)\left[B^{N-r}\left[R^r u_{-N} + \tau\sum_{s=-N}^{-N+r-1} R^{-N+r-s}g_s\right]\right.\right.$$
$$\left.\left. + \tau\sum_{s=-N+r}^{-1} B^{-1-s}Cg_s\right] + \tau g_0 + \frac{\tau^2}{2}f_0\right],$$

$$u_k = \left[\frac{1}{2}\left(B^k\left(\tau A^{1/2}\right) + B^k\left(-\tau A^{1/2}\right)\right) + i\sqrt{A}\frac{1}{2}\left(B^k(-\tau A^{1/2}) - B^k(\tau A^{1/2})\right)\right]$$
$$\left[B^{N-r}\left[R^r u_{-N} + \tau\sum_{s=-N}^{-N+r-1} R^{-N+r-s}g_s\right] + \tau\sum_{s=-N+r}^{-1} B^{-1-s}Cg_s\right] \tag{2.19}$$
$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}\left(\sqrt{A}\right)^{-1}\left[B^{k-s}\left(\tau\sqrt{A}\right) - B^{k-s}\left(-\tau\sqrt{A}\right)\right]f_s, \qquad 2 \le k \le N,$$

$$u_k = R^{N+k}u_{-N} + \tau\sum_{s=-N}^{k-1} R^{k-s}g_s, \qquad -N+1 \le k \le -N+r,$$

$$u_k = B^{N-r+k}\left[R^r u_{-N} + \tau\sum_{s=-N}^{-N+r-1} R^{-N+r-s}g_s\right] + \tau\sum_{s=-N+r}^{k-1} B^{k-1-s}Cg_s, \qquad -N+r+1 \le k \le 0,$$

for the solution of the problem,

$$\begin{gathered}
\tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + \frac{1}{2}Au_k + \frac{1}{4}A\left(u_{k+1} + u_{k-1}\right) = f_k,\\
f_k = f\left(t_k\right), \qquad t_k = k\tau, \qquad 1 \le k \le N-1,\\
\tau^{-1}\left(I + \frac{\tau^2}{4}A\right)\left(u_1 - u_0\right) = Z_1,\\
Z_1 = \frac{\tau}{2}\left(f_0 - Au_0\right) + \left(g_0 - Au_0\right), \qquad f_0 = f(0), \qquad g_0 = g(0),\\
\tau^{-1}\left(u_k - u_{k-1}\right) + \frac{1}{2}\left(Au_k + Au_{k-1}\right) = g_{k-1},\\
g_{k-1} = g\left(t_k - \frac{\tau}{2}\right), \qquad t_k = k\tau, \qquad -(N-r-1) \le k \le 0,\\
\tau^{-1}\left(u_k - u_{k-1}\right) + Au_k = g_{k-1}, \qquad u_{-N} \text{ is given},\\
g_{k-1} = g\left(t_k - \frac{\tau}{2}\right), \qquad t_k = k\tau, \qquad -N+1 \le k \le -(N-r), \qquad r = 2, \dots.
\end{gathered} \tag{2.20}$$

Third, using the condition $u_{-N} = \alpha u_N + \varphi$, we obtain

$$\left(I - \alpha\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right) + B^N\left(-\tau\sqrt{A}\right)\right)\right.\right.$$
$$\left.\left. + i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right) - B^N\left(\tau\sqrt{A}\right)\right)\right]B^{N-r}R^r\right)u_{-N}$$
$$= \alpha\left[\frac{1}{2}\left(B^N\left(\tau\sqrt{A}\right) + B^N\left(-\tau\sqrt{A}\right)\right) + i\sqrt{A}\frac{1}{2}\left(B^N\left(-\tau\sqrt{A}\right) - B^N\left(\tau\sqrt{A}\right)\right)\right]$$
$$\times\left[B^{N-r}\tau\sum_{s=-N}^{-N+r-1} R^{-N+r-s}g_s + \tau\sum_{s=-N+r}^{-1} B^{-1-s}Cg_s\right]$$

$$+\alpha/(2i)\left(\sqrt{A}\right)^{-1}\left(B^{N}\left(-\tau\sqrt{A}\right)-B^{N}(\tau\sqrt{A})\right)\left(g_{0}+\frac{\tau}{2}f_{0}\right)$$
$$-\sum_{s=1}^{N-1}\alpha\frac{\tau}{2i}\left(\sqrt{A}\right)^{-1}\left(B^{N-s}\left(\tau\sqrt{A}\right)-B^{N-s}\left(-\tau\sqrt{A}\right)\right)f_{s}+\varphi.$$

Since the operator

$$I-\frac{\alpha}{2}\left[\left(B^{N}\left(\tau\sqrt{A}\right)+B^{N}\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\left(B^{N}\left(-\tau\sqrt{A}\right)-B^{N}\left(\tau\sqrt{A}\right)\right)\right]B^{N-r}R^{r}$$

has an inverse, we have

$$\begin{aligned}u_{-N}=T\Bigg\{&\alpha\left[\frac{1}{2}\left(B^{N}\left(\tau\sqrt{A}\right)+B^{N}\left(-\tau\sqrt{A}\right)\right)+i\sqrt{A}\frac{1}{2}\left(B^{N}\left(-\tau\sqrt{A}\right)-B^{N}\left(\tau\sqrt{A}\right)\right)\right]\\&\left[B^{N-r}\tau\sum_{s=-N}^{-N+r-1}R^{-N+r-s}g_{s}+\tau\sum_{s=-N+r}^{-1}B^{-1-s}Cg_{s}\right]\\&+\alpha/(2i)\left(\sqrt{A}\right)^{-1}\left(B^{N}\left(-\tau\sqrt{A}\right)-B^{N}\left(\tau\sqrt{A}\right)\right)\left(g_{0}+\frac{\tau}{2}f_{0}\right)\\&-\sum_{s=1}^{N-1}\alpha\frac{\tau}{2i}\left(\sqrt{A}\right)^{-1}\left(B^{N-s}\left(\tau\sqrt{A}\right)-B^{N-s}\left(-\tau\sqrt{A}\right)\right)f_{s}+\varphi\Bigg\}.\end{aligned}\tag{2.21}$$

Hence, we have formulas (2.19) and (2.21), for the solution of the difference schemes (2.1). It is easy to show that the following estimate holds,

$$\max_{-N\leq k\leq N}\|u_{k}\|_{H}\leq M\left[\|u_{-N}\|_{H}+\max_{0\leq k\leq N-1}\left\|(\sqrt{A})^{-1}f_{k}\right\|_{H}+\max_{-N\leq k\leq 0}\|g_{k}\|_{H}+\|\varphi\|_{H}\right].\tag{2.22}$$

Using the formula (2.21) and the estimates (2.3), (2.7), (2.8), and (2.9), we obtain

$$\begin{aligned}\|u_{-N}\|_{H}\leq\|T\|\Bigg\{&\alpha\left[\left\|\frac{1}{2}\left(B^{N}\left(\tau\sqrt{A}\right)+B^{N}\left(-\tau\sqrt{A}\right)\right)\right\|\left\|(\sqrt{A})^{-1}\right\|\right.\\&\left.+\left\|\frac{1}{2}\left(B^{N}\left(-\tau\sqrt{A}\right)-B^{N}\left(\tau\sqrt{A}\right)\right)\right\|\right]\left[\tau\sum_{s=-N}^{-N+r-1}\left\|\sqrt{A}B^{N-r}R^{-N+r-s}\right\|\right.\\&\left.\times\|g_{s}\|_{H}+\tau\sum_{s=-N+r}^{-1}\left\|\sqrt{A}B^{-1-s}C\right\|\|g_{s}\|_{H}\right]\\&+\alpha/2\left\|B^{N}\left(-\tau\sqrt{A}\right)-B^{N}\left(\tau\sqrt{A}\right)\right\|\left[\left\|\left(\sqrt{A}\right)^{-1}\right\|\|g_{0}\|_{H}+\frac{\tau}{2}\left\|\left(\sqrt{A}\right)^{-1}f_{0}\right\|_{H}\right]\\&+\sum_{s=1}^{N-1}\alpha\frac{\tau}{2}\left\|B^{N-s}\left(\tau\sqrt{A}\right)-B^{N-s}\left(-\tau\sqrt{A}\right)\right\|\left\|\left(\sqrt{A}\right)^{-1}f_{s}\right\|_{H}+\|\varphi\|_{H}\Bigg\}\\\leq M&\left[\max_{0\leq k\leq N-1}\left\|\left(\sqrt{A}\right)^{-1}f_{k}\right\|_{H}+\max_{-N\leq k\leq 0}\|g_{k}\|_{H}+\|\varphi\|_{H}\right].\end{aligned}$$

From the last estimate and estimate (2.22) estimate (2.12) follows. The proofs of estimates (2.13) and (2.14) follow the scheme of estimate (2.12) and rely on estimates (2.3), (2.7), (2.9), and formula (2.21) and

$$\tau \sum_{s=-N}^{-N+r-1} B^{N-r} R^{-N+r-s} g_s = A^{-1} \left\{ B^{N-r} g_{-N+r-1} - B^{N-r} R^r g_{-N} + \sum_{s=-N+1}^{-N+r-1} B^{N-r} R^{-N+r-s} \left[g_{s-1} - g_s\right] \right\},$$

$$\tau \sum_{s=-N+r}^{-1} B^{-1-s} C g_s = A^{-1} \left\{ g_{-1} - B^{N-r} g_{-N+r} + \sum_{s=-N+r+1}^{-1} B^{-s} \left[g_{s-1} - g_s\right] \right\},$$

$$\sum_{s=1}^{N-1} \frac{\tau i}{2} \left(\sqrt{A}\right)^{-1} B^{N-s} \left(\pm\tau\sqrt{A}\right) f_s$$

$$= \pm A^{-1} \left(I \pm \frac{i\tau\sqrt{A}}{2} \right) \left\{ f_{N-1} - B^{N-1}\left(\pm\tau\sqrt{A}\right) f_1 + \sum_{s=2}^{N-1} B^{N-s} \left(\pm\tau\sqrt{A}\right) \left[f_{s-1} - f_s\right] \right\}. \quad \blacksquare$$

Note that the abstract Theorem 2.1 permits us to obtain the stability of the difference schemes for the approximate solution of boundary value problems of mixed type.

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