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Toroidalization of locally toroidal morphisms from N-folds to surfaces

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ABSTRACT

The toroidalization conjecture of D. Abramovich, K. Karu, K. Matsuki, and J. Wlodarczyk asks whether any given morphism of nonsingular varieties over an algebraically closed field of characteristic zero can be modified into a toroidal morphism. Following a suggestion by Dale Cutkosky, we define the notion of *locally toroidal* morphisms and ask whether any locally toroidal morphism can be modified into a toroidal morphism. In this paper, we answer the question in the affirmative when the morphism is between any arbitrary variety and a surface.

1. Introduction

Fix an algebraically closed field k of characteristic 0. A variety is an open subset of an irreducible proper k-scheme.

A *simple normal crossing* (SNC) divisor on a nonsingular variety is a divisor D on X, all of whose irreducible components are nonsingular and whenever r irreducible components $Z_1, \ldots, Z_r$ of D meet at a point p, then local equations $x_1, \ldots, x_r$ of Z_i form part of a regular system of parameters in $\mathcal{O}_{X,p}$.

If D is a SNC divisor and a point $p \in D$ belongs to exactly k components of D, then we say that p is a k point.

A *toroidal structure* on a nonsingular variety X is a SNC divisor D_X.

The divisor D_X specifies a *toric* chart (V_p, σ_p) at every closed point $p \in X$ where $p \in V_p \subset X$ is an open neighborhood and $\sigma_p : V_p \to X_p$ is an étale morphism to a toric variety X_p such that under σ_p the ideal of D_X at p corresponds to the ideal of the complement of the torus in X_p.

The idea of a toroidal structure is fundamental to algebraic geometry. It is developed in the classic book "Toroidal Embeddings I" [9] by G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat.

Definition 1.1 (*[9,1]*). Suppose that D_X and D_Y are toroidal structures on X and Y respectively. Let $p \in X$ be a closed point. A dominant morphism $f : X \to Y$ is *toroidal at* p (*with respect to the toroidal structures* D_X *and* D_Y) if the germ of f at p is formally isomorphic to a toric morphism between the toric charts at p and $f(p)$. f is *toroidal* if it is toroidal at all closed points in X.

A nonsingular subvariety V of X is a *possible center* for D_X if $V \subset D_X$ and V intersects D_X transversally. That is, V makes *SNCs* with D_X, as defined before Lemma 2.3. The blowup $\pi : X_1 \to X$ of a possible center is called a possible blowup. $D_{X_1} = \pi^{-1}(D_X)$ is then a toroidal structure on X_1.

Let *Sing*(f) be the set of points p in X where f is not smooth. It is a closed set.

The following "toroidalization conjecture" is the strongest possible general structure theorem for morphisms of varieties.

Conjecture 1.2. *Suppose that $f : X \longrightarrow Y$ is a dominant morphism of nonsingular varieties. Suppose also that there is a SNC divisor D_Y on Y such that $D_X = f^{-1}(D_Y)$ is a SNC divisor on X which contains the singular locus, Sing(f), of the map f.*

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Then there exists a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

where π, π_1 are possible blowups for the preimages of D_Y and D_X respectively, such that f_1 is toroidal with respect to $D_{Y_1} = \pi^{-1}(D_Y)$ and $D_{X_1} = {\pi_1}^{-1}(D_X)$

A slightly weaker version of the conjecture is stated in the paper [2] of D. Abramovich, K. Karu, K. Matsuki, and J. Wlodarczyk.

When Y is a curve, this conjecture follows easily from embedded resolution of hypersurface singularities, as shown in the introduction of [5]. The case when X and Y are surfaces has been known before (see Corollary 6.2.3 [2], [3,7]). The case when X has dimension 3 is completely resolved by Dale Cutkosky in [5,6]. A special case of $\dim(X)$ arbitrary and $\dim(Y) = 2$ is done in [8].

For detailed history and applications of this conjecture, see [6].

A related but weaker question asked by Dale Cutkosky is the following Question 1.4.

To state the question we need the following definition.

Definition 1.3. Let $f : X \to Y$ be a dominant morphism of nonsingular varieties. Suppose that the following are true.

1. There exist open coverings $\{U_1, \dots, U_m\}$ and $\{V_1, \dots, V_m\}$ of X and Y respectively such that the morphism f restricted to U_i maps into V_i for all $i = 1, \dots, m$.
2. There exist simple normal crossings divisors D_i and E_i in U_i and V_i respectively such that $f^{-1}(E_i) \cap U_i = D_i$ and $Sing(f|_{U_i}) \subset D_i$ for all $i = 1, \dots, m$.
3. The restriction of f to $U_i, f|_{U_i} : U_i \to V_i$, is toroidal with respect to D_i and E_i for all $i = 1, \dots, m$.

Then we say that f is *locally toroidal* with respect to the open coverings U_i and V_i and SNC divisors D_i and E_i.

For the remainder, when we say "f is locally toroidal", it is to be understood that f is locally toroidal with respect to the open coverings U_i and V_i and SNC divisors D_i and E_i as in the definition. We will usually not mention U_i, V_i, D_i and E_i.

We have the following.

Question 1.4. *Suppose that $f : X \longrightarrow Y$ is locally toroidal. Does there exist a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

where π, π_1 are blowups of nonsingular varieties such that there exist SNC divisors E, D on Y_1 and X_1 respectively such that $Sing(f_1) \subset D$, ${f_1}^{-1}(E) = D$ and f_1 is toroidal with respect to E and D?

The aim of this paper is to give a positive answer to this question when Y is a surface and X is arbitrary. The result is proved in Theorem 4.2.

Brief outline of the proof:

The core results (Theorems 4.1 and 4.2) are proved in Section 4. Sections 2 and 3 consist of preparatory material.

Let $f : X \to Y$ be a locally toroidal morphism with the notation as in Definition 1.3. The essential observation is this: if there is a SNC divisor E on Y such that $E_i \subset E$ for all i, then f is toroidal with respect to E and $f^{-1}(E)$. A proof of this observation is contained in the proof of Theorem 4.2.

The main task, then, is to construct the divisor E. This is not hard: consider the divisor $E' = \bar{E_1} + \cdots + \bar{E_m}$ where $\bar{E_i}$ is the Zariski closure of E_i in Y. By embedded resolution of singularities, there exists a finite sequence of blowups of points $\pi : Y_1 \to Y$ such that $\pi^{-1}(E')$ is a SNC divisor on Y_1.

The problem now reduces to constructing a sequence of blowups $\pi_1 : X_1 \to X$ such that there is a locally toroidal morphism $f_1 : X_1 \to Y_1$. This is done in Theorem 4.1.

Sections 2 and 3 prepare the ground for Theorem 4.1.

Given the sequence of blowups $\pi : Y_1 \to Y$ as above, there exist principalization algorithms which give a sequence of blowups $\pi_1 : X_1 \to X$ so that there exists a morphism $f_1 : X_1 \to Y_1$. The main difficulty we face is that such a morphism f_1

may not be locally toroidal. So a blanket appeal to existing principalizing algorithms can not be made. In Sections 2 and 3, we construct a specific algorithm that works in our situation.

Section 2 deals with the blowups that preserve the local toroidal structure. We call these *permissible blowups* (Definition 2.4). The main result of Section 2 is Lemma 2.5 which analyzes the effect of a permissible sequence of blowups.

In Section 3, we define invariants on nonprincipal locus of the morphism f. These invariants are positive integers and we prescribe permissible sequences of blowups under which these invariants drop (Theorems 3.3 and 3.4). Finally we achieve principalization in Theorem 3.6.

2. Permissible blowups

Let $f : X \longrightarrow Y$ be a locally toroidal morphism from a nonsingular n-fold X to a nonsingular surface Y with respect to open coverings $\{U_1, \dots, U_m\}$ and $\{V_1, \dots, V_m\}$ of X and Y respectively and SNC divisors D_i and E_i in U_i and V_i respectively. Then we have the following

Lemma 2.1. *Let $p \in D_i$. Then there exist regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X,p}$ and regular parameters u, v in $\mathcal{O}_{Y,q}$ such that one of the following forms holds:*

$1 \le k \le n-1 : u = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of D_i and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = x_{k+1}, \tag{1}$$

where $a_1, \dots, a_k > 0$.

$1 \le k \le n-1 : uv = 0$ is a local equation for E_i, $x_1 \cdots x_k = 0$ is a local equation of D_i and

$$u = ({x_1}^{a_1} \cdots {x_k}^{a_k})^m, \qquad v = ({x_1}^{a_1} \cdots {x_k}^{a_k})^t(\alpha + x_{k+1}), \tag{2}$$

where $a_1, \dots, a_k, m, t > 0$ and $\alpha \in K - \{0\}$.

$2 \le k \le n : uv = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of D_i and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k}, \tag{3}$$

where $a_1, \dots, a_k, b_1, \dots, b_k \ge 0$, $a_i + b_i > 0$ for all i and rank $\begin{bmatrix} a_1 & \cdot & \cdot & a_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} = 2$.

Proof. This follows from Lemma 4.2 in [8]. □

Definition 2.2. Suppose that D is a SNC divisor on a variety X, and V is a nonsingular subvariety of X. We say that V makes SNCs with D at $p \in X$ if there exist regular parameters $x_1, \dots, x_n$ in $\mathcal{O}_{X,p}$ and $e, r \le n$ such that $x_1 \cdots x_e = 0$ is a local equation of D at p and $x_{\sigma(1)} = \cdots = x_{\sigma(r)} = 0$ is a local equation of V at p for some injection $\sigma : \{1, \dots, r\} \to \{1, \dots, n\}$.

We say that V makes SNCs with D if V makes SNCs with D at all points $p \in X$.

Let $q \in Y$ and let m_q be the maximal ideal of $\mathcal{O}_{Y,q}$.

Define $W_q = \{p \in X \mid m_q\mathcal{O}_{X,p}$ is not principal$\}$. Note that the closed subset $W_q \subset f^{-1}(q)$ and that $m_q\mathcal{O}_{X,p}$ is principal if and only if $m_q\hat{\mathcal{O}}_{X,p}$ is principal.

Lemma 2.3. *For all $q \in Y$, W_q is a union of nonsingular codimension 2 subvarieties of X, which make SNCs with each divisor D_i on U_i.*

Proof. Let us fix a $q \in Y$ and denote $W = W_q$. Let $\mathfrak{I}_W$ be the reduced ideal sheaf of W in X, and let $\mathfrak{I}_q$ be the reduced ideal sheaf of q in Y.

Since the conditions that W is nonsingular and has codimension 2 in X are both local properties, we need only check that for all $p \in W$, $\mathfrak{I}_{W,p}$ is an intersection of height 2 prime ideals which are regular.

Since X is nonsingular, $\mathfrak{I}_q\mathcal{O}_X = \mathcal{O}_X(-F)\mathcal{I}$ where F is an effective Cartier divisor on X and $\mathcal{I}$ is an ideal sheaf such that the support of $\mathcal{O}_X/\mathcal{I}$ has codimension at least 2 on X. We have $W = \mathrm{supp}(\mathcal{O}_X/\mathcal{I})$. The ideal sheaf of W is $\mathfrak{I}_W = \sqrt{\mathcal{I}}$.

Let $p \in W$. We have that $p \in U_i$ for some $1 \le i \le m$.

Suppose first that $q \notin E_i$. Then f is smooth at p because it is locally toroidal. This means that there are regular parameters u, v at q which form a part of a regular sequence at p. So we have regular parameters $x_1, \dots, x_n$ in $\mathcal{O}_{X,p}$ such that $u = x_1$, $v = x_2$.

$\mathfrak{I}_q\mathcal{O}_{X,p} = (u, v)\mathcal{O}_{X,p} = (x_1, x_2)\mathcal{O}_{X,p}$. It follows that $\mathfrak{I}_{W,p} = (x_1, x_2)\mathcal{O}_{X,p}$. This gives us the lemma.

Suppose now that $q \in E_i$.

Since $p \in W_q$, there exist regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that one of the forms (1) or (3) holds.

Suppose that (1) holds. Since D_j is a SNC divisor, there exist regular parameters $y_1, \dots, y_n$ in $\mathcal{O}_{X,p}$ and some e such that $y_1 \cdots y_e = 0$ is a local equation of D_j.

Since $x_1 \cdots x_k = 0$ is a local equation for D_j in $\hat{\mathcal{O}}_{X,p}$, there exists a unit series $\delta \in \hat{\mathcal{O}}_{X,p}$ such that $y_1 \cdots y_e = \delta x_1 \cdots x_k$. Since the x_i and y_i are irreducible in $\hat{\mathcal{O}}_{X,p}$, it follows that $e = k$, and there exist unit series $\delta_i \in \hat{\mathcal{O}}_{X,p}$ such that $x_i = \delta_i y_i$ for $1 \le i \le k$, after possibly re-indexing the y_i.

Note that $y_1, \ldots, y_k, x_{k+1}, y_{k+2}, \ldots, y_n$ is a regular system of parameters in $\hat{\mathcal{O}}_{X,p}$, after possibly permuting $y_{k+1}, \ldots, y_n$.

So the ideal $(y_1, \ldots, y_k, x_{k+1}, y_{k+2}, \ldots, y_n)\hat{\mathcal{O}}_{X,p}$ is the maximal ideal of $\hat{\mathcal{O}}_{X,p}$. Since $x_{k+1} = v \in \mathcal{O}_{X,p}$, $y_1, \ldots, y_k, x_{k+1}, y_{k+2}, \ldots, y_n$ generate an ideal J in $\mathcal{O}_{X,p}$. Since $\hat{\mathcal{O}}_{X,p}$ is faithfully flat over $\mathcal{O}_{X,p}$, and $J\hat{\mathcal{O}}_{X,p}$ is maximal, it follows that J is the maximal ideal of $\mathcal{O}_{X,p}$. Hence $y_1, \ldots, y_k, x_{k+1}, y_{k+2}, \ldots, y_n$ is a regular system of parameters in $\mathcal{O}_{X,p}$.

Rewriting (1), we have $u = {y_1}^{a_1} \cdots {y_k}^{a_k}\bar{\delta}$, where $\bar{\delta}$ is a unit in $\hat{\mathcal{O}}_{X,p}$.

Since $\bar{\delta} = \frac{u}{{y_1}^{a_1} \cdots {y_k}^{a_k}}$, $\bar{\delta} \in \mathrm{QF}\,(\mathcal{O}_{X,p}) \cap \hat{\mathcal{O}}_{X,p}$, where $\mathrm{QF}\,(\mathcal{O}_{X,p})$ is the quotient field of $\mathcal{O}_{X,p}$. By Lemma 2.1 in [4], it follows that $\bar{\delta} \in \mathcal{O}_{X,p}$.

Since $\bar{\delta}$ is a unit in $\hat{\mathcal{O}}_{X,p}$, it is a unit in $\mathcal{O}_{X,p}$.

We have

$$\begin{aligned}\mathfrak{I}_{W,p} &= \sqrt{\mathfrak{I}_q\mathcal{O}_{X,p}} = \sqrt{(u,v)\mathcal{O}_{X,p}} = \sqrt{({y_1}^{a_1} \cdots {y_k}^{a_k}, x_{k+1})}\\ &= (y_1, x_{k+1}) \cap (y_2, x_{k+1}) \cap \cdots \cap (y_k, x_{k+1}),\end{aligned}$$

as required.

We argue similarly when (3) holds at p. $\square$

Let Z be a nonsingular codimension 2 subvariety of X such that $Z \subset W_q$ for some q. Let $\pi_1 : X_1 \to X$ be the blowup of Z. Denote by $(W_1)_q$ the set $\{p \in X_1 \mid m_q\hat{\mathcal{O}}_{X_1,p} \text{ is not invertible}\}$.

Given any sequence of blowups $X_n \to X_{n-1} \to \cdots \to X_1 \to X$, we define $(W_i)_q$ for each X_i as above.

Definition 2.4. Let $q \in Y$. A sequence of blowups $X_k \to X_{k-1} \to \cdots \to X_1 \to X$ is called a *permissible sequence with respect to q* if for all i, each blowup $X_{i+1} \to X_i$ is centered at a nonsingular codimension 2 subvariety Z of X_i such that $Z \subset (W_i)_q$.

We will often write simply permissible sequence without mentioning q if there is no scope for confusion.

Lemma 2.5. *Let $f : X \to Y$ be a locally toroidal morphism. Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to a $q \in Y$.*
I. *Suppose that $1 \le i \le m$ and $p \in (f \circ \pi_1)^{-1}(q) \cap {\pi_1}^{-1}(U_i)$ and $q \in E_i$. Then* **I.A** *and* **I.B** *as below hold.*
I.A. *There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and (u, v) in $\mathcal{O}_{Y,q}$ such that one of the following forms holds:*

$1 \le k \le n-1$: $u = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k} x_{k+1}, \tag{4}$$

where $b_i \le a_i$.
$1 \le k \le n-1$: $u = 0$ is a local equation of E_i, $x_1 \cdots x_k x_{k+1} = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k} {x_{k+1}}^{a_{k+1}}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k} {x_{k+1}}^{b_{k+1}}, \tag{5}$$

where $b_i \le a_i$ for $i = 1, \ldots, k$ and $b_{k+1} < a_{k+1}$.
$1 \le k \le n-1$: $u = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k}(x_{k+1} + \alpha), \tag{6}$$

where $b_i \le a_i$ for all i and $0 \ne \alpha \in K$.
$1 \le k \le n-1$: $uv = 0$ is a local equation for E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = ({x_1}^{a_1} \cdots {x_k}^{a_k})^m, \qquad v = ({x_1}^{a_1} \cdots {x_k}^{a_k})^t(\alpha + x_{k+1}), \tag{7}$$

where $a_1, \ldots, a_k, m, t > 0$ and $\alpha \in K - \{0\}$.
$2 \le k \le n$: $uv = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k}, \tag{8}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \ge 0$, $a_i + b_i > 0$ for all i and rank $\begin{bmatrix} a_1 & \cdot & \cdot & a_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} = 2$.

I.B. *Suppose that $p_1 \in (W_1)_q$. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and (u, v) in $\mathcal{O}_{Y,q}$ such that one of the following forms holds:*

$1 \le k \le n-1$: $u = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k} x_{k+1}, \tag{9}$$

where $b_i \le a_i$ and $b_i < a_i$ for some i. Moreover, the local equations of $(W_1)_q$ are $x_i = x_{k+1} = 0$ where $b_i < a_i$.

$2 \le k \le n$: $uv = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k}, \tag{10}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \ge 0$, $a_i + b_i > 0$ for all i, u does not divide v, v does not divide u, and rank $\begin{bmatrix} a_1 & \cdot & \cdot & a_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} = 2$. Moreover, the local equations of $(W_1)_q$ are $x_i = x_j = 0$ where $(a_i - b_i)(b_j - a_j) > 0$.

II. *Suppose that $1 \le i \le m$ and $p \in (f \circ \pi_1)^{-1}(q) \cap {\pi_1}^{-1}(U_i)$ and $q \notin E_i$. Then* **II.A** *and* **II.B** *as below hold.*

II.A. *There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and (u, v) in $\mathcal{O}_{Y,q}$ such that one of the following forms holds:*

$$u = x_1,\ v = x_2 \tag{11}$$

$$u = x_1,\ v = x_1(x_2 + \alpha) \quad \text{for some } \alpha \in K. \tag{12}$$

$$u = x_1x_2, \qquad v = x_2. \tag{13}$$

II.B. *Suppose that $p_1 \in (W_1)_q$. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and (u, v) in $\mathcal{O}_{Y,q}$ such that the following form holds:*

$$u = x_1, \qquad v = x_2. \tag{14}$$

The local equations of $(W_1)_q$ are $x_1 = x_2 = 0$.

III. *$(W_1)_q$ is a union of nonsingular codimension 2 subvarieties of X_1.*

Proof. **I**. We prove this part by induction on the number of blowups in the sequence $\pi_1 : X_1 \to X$. In X the conclusions hold because of Lemma 2.3 and f is locally toroidal. Suppose that the conclusions of the lemma hold after any sequence of l permissible blowups where $l \ge 0$.

Let $\pi_1 : X_1 \to X$ be a permissible sequence (with respect to q) of l blowups. Let $\pi_2 : X_2 \to X_1$ be the blowup of a nonsingular codimension 2 subvariety Z of X_1 such that $Z \subset (W_1)_q$.

Let $p \in {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (f \circ \pi_1 \circ \pi_2)^{-1}(q)$ for some $1 \le i \le m$.

If $p_1 = \pi_2(p) \notin Z$ then π_2 is an isomorphism at p and we have nothing to prove. Suppose then that $p_1 \in {\pi_1}^{-1}(U_i) \cap Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$.

Then by induction hypothesis (**I.B**) p_1 has the form (9) or (10). Suppose first that it has the form (9).

Then the local equations of Z at p_1 are $x_i = x_{k+1} = 0$ for some $1 \le i \le k$. Note that $b_i < a_i$.

As in the proof of Lemma 2.3, there exist regular parameters $y_1, \ldots, y_k, y_{k+1}, y_{k+2}, \ldots, y_n$ in $\mathcal{O}_{X_1,p_1}$ and unit series $\delta_i \in \hat{\mathcal{O}}_{X_1,p_1}$ such that $y_i = \delta_i x_i$ for $1 \le i \le k$.

Then $\mathcal{O}_{X_2,p}$ has one of the following two forms:

(a) $\mathcal{O}_{X_2,p} = \mathcal{O}_{X_1,p_1}[\frac{x_{k+1}}{y_i}]_{(y_i, \frac{x_{k+1}}{y_i} - \alpha)}$ for some $\alpha \in K$, or

(b) $\mathcal{O}_{X_2,p} = \mathcal{O}_{X_1,p_1}[\frac{y_i}{x_{k+1}}]_{(x_{k+1}, \frac{y_i}{x_{k+1}})}$

In case (a), set $\bar{y}_{k+1} = \frac{x_{k+1}}{y_i} - \alpha$. Then $y_1, \ldots, y_k, \bar{y}_{k+1}, y_{k+2}, \ldots, y_n$ are regular parameters in $\mathcal{O}_{X_2,p}$ and so $\hat{\mathcal{O}}_{X_2,p} = k[[y_1, \ldots, y_k, \bar{y}_{k+1}, y_{k+2}, \ldots, y_n]]$.

Let $c \ne 0$ be the constant term of the unit series δ_i.

Then evaluating δ_i in the local ring $\mathcal{O}_{X_2,p}$ we get,

$$\begin{aligned} \delta_i(y_1, \ldots, y_k, x_{k+1}, y_{k+2}, \ldots, y_n) &= \delta_i(y_1, \ldots, y_k, y_i(\bar{y}_{k+1} + \alpha), y_{k+1}, \ldots, y_n) \\ &= c + \Delta_1 y_1 + \cdots + \Delta_k y_k + \Delta_{k+2} y_{k+2} + \cdots + \Delta_n y_n \end{aligned}$$

for some $\Delta_i \in \mathcal{O}_{X_2,p}$.

Set $\bar{\alpha} = c\alpha$. Note that $\frac{x_{k+1}}{x_i} - \bar{\alpha} = \delta_i \frac{x_{k+1}}{y_k} - c\alpha = \delta_i(\bar{y}_{k+1} + \alpha) - c\alpha = \delta_i \bar{y}_{k+1} + (\delta_i - c)\alpha$.

Since $y_1, \ldots, y_k, \bar{y}_{k+1}, y_{k+2}, \ldots, y_n$ are regular parameters in $\hat{\mathcal{O}}_{X_2,p}$ the above calculations imply that $x_1, \ldots, x_k, \frac{x_{k+1}}{x_i} - \bar{\alpha}, y_{k+2}, \ldots, y_n$ are regular parameters in $\hat{\mathcal{O}}_{X_2,p}$.

Set $\bar{x}_{k+1} = \frac{x_{k+1}}{x_k} - \bar{\alpha}$.

We get $u = {x_1}^{a_1} \cdots {x_k}^{a_k},\ v = {x_1}^{b_1} \cdots {\bar{x}_i}^{b_i+1} \cdots {x_k}^{b_k}(\bar{x}_{k+1} + \alpha)$.

This is the form (6) if $\alpha \ne 0$ and form (4) if $\alpha = 0$.

In case (b), set $\bar{y}_{k+1} = \frac{y_i}{x_{k+1}}$. Then $y_1, \ldots, y_k, \bar{y}_{k+1}, y_{k+2}, \ldots, y_n$ are regular parameters in $\mathcal{O}_{X_2,p}$ and so $\hat{\mathcal{O}}_{X_2,p} = k[[y_1, \ldots, y_k, \bar{y}_{k+1}, y_{k+2}, \ldots, y_n]]$.

Then $x_1, \ldots, x_k, \frac{x_i}{x_{k+1}}, y_{k+2}, \ldots, y_n$ are regular parameters in $\hat{\mathcal{O}}_{X_2,p}$. Set $\bar{x}_i = \frac{x_i}{x_{k+1}}$.

$u = {x_1}^{a_1} \cdots {\bar{x}_i}^{a_i} \cdots {x_k}^{a_k} {x_{k+1}}^{a_i}$, $v = {x_1}^{b_1} \cdots {\bar{x}_i}^{b_i+1} \cdots {x_k}^{b_k} x_{k+1}$.

This is the form (5).

By the above analysis, when $p_1 = \pi_2(p)$ has form (9), if $p \in (W_2)_q$, then it also has to be of the form (9).

Suppose now that p_1 has the form (10). Then the local equations of Z at p_1 are $x_i = x_j = 0$ for some $1 \le i, j \le k$.

Then as in the above analysis there exist regular parameters $y_1, \ldots, \ldots, y_n$ in $\mathcal{O}_{X_1,p_1}$ and unit series $\delta_i \in \hat{\mathcal{O}}_{X_1,p_1}$ such that $y_i = \delta_i x_i$ for $1 \le i \le k$.

Then $\mathcal{O}_{X_2,p}$ has one of the following two forms:

(a) $\mathcal{O}_{X_2,p} = \mathcal{O}_{X_1,p_1}[\frac{y_i}{y_j}]_{(y_j, \frac{y_i}{y_j}-\alpha)}$ for some $\alpha \in K$, or

(b) $\mathcal{O}_{X_2,p} = \mathcal{O}_{X_1,p_1}[\frac{y_j}{y_i}]_{(y_i, \frac{y_j}{y_j})}$.

Arguing as above in case (a) we obtain regular parameters $x_1, \ldots, \bar{x}_i, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_2,p}$ so that

$$u = {x_1}^{a_1} \cdots (\bar{x}_i + \alpha)^{a_i} \cdots {x_j}^{a_i+a_j} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots (\bar{x}_i + \alpha)^{b_i} \cdots {x_j}^{b_i+b_j} \cdots {x_k}^{b_k}.$$

This is the form (8) if $\alpha = 0$.

If $\alpha \ne 0$, we obtain either the form (8) or the form (7) according as rank of
$\begin{bmatrix} a_1 & . & . & a_i+a_j & . & . & a_{j-1} & a_{j+1} & . & . & a_k \\ b_1 & . & . & b_i+b_j & . & . & b_{j-1} & b_{j+1} & . & . & b_k \end{bmatrix}$ is $= 2$ or < 2.

Again arguing as above in case (b) we obtain regular parameters $x_1, \ldots, \bar{x}_j, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_2,p}$ so that

$$u = {x_1}^{a_1} \cdots {x_i}^{a_i+a_j} \cdots \bar{x}_j^{a_j} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_i}^{b_i+b_j} \cdots \bar{x}_j^{b_j} \cdots {x_k}^{b_k}.$$

This is the form (8).

By the above analysis, when $p_1 = \pi_2(p)$ has the form (10), if $p \in (W_2)_q$, then it also has to be of the form (10).

This completes the proof of **I.A** for X_2. Now **I.B** is clear as the forms (9) and (10) are just the forms (4) and (8) from **I.A**.

II. We prove this part by induction on the number of blowups in the sequence $\pi_1 : X_1 \to X$.

Since $q \notin E_i$ and f is locally toroidal, f is smooth at any point $p_1 \in f^{-1}(q)$. This means that the regular parameters at q form a part of a regular sequence at p. So we have regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X,p_1}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = x_1$, $v = x_2$. This is the form (11). Thus the conclusions hold in X. Suppose that the conclusions of the lemma hold after any sequence of l permissible blowups where $l \ge 0$.

Let $\pi_1 : X_1 \to X$ be a permissible sequence (with respect to q) of l blowups. Let $\pi_2 : X_2 \to X_1$ be the blowup of a nonsingular codimension 2 subvariety Z of X_1 such that $Z \subset (W_1)_q$.

Let $p \in {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (f \circ \pi_1 \circ \pi_2)^{-1}(q)$ for some $1 \le i \le m$.

If $p_1 = \pi_2(p) \notin Z$ then π_2 is an isomorphism at p and we have nothing to prove. Suppose then that $p_1 \in {\pi_1}^{-1}(U_i) \cap Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$.

Then by induction hypothesis (**II.B**) p_1 has the form (14). Then the local equations of Z at p_1 are $x_1 = x_2 = 0$.

There exist regular parameters $\bar{x}_1, \bar{x}_2$ in $\hat{\mathcal{O}}_{X_2,p}$ such that one of the following forms holds:

$x_1 = \bar{x}_1$, $x_2 = \bar{x}_1(\bar{x}_2 + \alpha)$ for some $\alpha \in K$ or $x_1 = \bar{x}_1\bar{x}_2$, $x_2 = \bar{x}_2$. These two cases give the forms (12) and (13).

Now **II.B** is clear as the form (14) is just the form (11) from **II.A**.

III. Since $\{{\pi_1}^{-1}(U_i)\}$ for $1 \le i \le m$ is an open cover of X_1 and ${\pi_1}^{-1}(U_i) \cap (W_1)_q$ is a union of nonsingular codimension 2 subvarieties of X_1 for all i by **I** and **II**, $(W_1)_q$ is a union of nonsingular codimension 2 subvarieties of X_1. □

3. Principalization

Let $f : X \longrightarrow Y$ be a locally toroidal morphism from a nonsingular n-fold X to a nonsingular surface Y with respect to open coverings $\{U_1, \ldots, U_m\}$ and $\{V_1, \ldots, V_m\}$ of X and Y respectively and SNC divisors D_i and E_i in U_i and V_i respectively.

In this section we fix an i between 1 and m and a $q \in Y$.

Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to q. Our aim is to construct a permissible sequence $\pi_2 : X_2 \to X_1$ such that $\pi_2 \circ \pi_1 : X_2 \to X$ is a permissible sequence and ${\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$ is empty.

First suppose that $q \notin E_i$. If $p \in {\pi_1}^{-1}(U_i)$, then by Lemma 2.5 one of the forms (11) and (12) or (13) holds at p.

Theorem 3.1. *Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to $q \in Y$. Let i be such that $q \notin E_i$. Then there exists a permissible sequence $\pi_2 : X_2 \to X_1$ with respect to q such that ${\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$ is empty.*

Proof. If ${\pi_1}^{-1}(U_i) \cap (W_2)_q$ is empty, then there is nothing to prove. So suppose that ${\pi_1}^{-1}(U_i) \cap (W_2)_q \ne \emptyset$. By Lemma 2.3, it is a union of codimension 2 subvarieties of ${\pi_1}^{-1}(U_i)$.

Let $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a subvariety of ${\pi_1}^{-1}(U_i)$ of codimension 2.

Let $\pi_2 : X_2 \to X_1$ be the blowup of the Zariski closure $\bar{Z}$ of Z in X_1. Let $Z_1 \subset {\pi_2}^{-1}(Z)$ be a codimension 2 subvariety of ${\pi_2}^{-1}({\pi_1}^{-1}(U_i))$ such that $Z_1 \subset {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$.

By the proof of Lemma 2.5 it follows that $Z_1 \cap (W_2)_q = \emptyset$.

The theorem now follows by induction on the number of codimension 2 subvarieties Z in ${\pi_1}^{-1}(U_i) \cap (W_1)_q$. □

Now we suppose that $q \in E_i$.

Remark 3.2. Suppose that $\pi_1 : X_1 \to X$ is a permissible sequence with respect to some $q \in E_i$. Let $\pi_2 : X_2 \to X_1$ be a permissible blowup with respect to q. Let $p_1 \in {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$. Then clearly $p = \pi_2(p_1) \in {\pi_1}^{-1}(U_i) \cap (W_1)_q$.

Suppose that p_1 is a 1 point. Then the analysis in the proof of Lemma 2.5 shows that p also is a 1 point.

Suppose that p_1 is a 2 point where the form (10) holds. Then the analysis in the proof of Lemma 2.5 shows that p is a 2 or 3 point where the from (10) holds.

Suppose that $\pi_1 : X_1 \to X$ is a permissible sequence with respect to $q \in E_i$.

Let $p \in {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a 1 point. By Lemma 2.5, there exist regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^a,\ v = {x_1}^b x_2$ where $a > b$.

Define $\Omega_i(p) = a - b > 0$.

Let $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a codimension 2 subvariety of ${\pi_1}^{-1}(U_i)$.

Define $\Omega_i(Z) = \Omega_i(p)$ if there exists a 1 point $p \in Z$. This is well defined because $\Omega_i(p) = \Omega_i(p')$ for any two points $p, p' \in Z$.

If Z contains no 1 points, we define $\Omega_i(Z) = 0$.

Finally, define

$$\Omega_i(f \circ \pi_1) = \max\{\Omega_i(Z) | Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q \text{ is an irreducible subvariety of } {\pi_1}^{-1}(U_i) \text{ of codimension } 2\}$$

Theorem 3.3. *Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to $q \in E_i$. There exists a permissible sequence $\pi_2 : X_2 \to X_1$ with respect to q such that $\Omega_i(f \circ \pi_1 \circ \pi_2) = 0$.*

Proof. Suppose that $\Omega_i(f \circ \pi_1) > 0$. Let $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a subvariety of ${\pi_1}^{-1}(U_i)$ of codimension 2 such that $\Omega_i(f \circ \pi_1) = \Omega_i(Z)$.

Let $\pi_2 : X_2 \to X_1$ be the blowup of the Zariski closure $\bar{Z}$ of Z in X_1. Let $Z_1 \subset {\pi_2}^{-1}(Z)$ be a codimension 2 subvariety of ${\pi_2}^{-1}({\pi_1}^{-1}(U_i))$ such that $Z_1 \subset {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$. We claim that $\Omega_i(Z_1) < \Omega_i(Z)$.

If there are no 1 points of Z_1 then we have nothing to prove. Otherwise, let $p_1 \in Z_1$ be a 1 point. Then $\pi_1(p_1) = p$ is a 1 point of Z by Remark 3.2.

There are regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^a, v = {x_1}^b x_2$. There exist regular parameters $x_1, \bar{x_2}, \dots, x_n$ in $\hat{\mathcal{O}}_{X_2,p_1}$ such that $x_2 = x_1(x_2 + \alpha)$.

$u = {x_1}^a, v = {x_1}^{b+1}(x_2 + \alpha)$. Since $p_1 \in (W_2)_q, \alpha = 0$.

$\Omega_i(Z_1) = \Omega_i(p_1) = a - b - 1 < a - b = \Omega_i(Z)$.

The theorem now follows by induction on the number of codimension 2 subvarieties Z in ${\pi_1}^{-1}(U_i) \cap (W_1)_q$ such that $\Omega_i(f \circ \pi_1) = \Omega_i(Z)$ and induction on $\Omega_i(f \circ \pi_1)$. □

Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to $q \in E_i$.

Let $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a codimension 2 subvariety of ${\pi_1}^{-1}(U_i)$. Let $p \in Z$ be a 2 point where the form (10) holds.

There exist regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^{a_1}{x_2}^{a_2}$ and $v = {x_1}^{b_1}{x_2}^{b_2}$.

Define $\omega_i(p) = (a_1 - b_1)(b_2 - a_2)$. Then since $p \in (W_1)_q, \omega_i(p) > 0$.

Now define $\omega_i(Z) = \omega_i(p)$ if $p \in Z$ is a 2 point where the form (10) holds. If there are no 2 points of the form (10) in Z define $\omega_i(Z) = 0$. Then $\omega_i(Z)$ is well-defined.

Finally, define

$$\omega_i(f \circ \pi_1) = \max\{\omega_i(Z) | Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q \text{ is an irreducible subvariety of } {\pi_1}^{-1}(U_i) \text{ of codimension } 2\}.$$

Theorem 3.4. *Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to $q \in E_i$. Suppose that $\Omega_i(f \circ \pi_1) = 0$. There exists a permissible sequence $\pi_2 : X_2 \to X_1$ with respect to q such that $\Omega_i(f \circ \pi_1 \circ \pi_2) = 0$ and $\omega_i(f \circ \pi_1 \circ \pi_2) = 0$.*

Proof. Since $\Omega_i(f \circ \pi_1) = 0$, there are no 1 points in ${\pi_1}^{-1}(U_i) \cap (W_1)_q$. Let $X_2 \to X_1$ be any permissible blowup. Then by Remark 3.2 it follows that ${\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$ has no 1 points. Hence $\Omega_i(f \circ \pi_1 \circ \pi_2) = 0$.

Suppose that $\omega_i(f \circ \pi_1) > 0$. Let $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ be a codimension 2 irreducible subvariety of ${\pi_1}^{-1}(U_i)$ such that $\omega_i(f \circ \pi_1) = \omega_i(Z)$.

Let $\pi_2 : X_2 \to X_1$ be the blowup of the Zariski closure $\bar{Z}$ of Z in X_1. Let $Z_1 \subset {\pi_2}^{-1}(Z)$ be a codimension 2 subvariety of ${\pi_2}^{-1}({\pi_1}^{-1}(U_i))$ such that $Z_1 \subset {\pi_2}^{-1}({\pi_1}^{-1}(U_i)) \cap (W_2)_q$. We prove that $\omega_i(Z_1) < \omega_i(Z) = \omega_i(f \circ \pi_1)$.

If there are no 2 points of the form (10) in Z_1 then $\omega_i(Z_1) = 0$ and we have nothing to prove. Otherwise let $p_1 \in Z_1$ be a 2 point of the form (10).

By Remark 3.2, $p = \pi_2(p_1) \in Z$ is a 2 or 3 point of form (10).

Suppose that $p \in Z$ is a 2 point. There exist regular parameters $x_1, \dots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^{a_1}{x_2}^{a_2}$ and $v = {x_1}^{b_1}{x_2}^{b_2}$. Also the local equations of Z are $x_1 = x_2 = 0$.

Then there exist regular parameters $x_1, \bar{x_2}, x_3 \dots, x_n$ in $\hat{\mathcal{O}}_{X_2,p_1}$ such that $x_2 = x_1\bar{x_2}$ and $u = {x_1}^{a_1+a_2}\bar{x_2}^{a_2}$ and $v = {x_1}^{b_1+b_2}\bar{x_2}^{b_2}$.

$$\begin{aligned}\omega_i(Z_1) = \omega_i(p_1) &= (a_1 + a_2 - b_1 - b_2)(b_2 - a_2)\\ &= (a_1 - b_1)(b_2 - a_2) + (a_2 - b_2)(b_2 - a_2)\\ &< (a_1 - b_1)(b_2 - a_2) = \omega_i(p) = \omega_i(Z) = \omega_i(f \circ \pi_1).\end{aligned}$$

Suppose that $p \in Z$ is a 3 point. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^{a_1}{x_2}^{a_2}{x_3}^{a_3}$ and $v = {x_1}^{b_1}{x_2}^{b_2}{x_3}^{b_3}$. After permuting x_1, x_2, x_3 if necessary, we can suppose that the local equations of Z are $x_2 = x_3 = 0$.

Then there exist regular parameters $x_1, x_2, \bar{x_3} \cdots, x_n$ in $\hat{\mathcal{O}}_{X_2,p_1}$ such that $x_3 = x_2(\bar{x_3} + \alpha)$ and $u = {x_1}^{a_1}{x_2}^{a_2+a_3}(\bar{x_3} + \alpha)^{a_3}$ and $v = {x_1}^{b_1}{x_2}^{b_2+b_3}(\bar{x_3} + \alpha)^{b_3}$.

Since p_1 is a 2 point, we have $\alpha \neq 0$ and $a_1(b_2 + b_3) - b_1(a_2 + a_3) \neq 0$. After an appropriate change of variables x_1, x_2 we obtain regular parameters $\bar{x_1}, \bar{x_2}, \tilde{x_3}, x_4, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_2,p_1}$.

$u = \bar{x_1}^{a_1}\bar{x_2}^{a_2+a_3}$ and $v = \bar{x_1}^{b_1}\bar{x_2}^{b_2+b_3}$.

Since the local equations of $Z \subset {\pi_1}^{-1}(U_i) \cap (W_1)_q$ are $x_2 = x_3 = 0$, $b_2 - a_2$ and $b_3 - a_3$ have different signs. So $a_1 - b_1$ has the same sign as exactly one of $b_2 - a_2$ or $b_3 - a_3$. Without loss of generality suppose that $(a_1 - b_1)(b_2 - a_2) > 0$ and $(a_1 - b_1)(b_3 - a_3) < 0$.

Let Z' be the codimension 2 variety whose local equations are $x_1 = x_2 = 0$ defined in an appropriately small neighborhood in ${\pi_1}^{-1}(U_i)$. Then the closure $\bar{Z'}$ of Z' in ${\pi_1}^{-1}(U_i)$ is an irreducible codimension 2 subvariety contained in ${\pi_1}^{-1}(U_i) \cap (W_1)_q$.

$$\begin{aligned}\omega_i(Z_1) = \omega_i(p_1) &= (a_1 - b_1)(b_2 + b_3 - a_2 - a_3)\\ &= (a_1 - b_1)(b_2 - a_2) + (a_1 - b_1)(b_3 - a_3)\\ &< (a_1 - b_1)(b_2 - a_2) = \omega_i(\bar{Z'}) \leq \omega_i(f \circ \pi_1).\end{aligned}$$

The theorem now follows by induction on the number of codimension 2 subvarieties Z in ${\pi_1}^{-1}(U_i) \cap (W_1)_q$ such that $\omega_i(f \circ \pi_1) = \omega_i(Z)$ and induction on $\omega_i(f \circ \pi_1)$. □

Remark 3.5. Let $\pi_1 : X_1 \to X$ be a permissible sequence with respect to q. Let i be such that $1 \leq i \leq m$.

If $q \in E_i$, then it follows from Theorems 3.3 and 3.4 that there exists a permissible sequence $\pi_2 : X_2 \to X_1$ with respect to q such that $\Omega_i(f \circ \pi_1 \circ \pi_2) = 0$ and $\omega_i(f \circ \pi_1 \circ \pi_2) = 0$.

Theorem 3.6. *Let $f : X \longrightarrow Y$ be a locally toroidal morphism between a nonsingular n-fold X and a nonsingular surface Y. Let $q \in Y$.*

Then there exists a permissible sequence $\pi_1 : X_1 \to X$ with respect to q such that $(W_1)_q$ is empty.

Proof. First we apply Theorem 3.1 and Remark 3.5 to X and $i = 1$.

Suppose that $q \notin E_1$. Then by Theorem 3.1, there exists a permissible sequence $\pi_1 : X_1 \to X$ with respect to q such that ${\pi_1}^{-1}(U_1) \cap (W_1)_q = \emptyset$.

Now suppose that $q \in E_1$. It follows from Remark 3.5 that there exists a permissible sequence $\pi_1 : X_1 \to X$ with respect to q such that $\Omega_1(f \circ \pi_1) = 0$ and $\omega_1(f \circ \pi_1) = 0$. So there are no 1 points or 2 points of the form (10) in ${\pi_1}^{-1}(U_1) \cap (W_1)_q$. But if $Z \subset {\pi_1}^{-1}(U_1) \cap (W_1)_q$ is any codimension 2 irreducible subvariety of ${\pi_1}^{-1}(U_i)$, then a generic point of Z must either be a 1 point or a 2 point of the form (10). It follows then that ${\pi_1}^{-1}(U_1) \cap (W_1)_q$ is empty.

Now we apply Theorem 3.1 and Remark 3.5 to the permissible sequence $\pi_1 : X_1 \to X$ and $i = 2$.

If $q \notin E_2$, then by Theorem 3.1, there exists a permissible sequence $\pi_2 : X_2 \to X_1$ such that ${\pi_2}^{-1}({\pi_1}^{-1}(U_2)) \cap (W_2)_q = \emptyset$.

If $q \in E_2$, then as above there exists a permissible sequence $\pi_2 : X_2 \to X_1$ such that ${\pi_2}^{-1}({\pi_1}^{-1}(U_2)) \cap (W_2)_q$ is empty.

Notice that we also have ${\pi_2}^{-1}({\pi_1}^{-1}(U_1)) \cap (W_2)_q = \emptyset$.

Repeating the argument for $i = 3, 4, \ldots, m$ we obtain the desired permissible sequence. □

4. Toroidalization

Theorem 4.1. *Let $f : X \longrightarrow Y$ be a locally toroidal morphism from a nonsingular n-fold X to a nonsingular surface Y with respect to open coverings $\{U_1, \ldots, U_m\}$ and $\{V_1, \ldots, V_m\}$ of X and Y respectively and SNC divisors D_i and E_i in U_i and V_i respectively. Let $\pi : Y_1 \to Y$ be the blowup of a point $q \in Y$.*

Then there exists a permissible sequence $\pi_1 : X_1 \to X$ such that there is a locally toroidal morphism $f_1 : X_1 \to Y_1$ such that $\pi \circ f_1 = f \circ \pi_1$.

Proof. By Theorem 3.6 there is a permissible sequence $\pi_1 : X_1 \to X$ such that there exists a morphism $f_1 : X_1 \to Y_1$ and $\pi \circ f_1 = f \circ \pi_1$.

Let $p \in X_1$. Suppose that $p \in {\pi_1}^{-1}(U_i)$ for some i such that $1 \leq i \leq m$. If $\pi_1(p) \notin f^{-1}(q)$ then we have nothing to prove. So we assume that $\pi_1(p) \in f^{-1}(q)$.

Suppose first that $q \notin E_i$. Then by Lemma 2.5 one of the forms (12) or (13) holds at p. So there exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that

$$u = x_1, \qquad v = x_1(x_2 + \alpha) \quad \text{for some } \alpha \in K, \text{ or } u = x_1y_1, v = x_2.$$

Let $f_1(p) = q_1$. There exist regular parameters $u_1, v_1 \in \mathcal{O}_{Y_1,q_1}$ such that

$$u = u_1, \qquad v = u_1(v_1 + \alpha) \quad \text{or} \quad u = u_1v_1, \quad v = v_1$$

according as the form (12) or the form (13) holds. In either case, we have $u_1 = x_1$, $v_1 = x_2$, and f_1 is smooth at p.

Now suppose that $q \in E_i$.

By Lemma 2.5 there exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that one of the forms (4)–(7), or (8) of Lemma 2.5 holds.

Suppose first that the form (4) holds. Then since $m_q\hat{\mathcal{O}}_{X_1,p}$ is invertible, there exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ such that $u = {x_1}^{a_1} \cdots {x_k}^{a_k}$, $v = {x_1}^{a_1} \cdots {x_k}^{a_k} x_{k+1}$ for some $1 \le k \le n-1$.

Further $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and $u = 0$ is a local equation for E_i.

Let $f_1(p) = q_1$. There exist regular parameters (u_1, v_1) in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1$ and $v = u_1 v_1$. Hence the local equation of $\pi^{-1}(E_i)$ at q_1 is $u_1 = 0$.

$$u_1 = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v_1 = x_{k+1}.$$

This is the form (1).

Suppose now that the form (5) holds at p for $f \circ \pi_1$. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ and $1 \le k \le n-1$ such that $u = 0$ is a local equation of E_i, $x_1 \cdots x_k x_{k+1} = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k} {x_{k+1}}^{a_{k+1}}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k} {x_{k+1}}^{b_{k+1}},$$

where $b_i \le a_i$ for $i = 1, \ldots, k$ and $b_{k+1} < a_{k+1}$.

Let $f_1(p) = q_1$. There exist regular parameters u_1, v_1 in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1 v_1$ and $v = v_1$. Hence the local equation of $\pi^{-1}(E_i)$ at q_1 is $u_1 v_1 = 0$.

$$u_1 = {x_1}^{a_1-b_1} \cdots {x_k}^{a_k-b_k} {x_{k+1}}^{a_{k+1}-b_{k+1}}, \qquad v_1 = {x_1}^{b_1} \cdots {x_k}^{b_k} {x_{k+1}}^{b_{k+1}}.$$

This is the form (3). Note that the rank condition follows from the dominance of the map f_1.

Suppose now that the form (6) holds. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ and $1 \le k \le n-1$ such that $u = 0$ is a local equation of E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v = {x_1}^{b_1} \cdots {x_k}^{b_k}(x_{k+1} + \alpha),$$

where $b_i \le a_i$ for all i and $0 \ne \alpha \in K$.

Let $f_1(p) = q_1$. There exist regular parameters u_1, v_1 in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1 v_1$ and $v = v_1$. Hence the local equation of $\pi^{-1}(E_i)$ at q_1 is $u_1 v_1 = 0$.

$$u_1 = {x_1}^{a_1-b_1} \cdots {x_k}^{a_k-b_k}(x_{k+1} + \alpha)^{-1}, \qquad v_1 = {x_1}^{b_1} \cdots {x_k}^{b_k}(x_{k+1} + \alpha).$$

If rank $\begin{bmatrix} a_1 - b_1 & \cdot & \cdot & a_k - b_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} = 2$ then there exist regular parameters $\bar{x_1}, \ldots, \bar{x_n}$ in $\hat{\mathcal{O}}_{X_1,p}$ such that $u_1 = \bar{x_1}^{a_1-b_1} \cdots \bar{x_k}^{a_k-b_k}$, $v_1 = \bar{x_1}^{b_1} \cdots \bar{x_k}^{b_k}$. This is the form (3).

If rank $\begin{bmatrix} a_1 - b_1 & \cdot & \cdot & a_k - b_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} < 2$ then there exist regular parameters $\bar{x_1}, \ldots, \bar{x_n}$ in $\hat{\mathcal{O}}_{X_1,p}$ such that $u_1 = (\bar{x_1}^{a_1} \cdots \bar{x_k}^{a_k})^m$, $v = (\bar{x_1}^{a_1} \cdots \bar{x_k}^{a_k})^t(x_{k+1} + \beta)$, with $\beta \ne 0$. This is the form (2).

Suppose that the form (7) holds. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ and $1 \le k \le n-1$ such that $uv = 0$ is a local equation for E_i, $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and

$$u = ({x_1}^{a_1} \cdots {x_k}^{a_k})^m, \qquad v = ({x_1}^{a_1} \cdots {x_k}^{a_k})^t(\alpha + x_{k+1}),$$

where $a_1, \ldots, a_k, m, t > 0$ and $\alpha \in K - \{0\}$.

Suppose that $m \le t$. There exist regular parameters u_1, v_1 in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1$ and $v = u_1(v_1 + \beta)$ for some $\beta \in K$.

$$u_1 = ({x_1}^{a_1} \cdots {x_k}^{a_k})^m, \qquad v_1 = ({x_1}^{a_1} \cdots {x_k}^{a_k})^{t-m}(\alpha + x_{k+1}) - \beta.$$

If $m < t$ then $\beta = 0$. So $u_1 v_1 = 0$ is a local equation of $\pi^{-1}(E_i)$ and we have the form (2). If $m = t$ then $\alpha = \beta \ne 0$ and u_1 is a local equation of $\pi^{-1}(E_i)$. In this case we have the form (1).

Suppose that $m > t$. Then there exist regular parameters u_1, v_1 in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1 v_1$ and $v = v_1$.

$$u_1 = ({x_1}^{a_1} \cdots {x_k}^{a_k})^{m-t}(\alpha + x_{k+1})^{-1}, \qquad v_1 = ({x_1}^{a_1} \cdots {x_k}^{a_k})^t(\alpha + x_{k+1}).$$

We obtain the form (2).

Finally suppose that the form (8) holds. There exist regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ and u, v in $\mathcal{O}_{Y,q}$ and $2 \le k \le n$ such that $uv = 0$ is a local equation of E_i and $x_1 \cdots x_k = 0$ is a local equation of ${\pi_1}^{-1}(D_i)$ and $u = {x_1}^{a_1} \cdots {x_k}^{a_k}$, $v = {x_1}^{b_1} \cdots {x_k}^{b_k}$, where rank $\begin{bmatrix} a_1 & \cdot & \cdot & a_k \\ b_1 & \cdot & \cdot & b_k \end{bmatrix} = 2$.

We have either $a_i \ge b_i$ for all i or $a_i \le b_i$ for all i. Without loss of generality, suppose that $a_i \le b_i$ for all i.

Let $f_1(p) = q_1$. There exist regular parameters u_1, v_1 in $\mathcal{O}_{Y_1,q_1}$ such that $u = u_1$ and $v = u_1 v_1$. Hence the local equation of $\pi^{-1}(E_i)$ at q_1 is $u_1 v_1 = 0$.

$$u_1 = {x_1}^{a_1} \cdots {x_k}^{a_k}, \qquad v_1 = {x_1}^{b_1-a_1} \cdots {x_k}^{b_k-a_k}.$$

Further, rank $\begin{bmatrix} a_1 & \cdot & \cdot & a_k \\ b_1 - a_1 & \cdot & \cdot & b_k - a_k \end{bmatrix} = 2$. This is the form (1). □

Now we are ready to prove our main theorem.

Theorem 4.2. *Suppose that $f : X \longrightarrow Y$ is a locally toroidal morphism between a variety X and a surface Y. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

where π, π_1 are blowups of nonsingular varieties such that there exist SNC divisors E, D on Y_1 and X_1 respectively such that $Sing(f_1) \subset D$, $f_1{}^{-1}(E) = D$ and f_1 is toroidal with respect to E and D.

Proof. Let $E' = \bar{E_1} + \cdots + \bar{E_m}$ where $\bar{E_i}$ is the Zariski closure of E_i in Y. There exists a finite sequence of blowups of points $\pi : Y_1 \to Y$ such that $\pi^{-1}(E')$ is a SNC divisor on Y_1.

By Theorem 4.1, there exists a sequence of blowups $\pi_1 : X_1 \to X$ such that there is a locally toroidal morphism $f_1 : X_1 \to Y_1$ with $f \circ \pi_1 = \pi \circ f_1$.

Let $E = \pi^{-1}(E')$ and $D = f_1{}^{-1}(E)$.

We now verify that E and D are SNC divisors on Y_1 and X_1 respectively and that $f_1 : X_1 \to Y_1$ is toroidal with respect to D and E.

Let $p \in X_1$ and let $q = f_1(p)$.

Suppose that $p \notin D$, so that $q \notin E$. There exists i such that $1 \le i \le m$ and $p \in \pi_1{}^{-1}(U_i)$. Then $q \notin E = \pi^{-1}(E') \Rightarrow q \notin \pi^{-1}(E_i)$. So $p \notin f_1{}^{-1}(\pi^{-1}(E_i)) = \pi_1{}^{-1}(D_i)$. Then f_1 is smooth at p because $f_1|_{\pi_1{}^{-1}(U_i)}$ is toroidal.

Thus $Sing(f_1) \subset D$.

Suppose now that $p \in D$. Let $p \in \pi_1{}^{-1}(U_i)$ for some i between 1 and m. If $q \notin \pi^{-1}(E_i)$ then f_1 is smooth at p and then $D = f_1{}^{-1}(E)$ is a SNC divisor at p. We assume then that $q \in \pi^{-1}(E_i)$.

Case 1 $q \in E$ is a 1 point.

q is necessarily a 1 point of $\pi^{-1}(E_i)$.

Then $\pi^{-1}(E_i)$ and E are equal in a neighborhood of q. Hence $\pi_1{}^{-1}(D_i)$ and D are equal in a neighborhood of p. Since $\pi_1{}^{-1}(D_i)$ is a SNC divisor in a neighborhood of p, D is a SNC divisor in a neighborhood of p.

Since $f_1|_{\pi_1{}^{-1}(U_i)}$ is toroidal there exist regular parameters u, v in $\mathcal{O}_{Y_1,q}$ and regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ such that the the form (1) holds at p with respect to E and D.

Case 2 $q \in E$ is a 2 point.

q is either a 1 point or a 2 point of $\pi^{-1}(E_i)$.

Case 2(a) q is a 1 point of $\pi^{-1}(E_i)$.

There exists regular parameters u, v in $\mathcal{O}_{Y_1,q}$ and regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ such that the form (1) holds at p. There exists $\tilde{v} \in \mathcal{O}_{Y_1,q}$ such that $u, \tilde{v}$ are regular parameters in $\mathcal{O}_{Y_1,q}$, $u\tilde{v} = 0$ is a local equation for E at q, $u = 0$ is a local equation of $\pi^{-1}(E_i)$ at q, and

$$\tilde{v} = \alpha u + \beta v + \text{ higher degree terms in } u \text{ and } v,$$

for some $\beta \in K$ with $\beta \neq 0$.

Since $\pi_1{}^{-1}(D_i)$ is a SNC divisor in a neighborhood of p, there exist regular parameters $\bar{x}_1, \ldots, \bar{x}_n$ in $\mathcal{O}_{X_1,p}$ such that $\bar{x}_1 \cdots \bar{x}_k = 0$ is a local equation of $\pi_1{}^{-1}(D_i)$ at p. Since $x_1 \cdots x_k = 0$ is also a local equation of $\pi_1{}^{-1}(D_i)$ at p, there exist units $\delta_1, \ldots, \delta_k \in \hat{\mathcal{O}}_{X_1,p}$ such that, after possibly permuting the x_j, $x_j = \delta_j \bar{x}_j$ for $1 \le j \le k$.

$$\begin{aligned} \tilde{v} &= \alpha u + \beta v + \text{ higher degree terms in } u \text{ and } v \\ &= \alpha x_1{}^{a_1} \cdots x_k{}^{a_k} + \beta x_{k+1} + \text{ higher degree terms in } u \text{ and } v \\ &= \alpha \delta_1{}^{a_1} \cdots \delta_k{}^{a_k} \bar{x}_1^{a_1} \cdots \bar{x}_k^{a_k} + \beta x_{k+1} + \text{ higher degree terms in } u \text{ and } v \end{aligned}$$

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X_1,p}$ and let $\hat{\mathfrak{m}} = \mathfrak{m}\hat{\mathcal{O}}_{X_1,p}$ be the maximal ideal of $\hat{\mathcal{O}}_{X_1,p}$.

Since $\beta \neq 0$, $\bar{x_1}, \ldots, \bar{x_k}, \tilde{v}$ are linearly independent in $\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$, so that they extend to a system of regular parameters in $\mathcal{O}_{X_1,p}$.

Say $\bar{x_1}, \ldots, \bar{x_k}, \tilde{v}, \tilde{x}_{k+2}, \ldots, \tilde{x}_n$.

$u\tilde{v} = \bar{x_1} \cdots \bar{x_k}\tilde{v} = 0$ is a local equation of D at p, so D is a SNC divisor in a neighborhood of p, and $u, \tilde{v}$ give the form (3) with respect to the formal parameters $x_1, \ldots, x_k, \tilde{v}, \tilde{x}_{k+2}, \ldots, \tilde{x}_n$.

Case 2(b) q is a 2 point of $\pi^{-1}(E_i)$.

Then $\pi^{-1}(E_i)$ and E are equal in a neighborhood of q. Hence $\pi_1{}^{-1}(D_i)$ and D are equal in a neighborhood of p. Since $\pi_1{}^{-1}(D_i)$ is a SNC divisor in a neighborhood of p, D is a SNC divisor in a neighborhood of p.

Since $f_1|_{\pi_1{}^{-1}(U_i)}$ is toroidal there exist regular parameters u, v in $\mathcal{O}_{Y_1,q}$ and regular parameters $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X_1,p}$ such that the one of the forms (2) or (3) holds at p with respect to E and D. $\square$

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