On joint distributions of the maximum, minimum and terminal value of a continuous uniformly integrable martingale

Alexander M.G. Cox [a,*], Jan Obłój [b]

[a] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*
[b] *Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK*

Abstract

We study the joint laws of the maximum and minimum of a continuous, uniformly integrable martingale. In particular, we give explicit martingale inequalities which provide upper and lower bounds on the joint exit probabilities of a martingale, given its terminal law. Moreover, by constructing explicit and novel solutions to the Skorokhod embedding problem, we show that these bounds are tight. Together with previous results of Azéma & Yor, Perkins, Jacka and Cox & Obłój, this allows us to completely characterise the upper and lower bounds on all possible exit/no-exit probabilities, subject to a given terminal law of the martingale. In addition, we determine some further properties of these bounds, considered as functions of the maximum and minimum.
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1. Introduction

The study of the running maximum and minimum of a martingale has a prominent place in probability theory, starting with Doob's maximal and L^p inequalities. In seminal contributions, Blackwell and Dubins [4], Dubins and Gilat [11] and Azéma and Yor [2,3] established that the distribution of the maximum $\overline{M}_\infty := \sup_{t \le \infty} M_t$ of a uniformly integrable martingale M is

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* Corresponding author.
E-mail addresses: A.M.G.Cox@bath.ac.uk (A.M.G. Cox), jan.obloj@maths.ox.ac.uk (J. Obłój).
URLs: http://www.maths.bath.ac.uk/~mapamgc/ (A.M.G. Cox), http://www.maths.ox.ac.uk/people/jan.obloj
(J. Obłój).

bounded from above, in stochastic order, by the so called Hardy–Littlewood transform of the distribution of M_∞, and the bound is attained. This led to series of studies on the possible distributions of $(M_\infty, \overline{M}_\infty)$ including Gilat and Meilijson [13], Kertz and Rösler [17–19], Rogers [23], Vallois [24], see also Carraro, El Karoui and Obłój [6].

More recently, these problems have gained a new momentum from applications in the field of mathematical finance. The bounds on the distribution of the maximum, given the distribution of the terminal value, are interpreted as bounds on prices of barrier options given the prices of (vanilla) European options. Further, the bounds are often obtained by devising pathwise inequalities which then have the interpretation of (super) hedging strategies. This approach is referred to as robust pricing and hedging and goes back to Hobson [14], see also Obłój [21] and Hobson [15] for survey papers. More recently, for example in Acciaio et. al. [1], martingale inequalities have been used to study some classical probabilistic inequalities, and are of interest in their own right.

Here we propose to study the distribution of $(\overline{M}_\infty, \underline{M}_\infty)$, where $\underline{M}_\infty := \inf_{t \leq \infty} M_t$ is the infimum of the process, given the distribution of M_∞, for a uniformly integrable continuous martingale M. More precisely, we present sharp lower and upper bounds on all double exit/no-exit probabilities for M in terms of the distribution of M_∞, i.e. the probabilities that $\overline{M}_\infty$ is *greater/smaller* than $\overline{b}$ *and/or* that $\underline{M}_\infty$ is *greater/smaller* than $\underline{b}$, for some barriers $\underline{b} < \overline{b}$. This amounts to considering eight different events. They of course come in pairs, e.g. $\{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\}$ is the complement of $\{\overline{M}_\infty < \overline{b}$ or $\underline{M}_\infty \leq \underline{b}\}$ and, by symmetry, it suffices to consider only one of $\{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\}$ and $\{\overline{M}_\infty < \overline{b}, \underline{M}_\infty \leq \underline{b}\}$. It follows that to provide a complete description it suffices to consider the three events

$$\{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty \leq \underline{b}\}, \qquad \{\overline{M}_\infty < \overline{b}, \underline{M}_\infty > \underline{b}\} \quad \text{and} \quad \{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\}. \tag{1}$$

By continuity and time-change arguments, it follows that for a fixed distribution μ of M_∞, our problem is equivalent to studying these events for $M_t = B_{t \wedge \tau}$ where τ varies among all stopping times such that M is uniformly integrable and $M_\infty = B_\tau$ has distribution μ, i.e. solutions to the Skorokhod embedding problem for μ in B, see Obłój [20]. Sharp bounds on the probability of the first event in (1) follow from Perkins and tilted-Jacka solutions, see Section 4. The case of the second event was treated in Cox and Obłój [9] and is also recalled in Section 4.

Our contribution here is twofold. First, we derive lower and upper bounds on $\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b})$ in terms of the distribution of M_∞ and give explicit constructions of martingales which attain the bounds. We do this by devising pathwise inequalities which give upper and lower bounds and then by constructing two new solutions to the Skorokhod embedding problem for which equalities are attained in our pathwise inequalities. Second, we study universal qualitative properties of the probabilities of the events in (1) seen as surfaces in the parameters $\underline{b}, \overline{b}$. While the techniques used to derive the bounds on $\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b})$ are not new, the explicit constructions we need to use are novel, and our goal in the first part of the paper is to provide those bounds which are currently not known; in this sense, we complete previous work in the literature. The contribution in the second part of the paper is, to the best of our knowledge, the first attempt to address questions of this nature.

1.1. Motivation

We believe that there are two natural motivations for our results. First, we believe we solve an intrinsically interesting probabilistic question and second, our results correspond to robust pricing and hedging of certain double barrier options in finance. We elaborate now on both.

From the probabilistic point of view, we follow in the footsteps of seminal works mentioned above. The results therein were typically stated for a martingale and its maximum but naturally can be reformulated for a martingale and its minimum $\underline{M}_\infty$. They grant us a full understanding of possible joint distributions of couples $(M_\infty, \overline{M}_\infty)$ or $(M_\infty, \underline{M}_\infty)$. In contrast, much less is known about the joint distribution of $(M_\infty, \overline{M}_\infty, \underline{M}_\infty)$ and it proves much harder to study (although promising recent progress has been made in this direction in a discrete time setting, when one considers the joint law of a random walk, its maximum, minimum and *signature* by [12]). Indeed, already in the case of Brownian motion B, while the distribution of $(B_t, \overline{B}_t)$ is readily accessible with a simply and explicit density, the distribution of the triplet $(B_t, \overline{B}_t, \underline{B}_t)$ is described through an infinite series. Likewise, $\mathbb{P}(\overline{M}_\infty \geq \overline{b})$ is maximised among all martingales M with a fixed distribution of M_∞, by one extremal martingale simultaneously for all $\overline{b}$. In contrast, as we will show here, maximising $\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b})$ will require martingales with qualitatively different behaviour for different values of $(\underline{b}, \overline{b})$.

In terms of mathematical finance, the constructions presented here correspond to robust pricing (and hedging) of double touch/no-touch barrier options—for a detailed discussion of applications we refer to our earlier papers [10,9] where we studied the first two events in (1). Such an option would pay out 1 if and only if one barrier is attained *and* a second given barrier is not attained, i.e. we consider the payoff of the form $\{\overline{S}_T \geq \overline{b}, \underline{S}_T > \underline{b}\}$, where $(S_t : t \leq T)$ is a uniformly integrable martingale representing the stock price process. The double touch/no-touch options are partially a theoretical construct—(to the best of our knowledge) they are not commonly traded even in Foreign Exchange (FX) markets, where barriers options are most popular. However, they prove useful as they can be represented as a sum or difference of other barrier options. We can then interpret our results as super-/sub-hedges for sums and differences of barrier options. More precisely, we can write

$$\mathbf{1}_{\{\overline{S}_T \geq \overline{b}, \underline{S}_T > \underline{b}\}} = \mathbf{1}_{\{\overline{S}_T \geq \overline{b}\}} - \mathbf{1}_{\{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}\}} \tag{2}$$

$$= 1 - \left(\mathbf{1}_{\{\underline{S}_T \leq \underline{b}\}} + \mathbf{1}_{\{\overline{S}_T < \overline{b}, \underline{S}_T > \underline{b}\}} \right). \tag{3}$$

The first decomposition (2) writes the payoff of a double touch/no-touch option as a difference of a one-touch option (with payoff $\mathbf{1}_{\{\overline{S}_T \geq \overline{b}\}}$) and a double touch option. The second decomposition (3) writes the payoff of a double touch/no-touch option as one minus the portfolio of a one-touch option and a double no-touch (range) option with payoff $\mathbf{1}_{\{\overline{S}_T < \overline{b}, \underline{S}_T > \underline{b}\}}$. This is of particular interest as both one-touch and range options are liquidly traded in main currency pairs in FX markets. Effectively, using the no-arbitrage prices derived in Theorems 2.2 and 2.4, we obtain a way of checking for absence of arbitrage in the observed prices of European calls/puts, one-touch and range options. Furthermore, if one-touch options are liquidly traded, we can then exploit pathwise inequalities derived in this paper as super- or sub-hedging strategies for range options or double touch options. For certain barriers this will be sharper than the hedges derived in Cox and Obłój [10,9] which assumes only that vanilla options are liquid.

1.2. Notation

Throughout the paper M denotes a continuous uniformly integrable martingale and B a standard real-valued Brownian motion. The running maximum and minimum of a Brownian motion B or a martingale M are denoted respectively $\overline{B}_t = \sup_{u \leq t} B_u$ and $\underline{B}_t = \inf_{u \leq t} B_u$, and similarly $\overline{M}_t$ and $\underline{M}_t$. The first hitting times of levels are denoted $H_x(B) := \inf\{t \geq 0 : B_t = x\}$, $x \in \mathbb{R}$. Likewise we will consider $H_x(M)$ and $H_x(\omega)$, the first hitting times for a martingale M

and a continuous path ω. Most of the time we simply write H_x as it should be clear from the context which process/path we consider. We will use the hitting times primarily to express events involving the running maximum and minimum, e.g. note that $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b}, \underline{B}_\tau > \underline{b}\}} = \mathbf{1}_{\{H_{\overline{b}} \leq \tau < H_{\underline{b}}\}}$ a.s. We also introduce the following notation to indicate composition of stopping times: if τ_1, τ_2 are both stopping times, then the stopping time $(\tau_2 \circ \tau_1)(\omega) = \tau_1(\omega) + \tau_2(\theta_{\tau_1}(\omega))$, where $\theta_t(\omega)$ is the usual shift operator, $\theta_t : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by $(\theta_t(\omega))_s = \omega_{t+s}$.

We use the notation $a \ll b$ to indicate that a is *much smaller* than b—this is only used to give intuition and is not rigorous. The minimum and maximum of two numbers are denoted $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ respectively, and the positive part is denoted $a^+ = a \vee 0$.

Finally, for a probability measure μ on $\mathbb{R}$ we let $-\infty \leq \ell_\mu < r_\mu \leq \infty$ be the bounds of the support of μ, i.e. $[\ell_\mu, r_\mu]$ is the smallest interval with $\mu([\ell_\mu, r_\mu]) = 1$.

2. Bounds for the probability of double exit/no-exit

In this section we provide sharp bounds on the probability

$$\mathbb{P}\left(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\right)$$

where $\underline{b} < 0 < \overline{b}$, and $M = (M_t : t \leq \infty)$ is a continuous uniformly integrable martingale. Our approach will involve two steps: first we provide pathwise inequalities which induce upper and lower bounds on the given event. Second, we show that these bounds are attained. More specifically, consider a continuous path $(\omega_t : 0 \leq t \leq T)$, where $T \leq \infty$. We will introduce pathwise inequalities comparing $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$ to a sum of a "static term", some function $f(\omega_T)$, and a "dynamic term" of the generic form $\beta(\omega_T - b)\mathbf{1}_{\{H_b < T\}}$. Note that such a dynamic term is zero initially and, when b is hit, it introduces a β-rotation of $f(\omega_T)$ around b. Note also that when evaluated on paths of a martingale, it will be a martingale. Consequently, we will construct random variables which dominate (or are dominated by) the random variable $\mathbf{1}_{\{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\}}$ and which can be decomposed into a martingale term and a function of the terminal value M_∞. Bounds on the double exit/no-exit probability above will be obtained by taking expectations in these inequalities. We further claim that these bounds are tight. This is proven in the subsequent section, where we build extremal martingales by designing optimal solutions to the Skorokhod embedding problem for Brownian motion.

2.1. Pathwise inequalities: upper bounds

We need to consider three different inequalities. As we will see later, it is always optimal to use exactly one of them, and the choice depends on the distribution of M_∞ and the values of $\overline{b}, \underline{b}$. We give the cases intuitive labels, their meaning will become clearer when we subsequently construct extremal martingales. Throughout this and the next section we assume that $0 < T \leq \infty$ is fixed and $(\omega_t : 0 \leq t \leq T)$ is a given continuous function. The hitting times are relative to ω. To keep the notation simple we do not emphasise the dependence on ω, e.g. $H_{\underline{b}} = H_{\underline{b}}(\omega) := \inf\{t \leq T : \omega_t = \underline{b}\}$, or $\overline{G}^I(K) = \overline{G}^I(K, (\omega_t : t \leq T))$.

$\overline{G}^I$: upper bound for $\underline{b} \ll 0 < \overline{b}$.

The inequality is presented graphically in Fig. 1. We can write it as:

$$\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} \leq \frac{1}{(K - \underline{b})}\left((\omega_T - K)^+ - (\underline{b} - \omega_T)^+ - (\omega_T - \underline{b})\mathbf{1}_{\{H_{\underline{b}} < T\}}\right) + \mathbf{1}_{\{\omega_T > \underline{b}\}}$$
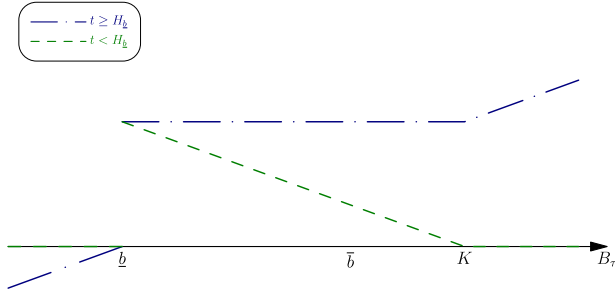
$$=: \overline{G}^I(K), \tag{4}$$

Fig. 1. $\overline{G}^I(K)$ in (4) providing an upper bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$.
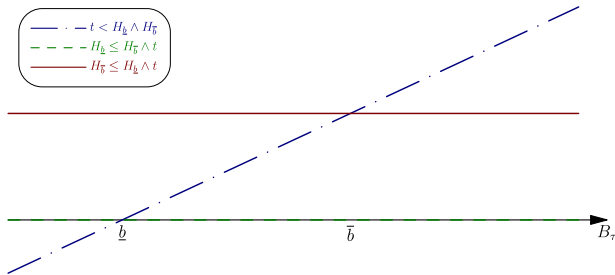

Fig. 2. $\overline{G}^{II}$ in (5) providing an upper bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$.

where we assume $K > \underline{b}$. We include here the special case where $K = \infty$, which corresponds to the upper bound $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} \leq \mathbf{1}_{\{\omega_T \geq \underline{b}\}}$. Note that the coefficient $1/(K - \underline{b})$ is taken so that the right-hand side after rotation at time $H_{\underline{b}}$ is zero above K.

$\overline{G}^{II}$: *upper bound for* $\underline{b} < 0 < \overline{b}$.

This is a fairly simple case: if we hit neither $\underline{b}$ nor $\overline{b}$, the inequality is simply $0 \leq \alpha_1(\omega_T - \underline{b})$ for some $\alpha_1 > 0$, so that the value is 1 if we strike $\overline{b}$ initially, and 0 if we strike $\underline{b}$ initially. This strategy is illustrated in Fig. 2. If the path hits either $\overline{b}$ or $\underline{b}$ we have a constant value of either 1 or 0 respectively:

$$
\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} \leq \alpha_1 \omega_T - \alpha_0 - \alpha_1(\omega_T - \overline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \wedge T\}} - \alpha_1(\omega_T - \underline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} \wedge T\}}
$$
$$
=: \overline{G}^{II}. \tag{5}
$$

The constraints on α_0, α_1 correspond to the need for the function to be zero if $\underline{b}$ is struck first, and 1 if $\overline{b}$ is struck first. We deduce that

$$
\begin{aligned}
\alpha_0 &= \underline{b}/(\overline{b} - \underline{b}) \\
\alpha_1 &= 1/(\overline{b} - \underline{b}).
\end{aligned} \tag{6}
$$

$\overline{G}^{III}$: *upper bound for* $\underline{b} < 0 \ll \overline{b}$.

The final inequality uses the fact that $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} \leq \mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}\}}$, and that the inequality for the latter also works for the former. We can then rewrite (2.2) from Brown, Hobson and Rogers [5]

as

$$\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} \leq \frac{(\omega_T - K)^+}{\overline{b} - K} + \frac{\overline{b} - \omega_T}{\overline{b} - K}\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}\}} =: \overline{G}^{III}(K), \tag{7}$$

where $K < \overline{b}$.

2.2. Pathwise inequalities: lower bounds

Observe that we have $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}} = 1 - \mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}}$ a.s. It follows that a pathwise upper bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$ corresponds to a pathwise lower bound of $\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}}$, and vice versa. We will use this below to rephrase some of the lower bounds as upper bounds.

$\underline{G}_I$: *lower bound for* $\underline{b} < 0 \ll \overline{b}$.

We let $\underline{G}_I$ to be the trivial inequality that the probability is bounded below by zero: $\underline{G}_I \equiv 0$.

$\underline{G}_{II}$: *lower bound for* $\underline{b} < 0 < \overline{b}$.

We describe an upper bound for $\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}}$ which, as argued above, is equivalent to a lower bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$. The inequality depends on two parameters K_1 and K_2 where $K_1 \geq \overline{b} > K_2 \geq \underline{b}$. The construction starts with equality on the region $[K_2, \overline{b})$ and inequality elsewhere. The first time the path hits $\overline{b}$, we rotate to get equality (with zero) on $[K_1, \infty)$ and so that the value is exactly 1 at $\underline{b}$. If the path later hits $\underline{b}$, we again rotate to gain equality (with 1) on $(-\infty, K_2]$ and $[\overline{b}, K_1]$. We write it as an inequality

$$\begin{aligned}
\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}} &\leq \alpha_2(K_2 - \omega_T)^+ + (1 - \alpha_4)\mathbf{1}_{\{\omega_T < \overline{b}\}} - \alpha_2(\omega_T - \overline{b})^+ \\
&\quad + \alpha_1(\omega_T - K_1)^+ + \alpha_4 + \beta_1(\omega_T - \overline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \wedge T\}} \\
&\quad + \beta_2(\omega_T - \underline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \leq T\}} + \beta_3(\omega_T - \underline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} \wedge T\}} \\
&=: 1 - \underline{G}_{II}(K_1, K_2),
\end{aligned} \tag{8}$$

which we present graphically in Fig. 3. It follows that $\underline{G}_{II}(K_1, K_2)$ is a lower bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$. We deduce immediately from the rotation conditions that $\beta_1 = \alpha_2 - \alpha_1$, $\beta_2 = \alpha_1$ and $\beta_3 = \alpha_2$. We have to satisfy two more constraints, namely that after hitting $\overline{b}$ and rotating the function is zero on $[K_1, \infty)$ and one at $\underline{b}$. Working out the values we have

$$\begin{cases}
\alpha_1 = \dfrac{1}{K_1 - \underline{b}} \\[2mm]
\alpha_2 = \dfrac{\overline{b} - \underline{b}}{(K_1 - \underline{b})(\overline{b} - K_2)} \\[2mm]
\alpha_4 = \dfrac{K_1 - \overline{b}}{K_1 - \underline{b}}
\end{cases}
\qquad
\begin{cases}
\beta_1 = \alpha_2 - \alpha_1 \\
\beta_2 = \alpha_1 \\
\beta_3 = \alpha_2.
\end{cases} \tag{9}$$

Observe that $\alpha_4 \in (0, 1]$ and $0 < \alpha_1 \leq \alpha_2$. We note that if we hit $\underline{b}$ before $\overline{b}$ we have a strict inequality in (8). Also, in the case where $K_2 = \underline{b}$ a number of the terms simplify: in particular, the construction initially gives $\underline{G}_{III} = 1$ for $\omega_T \in [\underline{b}, \overline{b})$ for $T < H_{\overline{b}}$. More generally, we can also have $K_1 = \overline{b}$ (with or without also $K_2 = \underline{b}$) and all the claims remain true.

$\underline{G}_{III}$: *lower bound for* $\underline{b} \ll 0 < \overline{b}$.

As previously, we describe an upper bound for $\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}}$. The inequality is represented in Fig. 4 and depends on two values K_1 and K_2 such that $\underline{b} < K_2 < K_1 < \overline{b}$. The inequality
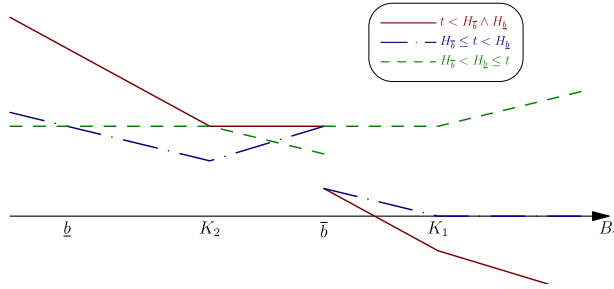
Fig. 3. $(1 - \underline{G}_{II}(K_1, K_2))$ in (8)–(9) providing an upper bound for $\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}} = 1 - \mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$. The case where we hit $\underline{b}$ before $\overline{b}$ is not shown.
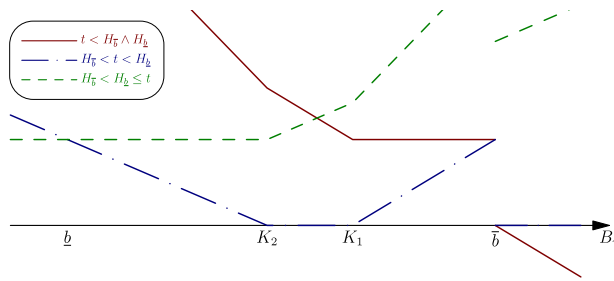

Fig. 4. $(1 - \underline{G}_{III}(K_1, K_2))$ in (10)–(11) providing an upper bound for $\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}} = 1 - \mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$. The case when we hit $\underline{b}$ before $\overline{b}$ is not shown.

starts with equality (equal to 1) between K_1 and $\overline{b}$, and if we hit $\overline{b}$ initially, we rotate to get equality (to 0) between K_2 and K_1. If we hit $\underline{b}$ after this, we rotate again to ensure the function is equal to 1 below K_2. If we initially hit $\underline{b}$ rather than $\overline{b}$, we rotate to get a function that is generally strictly greater than one. We write it as

$$
\begin{aligned}
\mathbf{1}_{\{\overline{\omega}_T < \overline{b} \text{ or } \underline{\omega}_T \leq \underline{b}\}} &\leq \alpha_2 (K_2 - \omega_T)^+ + \alpha_1 (K_1 - \omega_T)^+ + \mathbf{1}_{\{\omega_T < \overline{b}\}} - \alpha_1 (\omega_T - \overline{b})^+ \\
&\quad + \beta_1 (\omega_T - \overline{b}) \mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \wedge T\}} + \beta_2 (\omega_T - \underline{b}) \mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \leq T\}} + \beta_3 (\omega_T - \underline{b}) \mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} \wedge T\}} \\
&=: 1 - \underline{G}_{III}(K_1, K_2),
\end{aligned}
\tag{10}
$$

and it follows that $\underline{G}_{III}(K_1, K_2)$ is a lower bound for $\mathbf{1}_{\{\overline{\omega}_T \geq \overline{b}, \underline{\omega}_T > \underline{b}\}}$. We deduce immediately from the rotation conditions that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ and $\beta_3 = \alpha_1 + \alpha_2$. We have to satisfy two more constraints, namely that after hitting $\overline{b}$ and rotating, the function is zero on (K_2, K_1) and one in $\underline{b}$. Working out the values we have

$$
\begin{cases}
\alpha_1 = \dfrac{1}{\overline{b} - K_1} \\[2mm]
\alpha_2 = \dfrac{1}{K_2 - \underline{b}}
\end{cases}
\qquad
\begin{cases}
\beta_1 = \alpha_1 \\
\beta_2 = \alpha_2 \\
\beta_3 = \alpha_1 + \alpha_2.
\end{cases}
\tag{11}
$$

As in the previous case, we have a strict inequality in (10) if the path hits $\underline{b}$ before $\overline{b}$.

2.3. Probabilistic bounds

We now consider the pathwise inequalities above evaluated on a path of a continuous uniformly integrable martingale $M = (M_t : 0 \leq t \leq \infty)$. This gives a.s. bounds on $\mathbf{1}_{\{\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\}}$. By taking expectations we obtain bounds on the double exit/no-exit probabilities in terms of the distribution of M_∞. Indeed, observe that each of the bounds we get can be decomposed into two terms. The first of these depends on M_∞ alone, for example, in (8), the sum of the four quantities preceded by an α. The second corresponds to a martingale and disappears when taking expectations, e.g. considering again (8), the three terms which are preceded by a β sum to give a term with expected value zero.

Proposition 2.1. *Suppose* $M = (M_t : 0 \leq t \leq \infty)$ *is a continuous uniformly integrable martingale. Then*

$$\mathbb{P}\left(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\right) \leq \inf\left\{\mathbb{E}\left[\overline{G}^I(K)\right], \mathbb{E}\left[\overline{G}^{II}\right], \mathbb{E}\left[\overline{G}^{III}(K')\right]\right\}, \tag{12}$$

where the infimum is taken over $0 < K' < \overline{b} < K$ *and where* $\overline{G}^I, \overline{G}^{II}, \overline{G}^{III}$ *are given by* (4), (5), (6), *and* (7) *respectively, evaluated on paths of* M.

Our goal is to show that the above bound is optimal. A key aspect of the above result is that the right hand-side of (12) depends only on the distribution of M_∞ and not on the law of the martingale M. We let μ be a probability measure on $\mathbb{R}$ with finite first moment. It is clear that we may then assume (subject to a suitable shift of the martingale) that the measure μ is centred. We also exclude the trivial case where $\mu = \delta_0$ from our arguments, so necessarily $\mu((-\infty, 0))$ and $\mu((0, \infty))$ are both strictly positive. We write $M \in \mathcal{M}_\mu$ to denote that M is a continuous uniformly integrable martingale with $M_\infty \sim \mu$.

In the arguments below, we will commonly want to discuss the measure μ restricted to some interval. Moreover, in the case where there is an atom of μ at a point y, it may become necessary to split the atom into more than one part. It will be convenient therefore to split the measure μ according to its quantiles. We therefore introduce the notation $F(x) = \mu((-\infty, x])$ for the usual distribution function of the measure μ, and write $F^{-1}(q) = \inf\{x \in \mathbb{R} : F(x) \geq q\} \vee \ell_\mu$. Then for $p, q \in [0, 1]$ with $p \leq q$ we define the sub-probability measures

$$\mu_p^q((-\infty, x]) = (F(x) \wedge q - p) \vee 0 =: F_p^q(x). \tag{13}$$

In addition, we will write $\mu^q = \mu_0^q$ and $\mu_p = \mu_p^1$. Observe that $\mu_p^q(\mathbb{R}) = q - p$.

The *barycentre* of μ associates to a non-empty Borel set $\Gamma \subset \mathbb{R}$ the mean of μ over Γ via

$$\mu_B(\Gamma) = \frac{\int_\Gamma u\, \mu(\mathrm{d}u)}{\int_\Gamma \mu(\mathrm{d}u)}. \tag{14}$$

An obvious extension is to consider the barycentre of the measure μ when restricted to μ_p^q, which we denote by m_p^q, so

$$m_p^q = \begin{cases} (q - p)^{-1} \int x\, \mu_p^q(\mathrm{d}x) & \text{if } q > p \\ F^{-1}(q) & \text{otherwise.} \end{cases} \tag{15}$$

Now fix $\underline{b}, \overline{b} \in \mathbb{R}$ with $\underline{b} < 0 < \overline{b}$. Of importance in our constructions will be the following notions. Given p with $p \leq F(\underline{b}-)$, we want to find the probability q such that $m_p^q = \underline{b}$.

Specifically, define a function $\rho_- : [0, F(\underline{b}-)] \to [F(\underline{b}), 1]$ by

$$\rho_-(p) = \inf\{q \geq F(\underline{b}) : m_p^q \geq \underline{b}\}. \tag{16}$$

Similarly, we can define $\rho_+ : [F(\overline{b}), 1] \to [0, F(\overline{b}-)]$ by

$$\rho_+(q) = \sup\{p \leq F(\overline{b}-) : m_p^q \leq \overline{b}\}. \tag{17}$$

It is straightforward to see that $\rho_-(p)$ and $\rho_+(q)$ are both continuous, strictly decreasing functions, and are well defined since $\underline{b} < 0 = \int x\, \mu(dx) < \overline{b}$, so that the infimum in (16) and the supremum in (17) are both over non-empty sets. Further, note that we get:

$$m_p^{\rho_-(p)} = \underline{b}, \qquad m_{\rho_+(q)}^q = \overline{b} \tag{18}$$

for all $p \leq F(\underline{b}-)$ and all $q \geq F(\overline{b})$. Observe that the barycentre has two nice properties: first, if we rescale the measure μ by a constant, then the barycentre is unchanged. Second, if we wish to show that a measure μ has barycentre b, it is sufficient to show that

$$\int (x - b)\, \mu(dx) = 0,$$

independent of whether μ is a probability measure. In the case where μ is a probability measure $\mu_B(\mathbb{R})$ is just the mean of the measure. Finally, we introduce the additional useful notation

$$\tilde{m}_p^q = (q - p)m_p^q.$$

Since the functions ρ_+ and ρ_- are both continuous and strictly decreasing, their inverses are also continuous and strictly decreasing where defined—for example, ρ_+^{-1} maps $[\rho_+(1), F(\overline{b}-)] \to [F(\overline{b}), 1]$.

A critical role in the construction of embeddings will be played by the following definition. Set

$$\pi^* = \inf\left\{p \in [\rho_+(1) \vee F(\underline{b}), F(\overline{b}-)] : \rho_+^{-1}(p) - p \leq \frac{-\underline{b}}{\overline{b} - \underline{b}}\right\} \wedge F(\overline{b}-), \tag{19}$$

where we use the standard convention that the infimum of an empty set is ∞. Since $\rho_+^{-1}(F(\overline{b}-)) = F(\overline{b})$, $\rho_+^{-1}(p)$ is continuous and $\underline{b} < 0$, it follows that $\pi^* \in [\rho_+(1) \vee F(\underline{b}), F(\overline{b}-)]$. Then we have the following theorem.

Theorem 2.2 (*Upper Bound*). *The bound in* (12) *is sharp. More precisely, let μ be a given centred probability measure on $\mathbb{R}$. Then exactly one of the following is true*

I. *'$\underline{b} \ll 0 < \overline{b}$': we have $\pi^* = F(\underline{b})$ and $\rho_+^{-1}(\pi^*) - \pi^* < -\underline{b}(\overline{b} - \underline{b})^{-1}$.*
   Then there is a martingale $M \in \mathcal{M}_\mu$ such that

$$\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = \mathbb{E}\left[\overline{G}^I(z^*)\right],$$

   where $\overline{G}^I$ is given by (4) *evaluated on paths of M, and $z^* = F^{-1}(\xi)$ where ξ solves*

$$\int (x - \underline{b})\, \mu_{F(\underline{b})}^\xi = -\underline{b}. \tag{20}$$

II. *'$\underline{b} < 0 < \overline{b}$': we have $\rho_+^{-1}(\pi^*) - \pi^* \geq -\underline{b}(\overline{b} - \underline{b})^{-1}$.*

Then there is a martingale $M \in \mathcal{M}_\mu$ such that

$$\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = \mathbb{E}\left[\overline{G}^{II}\right] = -\underline{b}(\overline{b} - \underline{b})^{-1},$$

where $\overline{G}^{II}$ is given by (5)–(6) evaluated on paths of M.

III. *'$\underline{b} < 0 \ll \overline{b}$': we have $\pi^* = \rho_+(1)$ and $1 - \pi^* < -\underline{b}(\overline{b} - \underline{b})^{-1}$.*
Then there is a martingale $M \in \mathcal{M}_\mu$ such that

$$\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = \mathbb{E}\left[\overline{G}^{III}(F^{-1}(\pi^*))\right],$$

where $\overline{G}^{III}$ is given by (7) evaluated on paths of M.

In a similar manner to Proposition 2.1, the pathwise inequalities described in Section 2.2 instantly imply a lower bound on the double exit/no-exit probabilities:

Proposition 2.3. *Suppose $M = (M_t : 0 \leq t \leq \infty)$ is a continuous uniformly integrable martingale. Then*

$$\mathbb{P}\left(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}\right) \geq \sup\left\{0, \mathbb{E}\left[\underline{G}_{II}(K_1', K_2)\right], \mathbb{E}\left[\underline{G}_{III}(K_1, K_2)\right]\right\}, \tag{21}$$

where the supremum is taken over $\underline{b} < K_2 < K_1 < \overline{b} < K_1'$ and where $\underline{G}_{II}, \underline{G}_{III}$ are given by (8), (9) and (10), (11) respectively, evaluated on paths of M.

We proceed to show that this lower bound is optimal. Write

$$\gamma = 1 - F(\overline{b}-) + F(\underline{b}), \tag{22}$$

and consider the condition

$$\tilde{m}_{F(\underline{b})}^{F(\overline{b}-)} + \gamma\underline{b} \geq 0. \tag{23}$$

If this holds, then we can find $\lambda \in (F(\underline{b}), F(\overline{b}-)]$ such that

$$\tilde{m}_{F(\underline{b})}^\lambda + (1 - \lambda + F(\underline{b}))\underline{b} = 0 \tag{24}$$

since the left-hand side is increasing in λ and runs between $\underline{b}$ and a term which is positive by (23). If (23) fails, we can imagine moving mass from an atom at $\underline{b}$, to the right, in the process moving the average of the mass upwards. In this case, consider the condition

$$\tilde{m}_{F(\underline{b})}^{F(\overline{b}-)} + \gamma\overline{b} \leq 0. \tag{25}$$

If (23) fails, and (25) holds, then we set $\xi = F(\underline{b})$ and we can find $\lambda \in (0, \gamma]$ such that

$$\tilde{m}_{F(\underline{b})}^{F(\overline{b}-)} + \lambda\underline{b} + (\gamma - \lambda)\overline{b} = 0. \tag{26}$$

Given such a λ, we will show that there exists $\pi^* \in [F(\overline{b}-), 1)$ such that

$$\tilde{m}^\xi + \tilde{m}_{F(\overline{b}-)}^{\pi^*} = \underline{b}(\xi + \pi^* - F(\overline{b}-)). \tag{27}$$

If (25) also fails, and

either $\rho_-(0) \geq F(\overline{b}-)$ or $\rho_-(0) < F(\overline{b}-)$ and

$$\tilde{m}_{\rho_-(0)}^{F(\overline{b}-)} + \overline{b}(1 - F(\overline{b}-) + \rho_-(0)) > 0 \tag{28}$$

then there exists $\xi \in (F(\underline{b}), \rho_-(0) \wedge F(\overline{b}-))$ such that

$$\tilde{m}_\xi^{F(\overline{b}-)} + \overline{b}(1 - F(\overline{b}-) + \xi) = 0. \tag{29}$$

Then we define π^* as the solution to (27) again.

Finally, if (23), (25) and (28) all fail, then there exists $\pi^* \in [\rho_-(0), F(\overline{b}-))$ such that

$$\tilde{m}_{\pi^*}^{F(\overline{b}-)} + \overline{b}(1 - F(\overline{b}-) + \pi^*) = 0. \tag{30}$$

Theorem 2.4 (*Lower Bound*). *The bound in* (21) *is sharp. More precisely, let μ be a given centred probability measure on $\mathbb{R}$. Then exactly one of the following is true:*

  I. *'$\underline{b} < 0 \ll \overline{b}$': condition* (23) *holds.*
     Then there is a martingale $M \in \mathcal{M}_\mu$ such that $\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = 0 = \mathbb{E}\left[\underline{G}_I\right]$.
 II. *'$\underline{b} < 0 < \overline{b}$': condition* (23) *fails, and either* (25) *holds or* (25) *fails and* (28) *holds.*
     Then there is a martingale $M \in \mathcal{M}_\mu$ such that

$$\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = \mathbb{E}\left[\underline{G}_{II}(\pi^*, \xi)\right], \tag{31}$$

  where $\underline{G}_{II}$ is given via (8) *and* (9)*, evaluated on paths of M, and π^* solves* (27)*.*
III. *'$\underline{b} \ll 0 < \overline{b}$': conditions* (23)*,* (25) *and* (28) *fail.*
     Then there is a martingale $M \in \mathcal{M}_\mu$ such that

$$\mathbb{P}(\overline{M}_\infty \geq \overline{b}, \underline{M}_\infty > \underline{b}) = \mathbb{E}\left[\underline{G}_{III}(\pi^*, \rho_-(0))\right] \tag{32}$$

  where $\underline{G}_{II}$ is given via (10) *and* (11)*, evaluated on paths of M, and π^* is given by* (30)*.*

Remark 2.5. Throughout the paper, we have assumed that $(M_t)_{t\geq 0}$ has continuous paths. This assumption can be relaxed. It is relatively simple to see that if we only assume that barriers $\underline{b}, \overline{b}$ are crossed in a continuous manner then all of our results remain true. If we only assume that (M_t) has càdlàg paths then the situation is more complex. The optimal behaviour will essentially be as before, but we can use jumps to hide some of the occasions where a barrier is hit. More precisely, consider the continuous martingale M given in Theorem 2.2 and, for $\varepsilon > 0$, consider the time-change:

$$\rho_t^\varepsilon = \inf\{u \geq t : M_u \in [\underline{b} + \varepsilon, \infty)\}.$$

Then $N_t = M_{\rho_t^\varepsilon}$ is a UI martingale which excludes paths of M_t where the minimum goes below $\underline{b} + \varepsilon$, but which later return above $\underline{b} + \varepsilon$. In general, any possible martingale M_t can be improved by performing such an operation, and so this suggests that an optimal discontinuous model can be chosen in such a manner that it is continuous on $[\underline{b} + \varepsilon, \infty)$ and only takes values on $(-\infty, \underline{b}]$ if it is the final value of the martingale. This observation can be used as a starting point for an analysis similar to that given above to determine the optimal martingale models for a given measure. We do not pursue the details here.

3. Proofs that the bounds are sharp via new solutions to the Skorokhod embedding problem

In this section we prove Theorems 2.2 and 2.4. We do this by constructing new solutions to the Skorokhod embedding problem for a Brownian motion B. Specifically, we will construct

stopping times τ such that $B_\tau \sim \mu$, $(B_{t \wedge \tau} : t \geq 0)$ is UI and equalities are attained almost surely in the inequalities of Sections 2.1–2.2. It is then straightforward to see that martingales required in Theorems 2.2 and 2.4 are given by $M_t := B_{t \wedge \tau}$.

We will use below some well known facts about the existence of Skorokhod embeddings. Specifically, given a measure μ with mean m and a Brownian motion B with $B_0 = m$, then there exists a stopping time τ such that $B_\tau \sim \mu$ and $(B_{t \wedge \tau} : t \geq 0)$ is uniformly integrable. Moreover, it follows from uniform integrability that if the measure μ is supported on a bounded interval, then the process will stop before the first exit time of the interval.

Proof of Theorem 2.2. We take $B = (B_t : t \geq 0)$ a standard real-valued Brownian motion. All the hitting times $H_\bullet$ below are for B. As described above, we will prove this result by constructing a stopping time τ such that B_τ has the distribution μ, and such that the conjectured bounds hold for the corresponding continuous time martingale which is the stopped process.

From the definition of π^* in (19) it is clear that at least one of the cases holds. Clearly II excludes the other two. To show that I and III are exclusive, as $\rho_+^{-1}(\rho_+(1)) = 1$, it suffices to argue that the following is impossible

$$\pi^* = \rho_+(1) = F(\underline{b}) > \overline{b}(\overline{b} - \underline{b})^{-1}. \tag{33}$$

Assume (33) holds. From the last condition we get $\overline{b}(1 - \pi^*) < -\underline{b}\pi^*$, and using the fact that $\pi^* = \rho_+(1)$, this can be expressed as $\int x \, \mu_{\pi^*}(dx) + \underline{b}\pi^* < 0$. However $\pi^* \geq F(\underline{b})$ implies that this is greater than or equal to $\int x \, \mu(dx) = 0$ giving a contradiction. We conclude that the cases I–III are exclusive.

We now show the existence of a suitable embedding. We consider initially the case I. We first note that the solution ξ of (20) is in $(\rho_+^{-1}(\pi^*), 1]$. Since

$$\int (x - \underline{b}) \, \mu_{F(\underline{b})}^{\rho_+^{-1}(F(\underline{b}))}(dx) = \int (x - \underline{b}) \mu_{F(\underline{b})}^{\rho_+^{-1}(F(\underline{b}))}(dx) + \int (\overline{b} - \underline{b}) \, \mu_{F(\underline{b})}^{\rho_+^{-1}(F(\underline{b}))}(dx)$$

$$= (\overline{b} - \underline{b}) \left(\rho_+^{-1}(F(\underline{b})) - F(\underline{b}) \right)$$

$$< -\underline{b},$$

we conclude that $\xi > \rho_+^{-1}(\pi^*)$. To see that $\xi \leq 1$, we note:

$$\int (x - \underline{b}) \, \mu_{F(\underline{b})}(dx) \geq \int (x - \underline{b}) \, \mu(dx) = -\underline{b}.$$

Since the expression $\int (x - \underline{b}) \, \mu_{F(\underline{b})}^\xi(dx)$ is strictly increasing and continuous in ξ, there is a unique ξ. For this value of ξ, we now define a measure ν by

$$\nu = \left[-\frac{\underline{b}}{\overline{b} - \underline{b}} - (\xi - F(\underline{b})) \right] \delta_{\underline{b}} + \mu_{F(\underline{b})}^\xi.$$

Observe that the atom at $\underline{b}$ has mass greater than or equal to zero, and by construction, ν has total mass $-\underline{b}(\overline{b} - \underline{b})^{-1}$ and barycentre $\overline{b}$ since

$$\int (x - \overline{b}) \, \nu(dx) = \int (x - \overline{b}) \, \mu_{F(\underline{b})}^\xi(dx) + \left[-\frac{\underline{b}}{\overline{b} - \underline{b}} - (\xi - F(\underline{b})) \right] (\underline{b} - \overline{b})$$

$$= (\underline{b} - \overline{b})(\xi - F(\underline{b})) - \underline{b} + \left[-\frac{\underline{b}}{\overline{b} - \underline{b}} - (\xi - F(\underline{b})) \right] (\underline{b} - \overline{b})$$

$$= 0.$$

We now show that this means we can construct a suitable embedding. The idea will be initially to run until the first time we hit either of $\overline{b}$ or $\underline{b}$. The mass that hits $\overline{b}$ first will then be used to embed ν, and all the mass that hits $\underline{b}$ (which will include the atomic term from ν) can then be embedded in the remaining areas, $(0, \underline{b}] \cup [F^{-1}(\xi), \infty)$. So suppose we are in case I, and let τ_1 be first time we hit one of $\underline{b}$ or $\overline{b}$, so $\tau_1 = H_{\overline{b}} \wedge H_{\underline{b}}$. Then $\mathbb{P}(B_{\tau_1} = \overline{b}) = -\underline{b}(\overline{b} - \underline{b})^{-1}$. Let τ_2 be a UI embedding of the probability measure $-\frac{\overline{b}-\underline{b}}{\overline{b}}\nu$ given $B_0 = \overline{b}$ and let τ_3 be a UI embedding of σ given $B_0 = \underline{b}$, where

$$\sigma = \frac{\left(\mu^{F(\underline{b})} + \mu_\xi\right)}{F(\underline{b}) + 1 - \xi}.$$

It can be verified that σ has barycentre $\underline{b}$ since

$$\int (x - \underline{b}) \left(\mu^{F(\underline{b})} + \mu_\xi\right)(dx) = \int (x - \underline{b})\,\mu(dx) - \int (x - \underline{b})\,\mu_{F(\underline{b})}^\xi(dx) = 0.$$

Then (recalling the definition in Section 1.2) we set

$$\tau := \tau_2 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\overline{b}}\}} \mathbf{1}_{\{\tau_2 \circ \tau_1 < H_{\underline{b}}\}} + \tau_3 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\underline{b}}\}} + \tau_3 \circ \tau_2 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\overline{b}}\}} \mathbf{1}_{\{\tau_2 \circ \tau_1 = H_{\underline{b}}\}}.$$

We see that τ is a UI embedding of μ, and moreover τ is such that $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b},\ \underline{B}_\tau > \underline{b}\}} = \overline{G}^I(F^{-1}(\xi))$ a.s.

Consider now case II. Suppose initially that in addition, $\rho_+^{-1}(\pi^*) - \pi^* = -\underline{b}(\overline{b} - \underline{b})^{-1}$. We define measures ν and σ by:

$$\nu = \frac{1}{\rho_+^{-1}(\pi^*) - \pi^*} \mu_{\pi^*}^{\rho_+^{-1}(\pi^*)}$$

$$\sigma = \frac{1}{1 + \pi^* - \rho_+^{-1}(\pi^*)} \left(\mu^{\pi^*} + \mu_{\rho_+^{-1}(\pi^*)}\right).$$

Then ν has barycentre $\overline{b}$, while σ has barycentre $\underline{b}$. Let τ_1 be as above, τ_2 be a UI embedding of ν given $B_0 = \overline{b}$ and τ_3 be a UI embedding of σ given $B_0 = \underline{b}$. Then the stopping time

$$\tau := \tau_2 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\overline{b}}\}} + \tau_3 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\underline{b}}\}}$$

is a UI embedding of μ, and $B_{t \wedge \tau}$ satisfies $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b},\ \underline{B}_\tau > \underline{b}\}} = \overline{G}^{II}$ a.s. where $\overline{G}^{II}$ is the random variable defined in (5), evaluated on paths of B.

The case where $\rho_+^{-1}(\pi^*) - \pi^* > -\underline{b}(\overline{b} - \underline{b})^{-1}$ is almost identical—observe that in this case, there must be an atom of μ at $\overline{b}$ with $F(\overline{b}) - F(\overline{b}-) > -\underline{b}(\overline{b} - \underline{b})^{-1}$. However, the argument above works without alteration if we take:

$$\nu = \frac{1}{-\underline{b}(\overline{b} - \underline{b})^{-1}} \delta_{\overline{b}}$$

$$\sigma = \frac{1}{1 + \underline{b}(\overline{b} - \underline{b})^{-1}} (\mu - \nu).$$

Finally we consider III. Then define measures ν and σ by:

$$\nu = \frac{1}{1 - \pi^*} \mu_{\pi^*}$$

$$\sigma = \frac{1}{\pi^*} \mu^{\pi^*}.$$

So the barycentre of ν is $\overline{b}$, and the barycentre of σ is m^{π^*}. Define τ_1 to be the first hitting time of $\{m^{\pi^*}, \overline{b}\}$, so $\tau_1 = H_{m^{\pi^*}} \wedge H_{\overline{b}}$, then $\mathbb{P}(B_{\tau_1} = \overline{b}) = \pi^* = -m^{\pi^*}(\overline{b} - m^{\pi^*})^{-1}$. We may then proceed as above, so we define τ_2 to be a UI embedding of ν given $B_0 = \overline{b}$ and τ_3 to be a UI embedding of σ given $B_0 = m^{\pi^*}$. Then the stopping time

$$\tau := \tau_2 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{\overline{b}}\}} + \tau_3 \circ \tau_1 \mathbf{1}_{\{\tau_1 = H_{m^{\pi^*}}\}}$$

is a UI embedding of μ, and satisfies $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b}, \, \underline{B}_\tau > \underline{b}\}} = \overline{G}^{III}(F^{-1}(\pi^*))$ a.s. where $\overline{G}^{III}(\cdot)$ is the random variable defined in (7), evaluated on paths of B. $\quad\square$

Proof of Theorem 2.4. The setup, and general methodology, is analogous to the proof of Theorem 2.2.

It follows from their respective definitions that exactly one of I–III holds.

Suppose I holds, so that (23) is true. Then, by continuity, there exists $\lambda \in (F(\underline{b}), F(\overline{b}-)]$ such that (24) holds (taking $\lambda = F(\underline{b})$ gives $\underline{b}$ on the left hand side of (24)). Let τ_1 be a UI embedding of

$$\chi = \mu_{F(\underline{b})}^{\lambda} + (1 - \lambda + F(\underline{b}))\delta_{\underline{b}} \tag{34}$$

in the Brownian motion starting at 0, and observe that the measure

$$\nu = \frac{\mu^{F(\underline{b})} + \mu_\lambda}{1 - \lambda + F(\underline{b})}$$

has mean $\underline{b}$, which follows since:

$$(1 - \lambda + F(\underline{b})) \int x\nu(\mathrm{d}x) = \tilde{m}^{F(\underline{b})} + \tilde{m}_\lambda$$

$$= -\tilde{m}_{F(\underline{b})}^{\lambda} = \underline{b}(1 - \lambda + F(\underline{b})).$$

Let τ_2 be a UI embedding of ν in a Brownian motion starting from $B_0 = \underline{b}$. Finally define

$$\tau := \tau_1 \mathbf{1}_{\{B_{\tau_1} \neq \underline{b}\}} + \tau_2 \circ \tau_1 \mathbf{1}_{\{B_{\tau_1} = \underline{b}\}},$$

which is a UI embedding of μ in the Brownian motion B. Note that $\overline{B}_\tau \geq \overline{b}$ only if $\underline{B}_\tau \leq \underline{b}$. It follows that $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b}, \, \underline{B}_\tau > \underline{b}\}} = 0 = \underline{G}_I$ a.s.

Suppose now that II holds. We consider separately the case where (23) fails and (25) holds, and the case where both (23) and (25) fail, but (28) holds. First suppose (25) holds. Then

$$\lambda \mapsto \tilde{m}_{F(\underline{b})}^{F(\overline{b}-)} + \lambda \underline{b} + (\gamma - \lambda)\overline{b}$$

is continuous, and strictly negative for $\lambda = 0$ and positive for $\lambda = \gamma$. Hence there exists $\lambda \in (0, \gamma]$ such that (26) holds. Fix $\xi = F(\underline{b})$ and consider

$$[F(\overline{b}-), 1) \ni \pi^* \mapsto \tilde{m}^\xi + \tilde{m}_{F(\overline{b}-)}^{\pi^*} - \underline{b}(\xi + \pi^* - F(\overline{b}-)).$$

In the limit as $\pi^* \to 1$, the expression simplifies to $-\tilde{m}_{F(\underline{b})}^{F(\overline{b}-)} - \gamma \underline{b}$ which is strictly positive since (23) is assumed to fail, while if $\pi^* = F(\overline{b}-)$ the expression simplifies to $\tilde{m}^{F(\underline{b})} - \underline{b}F(\underline{b})$, which is non-positive, since $\tilde{m}^{F(\underline{b})} = \int x \, \mu^{F(\underline{b})}(\mathrm{d}x) \leq \int \underline{b} \, \mu^{F(\underline{b})}(\mathrm{d}x)$. Hence there is a unique π^* satisfying (27).

Now define a measure

$$\chi = \mu_{F(\underline{b})}^{F(\overline{b}-)} + \lambda\delta_{\underline{b}} + (\gamma - \lambda)\delta_{\overline{b}}.$$

From (26) it follows that χ is centred, and we embed this initially. The mass which arrives at $\bar{b}$ will then run to the measure

$$\nu = \frac{(\gamma - \lambda - (1 - \pi^*))\delta_{\underline{b}} + \mu_{\pi^*}}{\gamma - \lambda}$$

which has mean $\bar{b}$ by the following computation:

$$(\gamma - \lambda) \int x \, \nu(\mathrm{d}x) = \underline{b}(\gamma - \lambda - (1 - \pi^*)) + \tilde{m}_{\pi^*}$$

$$= \underline{b}(\gamma - \lambda - (1 - \pi^*)) - \tilde{m}_{F(\bar{b}-)}^{\pi^*} - \tilde{m}_{F(\underline{b})}^{F(\bar{b}-)} - \tilde{m}^{F(\underline{b})}$$

$$= \underline{b}(\gamma - \lambda - (1 - \pi^*)) - \underline{b}(\xi + \pi^* - F(\bar{b}-)) + \lambda \underline{b} + (\gamma - \lambda)\bar{b}$$

$$= \underline{b}(\gamma - 1 - \xi + F(\bar{b}-)) + \bar{b}(\gamma - \lambda).$$

Here we have used (26), (27) and the fact that $\xi = F(\underline{b})$. From the definition of γ in (22), the desired conclusion follows.

Finally, we embed the remaining part of μ from the mass that finishes at $\underline{b}$ after either the first or second step, which has total probability $\gamma - \lambda + \pi^* - 1 + \lambda = \xi + \pi^* - F(\bar{b}-)$. Set

$$\sigma = \frac{\mu^\xi + \mu_{F(\bar{b}-)}^{\pi^*}}{\xi + \pi^* - F(\bar{b}-)}, \tag{35}$$

and σ has mean $\underline{b}$:

$$(\xi + \pi^* - F(\bar{b}-)) \int x \, \sigma(\mathrm{d}x) = \tilde{m}^\xi + \tilde{m}_{F(\bar{b}-)}^{\pi^*}$$

$$= \underline{b}(\xi + \pi^* - F(\bar{b}-))$$

by (27). The final stopping time will be of the same form both in this case and in the case where (25) holds, and when (25) fails but (28) holds. So before constructing the embedding, we give a description of the relevant measures in the second case.

Suppose (25) fails, but (28) holds. Then in a similar manner to above, we can find $\xi \in (F(\underline{b}), \rho_-(0) \wedge F(\bar{b}-))$ such that (29) holds. Define

$$\chi = \mu_\xi^{F(\bar{b}-)} + (1 - F(\bar{b}-) + \xi)\delta_{\bar{b}}$$

and choose π^* as before as the solution to (27). Then set

$$\nu = \frac{(\pi^* - F(\bar{b}-) + \xi)\delta_{\underline{b}} + \mu_{\pi^*}}{1 - F(\bar{b}-) + \xi}$$

and we verify that ν has mean $\bar{b}$:

$$(1 - F(\bar{b}-) + \xi) \int x \, \nu(\mathrm{d}x) = \underline{b}(\pi^* - F(\bar{b}-) + \xi) + \tilde{m}_{\pi^*}$$

$$= \underline{b}(\pi^* - F(\bar{b}-) + \xi) - \tilde{m}_{F(\bar{b}-)}^{\pi^*} - \tilde{m}_\xi^{F(\bar{b}-)} - \tilde{m}^\xi$$

$$= \underline{b}(\pi^* - F(\bar{b}-) + \xi) - \underline{b}(\xi + \pi^* - F(\bar{b}-)) + \bar{b}(1 - F(\bar{b}-) + \xi)$$

$$= \bar{b}(1 - F(\bar{b}-) + \xi).$$

Finally, setting σ as in (35) we again have σ with mean $\underline{b}$.

In both cases, we construct an embedding as follows: let τ_1 be a UI embedding of χ (starting from 0). Then let τ_2 be a UI embedding of ν (starting from $\overline{b}$). Finally, we let τ^3 be a UI embedding of σ (starting from $\underline{b}$). We then define the complete embedding by:

$$\tau := \tau_1 \mathbf{1}_{\{B_{\tau_1} \in (\underline{b}, \overline{b})\}} + \tau_2 \circ \tau_1 \mathbf{1}_{\{B_{\tau_1} = \overline{b}\}} \mathbf{1}_{\{B_{\tau_2 \circ \tau_1} > \underline{b}\}}$$
$$+ \tau_3 \circ \left(\tau_1 \mathbf{1}_{\{B_{\tau_1} = \underline{b}\}} + \tau_2 \circ \tau_1 \mathbf{1}_{\{B_{\tau_1} = \overline{b}\}} \mathbf{1}_{\{B_{\tau_2 \circ \tau_1} = \underline{b}\}} \right),$$

and it follows from our construction that τ is a UI embedding of μ which moreover satisfies $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b}, \, \underline{B}_\tau > \underline{b}\}} = \underline{G}_{II}(\pi^*, \xi)$.

Suppose finally we are in case III, so that (23), (25) and (28) all fail. Then there exists $\pi^* \in [\rho_-(0), F(\overline{b}-))$ such that (30) holds.

Define the probability measure

$$\chi = \mu_{\pi^*}^{F(\overline{b}-)} + (1 - F(\overline{b}-) - \pi^*) \delta_{\overline{b}},$$

which has mean 0 by the definition of π^*. Define also

$$\nu = \frac{\rho_-(0) \delta_{\underline{b}} + \mu_{\rho_-(0)}^{\pi^*} + \mu_{F(\overline{b}-)}}{1 - F(\overline{b}-) + \pi^*}$$

and we confirm that ν has mean $\overline{b}$:

$$(1 - F(\overline{b}-) + \pi^*) \int x \, \nu(dx) = \tilde{m}^{\rho_-(0)} + \tilde{m}_{\rho_-(0)}^{\pi^*} + \tilde{m}_{F(\overline{b}-)}$$
$$= \tilde{m}^{\pi^*} + \tilde{m}_{F(\overline{b}-)}$$
$$= -\tilde{m}_{\pi^*}^{F(\overline{b}-)}$$
$$= \overline{b}(1 - F(\overline{b}-) + \pi^*).$$

Finally, any mass which is at $\underline{b}$ we finally embed to the measure $\sigma = (\rho_-(0))^{-1} \mu^{\rho_-(0)}$. That is, we define the stopping times τ_1 which is a UI embedding of χ starting at 0. Then let τ_2 be a UI embedding of ν, given initial value $\overline{b}$, and τ_3 an embedding of σ given initial value $\underline{b}$. Finally, we define

$$\tau := \tau_1 \mathbf{1}_{\{B_{\tau_1} \neq \overline{b}\}} + \tau_2 \circ \tau_1 \mathbf{1}_{\{B_{\tau_1} = \overline{b}\}} \mathbf{1}_{\{B_{\tau_2 \circ \tau_1} > \underline{b}\}} + \tau_3 \circ \tau_2 \circ \tau_1 \mathbf{1}_{\{B_{\tau_1} = \overline{b}\}} \mathbf{1}_{\{B_{\tau_2 \circ \tau_1} = \underline{b}\}},$$

to get a UI embedding of μ in B. Furthermore, it follows from the construction that $\mathbf{1}_{\{\overline{B}_\tau \geq \overline{b}, \, \underline{B}_\tau > \underline{b}\}} = \underline{G}_{III}(\pi^*, \rho_-(0))$. □

4. On joint distribution of the maximum and minimum of a continuous UI martingale

We turn now to studying the properties of joint distribution of the maximum and minimum of a continuous UI martingale. As previously, $(M_t : 0 \leq t \leq \infty)$ is a uniformly integrable continuous martingale. We let μ be its terminal distribution, $\mu \sim M_\infty$, and recall that $-\infty \leq \ell_\mu < r_\mu \leq \infty$ are the bounds of the support of μ, i.e. $[\ell_\mu, r_\mu]$ is the smallest interval with $\mu([\ell_\mu, r_\mu]) = 1$. Using Theorems 2.2 and 2.4, as well as existing results, we study the functions

$$p(\underline{b}, \overline{b}) = \mathbb{P} \left(\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty < \overline{b} \right) \tag{36}$$

$$q(\underline{b}, \overline{b}) = \mathbb{P} \left(\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty \geq \overline{b} \right) \tag{37}$$

$$r(\underline{b}, \overline{b}) = \mathbb{P} \left(\underline{M}_\infty \leq \underline{b} \text{ and } \overline{M}_\infty \geq \overline{b} \right) \tag{38}$$

for $\underline{b} \le 0 \le \overline{b}$. Note that with no restrictions on M_0, when looking at extrema of the functions above, it is enough to consider M_0 a constant (e.g. when maximising r) or $M_0 \equiv M_\infty$ (e.g. when minimising r). The latter is degenerate and henceforth we assume M_0 is a constant a.s. Further, as our results are translation invariant, we may and will take $M_0 = 0$ a.s. It follows that μ is centred.

It follows from Dambis, Dubins–Schwarz Theorem that M is a (continuous) time change of Brownian motion, i.e. we can write $M_t = B_{\tau_t}$, $t \le \infty$, for some Brownian motion and an increasing family of stopping times (τ_t) with $B_{\tau_\infty} \sim M_\infty$, $(B_{t \wedge \tau_\infty} : t \ge 0)$ UI and $\overline{M}_\infty = \overline{B}_{\tau_\infty}$, $\underline{M}_\infty = \underline{B}_{\tau_\infty}$. In consequence, the problem reduces to studying the maximum and minimum of Brownian motion stopped at $\tau = \tau_\infty$, which is a solution the Skorokhod embedding problem. We can deduce results about the optimal properties of the martingales from corresponding results about Skorokhod embeddings. Our first result concerns the embeddings of Perkins and the 'tilted-Jacka' construction, which we now recall using the notation established previously. These constructions have been considered in [10], and we will need some results from this paper; however both constructions have a long history—see for example [22,7,16,8]. For the Perkins embedding we define[1]

$$\gamma_+(p) = q \text{ where } q \text{ solves } \tilde{m}^q + \tilde{m}_p = (1 - p + q)F(p), \quad p > F(0)$$
$$\gamma_+(q) = p \text{ where } p \text{ solves } \tilde{m}^q + \tilde{m}_p = (1 - p + q)F(q), \quad q < F(0-).$$
(39)

The stopping time τ_P is then defined via:

$$\tau_P = \inf\{t \ge 0 : F(B_t) \notin (\gamma_+(F(\overline{B}_t)), \gamma_-(F(\underline{B}_t)))\}.$$
(40)

In a similar spirit, the tilted-Jacka construction is given as follows. Choose $\pi^* \in [0, 1]$ such that $(\underline{b} - m^{\pi^*})(m_{\pi^*} - \overline{b}) \ge 0$—this is always possible, since we can always find π^* such that $m^{\pi^*} = \underline{b}$ say. Then set $\chi = \pi^* \delta_{m^{\pi^*}} + (1 - \pi^*) \delta_{m_{\pi^*}}$. The construction is as follows: we first embed the distribution χ, then, given we hit m^{π^*}, we embed μ^{π^*} using the reversed Azéma–Yor construction (cf. [20]); if we hit m_{π^*} then we embed μ_{π^*} using the Azéma–Yor construction.

Finally, we observe that both cases give rise to martingales with certain optimality properties using the fact that the stopped Brownian motion is a continuous martingale.

Proposition 4.1. *We have the following properties:*

(i) $p(0, \overline{b}) = 0 = p(\underline{b}, 0)$, $q(0, \overline{b}) = 0 = q(\underline{b}, r_\mu)$ and $r(\ell_\mu, \overline{b}) = 0 = r(\underline{b}, r_\mu)$;

(ii) $p(\underline{b}, \overline{b}) = 1$ on $[-\infty, \ell_\mu] \times (r_\mu, \infty]$, $q(\underline{b}, \overline{b}) = 1$ on $[-\infty, \ell_\mu] \times \{0\}$, and $r(0, 0) = 1$;

(iii) p and q are non-increasing in $\underline{b} \in (\ell_\mu, 0)$ and p is non-decreasing in $\overline{b} \in (0, r_\mu)$; r is non-decreasing in $\underline{b} \in (\ell_\mu, 0)$ and q and r are non-decreasing in $\overline{b} \in (0, r_\mu)$;

(iv) *for* $\ell_\mu \le \underline{b} < 0 < \overline{b} \le r_\mu$ *we have*

$$\mathbb{P}\big(\underline{B}_{\tau_J} > \underline{b} \text{ and } \overline{B}_{\tau_J} < \overline{b}\big) \le p(\underline{b}, \overline{b}) \le \mathbb{P}\big(\underline{B}_{\tau_P} > \underline{b} \text{ and } \overline{B}_{\tau_P} < \overline{b}\big),$$
(41)

where (B_t) *is a standard Brownian motion with* $B_0 = 0$, τ_P *is the Perkins stopping time* [10, (4.4)] *embedding* μ *and* τ_J *is the 'tilted-Jacka' stopping time* [10, (4.6)], *for barriers* $(\underline{b}, \overline{b})$, *embedding* μ;

(v) *for* $\ell_\mu \le \underline{b} < 0 < \overline{b} \le r_\mu$, *the lower bound on* $q(\underline{b}, \overline{b})$ *is given by* (12), *and the upper bound is given by* (21). *Moreover these bounds are attained by the constructions in Theorems* 2.2 *and* 2.4 *respectively;*

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[1] Strictly, we only consider the case where $\mu(\{0\}) = 0$. If this is not the case, then the optimal embedding requires independent randomisation to stop some mass at zero initially.

(vi) *for $\ell_\mu \le \underline{b} < 0 < \overline{b} \le r_\mu$, the lower bound on $r(\underline{b}, \overline{b})$ is given by Proposition 2.3 of [9], and the upper bound is given by Proposition 2.1 of [9]. Moreover these bounds are attained by the constructions in Theorems 2.4 and 2.2 of [9] respectively.*

The first three assertions of the proposition are clear. Assertion (iv) is a reformulation of Lemmas 4.2 and 4.3 of [10]—it suffices to note that $(B_{t \wedge \tau_J})$, $(B_{t \wedge \tau_P})$, (M_t) are all UI martingales starting at 0 and with the same terminal law μ for $t = \infty$. Likewise, part (vi) is a reinterpretation of the results of [9]. We note that therein the results were formulated for the case of non-atomic μ. They extend readily, with methods used in Section 3, specifically by characterising the stopping distributions via quantiles of the underlying measures, to the general case.

We can think of any of the functions $p(\cdot, \cdot)$, $q(\cdot, \cdot)$, and $r(\cdot, \cdot)$ as a surface defined over the quarter-plane $[-\infty, 0] \times [0, \infty]$. Proposition 4.1 describes boundary values of the surface, monotonicity properties and gives an upper and a lower bound on the surface. However we note that – most obviously in (iv) – there is a substantial difference between the bounds linked to the fact that τ_P does not depend on $(\underline{b}, \overline{b})$ while τ_J does. In consequence, the upper bound is attainable: there is a martingale (M_t), namely $M_t = (B_{t \wedge \tau_P})$, for which p is equal to the upper bound for all $(\underline{b}, \overline{b})$. In contrast a martingale (M_t) for which p would be equal to the lower bound does not exist. For the martingale $M_t = (B_{t \wedge \tau_J})$, where τ_J is defined for some pair $(\underline{b}, \overline{b})$, p will attain the lower bound in some neighbourhood of $(\underline{b}, \overline{b})$ which will be strictly contained in $(\ell_\mu, 0) \times (0, r_\mu)$. More generally, the latter case is more typical of all the constructions which are used in the result; however, with some careful construction, it seems likely that one can usually find a construction which will be optimal for all values of $(\underline{b}, \overline{b})$ which lie in some small open set (for example, this is true of the tilted-Jacka construction), but there will be limits on how large the region on which a given construction is optimal can be made.

We now give a result which provides some further insight into the structure of the bounds discussed above. In particular, we can show some finer properties of the functions p, q, r and their upper and lower bounds. We state and prove the result for the function p, but the corresponding versions for q and r will follow in a clear manner.

Theorem 4.2. *The function $p(\underline{b}, \overline{b})$ is càglàd in $\overline{b}$ and càdlàg in $\underline{b}$. Moreover, if p is discontinuous at $(\underline{b}, \overline{b})$, then μ must have an atom at one of $\underline{b}$ or $\overline{b}$. Further:*

(i) *if there is a discontinuity at $(\underline{b}, \overline{b})$ of the form:*

$$\limsup_{w \to \overline{b}} p(\underline{b}, w) > p(\underline{b}, \overline{b})$$

then the function g defined by

$$g(u) = \limsup_{w \to \overline{b}} p(u, w) - p(u, \overline{b}), \quad u \le \underline{b}$$

is non-increasing.

(ii) *if there is a discontinuity at $(\underline{b}, \overline{b})$ of the form:*

$$\limsup_{u \to \underline{b}} p(u, \overline{b}) > p(\underline{b}, \overline{b})$$

then the function h defined by

$$h(w) = \limsup_{u \to \underline{b}} p(u, w) - p(\underline{b}, w), \quad w \ge \overline{b}$$

is non-decreasing.

And, at any discontinuity, we will be in at least one of the above cases.

In addition the lower bound (corresponding to the tilted-Jacka construction) is continuous in $(\ell_\mu, 0) \times (0, r_\mu)$, and continuous at the boundary ($\overline{b} = r_\mu$ and $\underline{b} = \ell_\mu$) unless there is an atom of μ at either r_μ or ℓ_μ, while the upper bound (which corresponds to the Perkins construction) has a discontinuity corresponding to every atom of μ.

Remark 4.3. (i) Considering q instead of p, the function will be càdlàg in both arguments, and the directions of the convergence results needs to be adapted suitably. We also observe that discontinuities in the upper bound occur only if there is an atom of μ at $\underline{b}$, *and* we are in case I of Theorem 2.2. Similarly, there is a discontinuity in the lower bound at $\overline{b}$ if there is an atom of μ at $\overline{b}$, and we are in either of cases II or III of Theorem 2.4.

(ii) Considering r instead of p, the function will be càglàd in $\underline{b}$ and càdlàg in $\overline{b}$. We also observe that discontinuities in the upper bound never occur, while there are discontinuities in the lower bound at $\overline{b}$ and/or $\underline{b}$ if there is an atom of μ at either of these values.

Before we prove the above result, we note the following useful result, which is a simple consequence of the martingale property:

Proposition 4.4. *Suppose that $(M_t)_{t \geq 0}$ is a UI martingale with $M_\infty \sim \mu$. Then $\mathbb{P}(\overline{M}_\infty = \overline{b}) > 0$ implies $\mu(\{\overline{b}\}) \geq \mathbb{P}(\overline{M}_\infty = \overline{b})$ and*

$$\{\overline{M}_\infty = \overline{b}\} = \{M_t = \overline{b}, \ \forall t \geq H_{\overline{b}}\} \subseteq \{M_\infty = \overline{b}\} \quad a.s.$$

Proof of Theorem 4.2. We begin by noting that by definition of $p(\underline{b}, \overline{b})$, we necessarily have the claimed continuity and limiting properties. Further,

$$\liminf_{(s,v) \to (u,w)} p(s, v) \geq \mathbb{P}(\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty < \overline{b})$$

and

$$\limsup_{(s,v) \to (u,w)} p(s, v) \leq \mathbb{P}(\underline{M}_\infty \geq \underline{b} \text{ and } \overline{M}_\infty \leq \overline{b}).$$

It follows that the function p is continuous at $(\underline{b}, \overline{b})$ if $\mathbb{P}(\underline{M}_\infty = \underline{b}) = \mathbb{P}(\overline{M}_\infty = \overline{b}) = 0$. By Proposition 4.4, this is true when $\mu(\{\overline{b}, \underline{b}\}) = 0$.

Note that we can now see that at a discontinuity of p, we must be in at least one of the cases (i) or (ii). This is because discontinuity at $(\underline{b}, \overline{b})$ is equivalent to

$$\mathbb{P}(\underline{M}_\infty \geq \underline{b} \text{ and } \overline{M}_\infty \leq \overline{b}) > \mathbb{P}(\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty < \overline{b}),$$

from which we can deduce that at least one of the events

$$\{\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty = \overline{b}\}, \qquad \{\underline{M}_\infty = \underline{b} \text{ and } \overline{M}_\infty < \overline{b}\}, \qquad \{\underline{M}_\infty = \underline{b} \text{ and } \overline{M}_\infty = \overline{b}\}$$

is assigned positive mass. However, by Proposition 4.4 the final event implies both $M_\infty = \underline{b}$ and $M_\infty = \overline{b}$ which is impossible. Consequently, at least one of the first two events must be assigned positive mass, and these are precisely the cases (i) and (ii).

Consider now case (i). We can rewrite the statement as: if $g(\underline{b}) > 0$, then $g(u)$ is decreasing for $u < \underline{b}$. Note however that

$$g(u) = \mathbb{P}(\underline{M}_\infty > u \text{ and } \overline{M}_\infty \leq \overline{b}) - \mathbb{P}(\underline{M}_\infty > u \text{ and } \overline{M}_\infty < \overline{b})$$

$$= \mathbb{P}(\underline{M}_\infty > u \text{ and } \overline{M}_\infty = \overline{b})$$

which is clearly non-increasing in u. In fact, provided that $g(\underline{b}) < \mathbb{P}(\overline{M}_\infty = \overline{b})$, it follows from e.g. [23, Theorem 4.1] that g is strictly decreasing for $\underline{b} > u > \sup\{u \geq -\infty : g(u) = \mathbb{P}(\overline{M}_\infty = \overline{b})\}$. A similar proof holds in case (ii).

We now consider the lower bounds corresponding to the tilted-Jacka construction. We wish to show that

$$\mathbb{P}(\underline{M}_\infty \geq \underline{b} \text{ and } \overline{M}_\infty \leq \overline{b}) = \mathbb{P}(\underline{M}_\infty > \underline{b} \text{ and } \overline{M}_\infty < \overline{b}),$$

for any $(\overline{b}, \underline{b})$ except those excluded in the statement of the theorem. We note that it is sufficient to show that $\mathbb{P}(\underline{M}_\infty = \underline{b}) = \mathbb{P}(\overline{M}_\infty = \overline{b}) = 0$, and by Proposition 4.4 it is only possible to have an atom in the law of the maximum or the minimum if the process stops at the maximum with positive probability; we note however that the stopping time τ_J, due to properties of the Azéma–Yor embedding precludes such behaviour except at the points ℓ_μ, r_μ.

Considering now the Perkins construction, we note from (40) and the fact that the function γ_+ is decreasing, that we will stop at $\underline{b}$ only if $\gamma_+(F(\overline{M}_t)) = \underline{b}$ and $M_t = \underline{M}_t = \underline{b}$. It follows from (39) that there is a range of values $(\overline{b}_*, \overline{b}^*)$ for which $\gamma_+(F(b)) = \underline{b}$, and consequently, we must have $h(b) = \mathbb{P}(\underline{M}_\infty = \underline{b}, \overline{M}_\infty < b)$ increasing in b as b goes from $\overline{b}_*$ to $\overline{b}^*$, with $h(\overline{b}_*) = \mathbb{P}(\underline{M}_\infty = \underline{b}, \overline{M}_\infty < \overline{b}_*) = 0$ and $h(\overline{b}^*) = \mathbb{P}(\underline{M}_\infty = \underline{b}, \overline{M}_\infty < \overline{b}^*) = \mu(\{\underline{b}\})$.[2] Similar results for the function g also follow. □

Conclusions

In this paper, we studied the possible joint distributions of $(\overline{M}_\infty, \underline{M}_\infty)$ given the law of M_∞, and were able to obtain number of qualitative properties and sharp quantitative bounds. It follows from our results that the interaction between the maximum and minimum is highly non-trivial which makes the pair above much harder to study than $\overline{M}_\infty$ and $\underline{M}_\infty$ on their own. This is best seen in the case of Brownian motion where $\overline{B}_t$ has an easily accessible distribution while the description of the joint distribution of $(\underline{B}_t, \overline{B}_t)$ is much more involved. A further natural question arising from our work is to characterise the joint distributions of the triple $(M_\infty, \overline{M}_\infty, \underline{M}_\infty)$. At present it is not clear to us if, and to what extent, a complete characterisation of the possible joint distributions of this triple, in the spirit of Rogers [23] and Vallois [24], is feasible. It remains an open and challenging problem.

Acknowledgements

References

[1] B. Acciaio, M. Beiglbock, F. Penkner, W. Schachermayer, J. Temme, A trajectorial interpretation of Doob's martingale inequalities, Ann. Appl. Probab. 23 (4) (2013) 1494–1505.
[2] J. Azéma, M. Yor, Une solution simple au problème de Skorokhod, in: Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), in: Lecture Notes in Math., vol. 721, Springer, Berlin, 1979, pp. 90–115.

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[2] In fact, as above, it follows from e.g. [23, Theorem 2.2] that the maximum must have a strictly positive density with respect to Lebesgue measure, and therefore that the function h is strictly increasing between the points $\overline{b}_*$ and $\overline{b}^*$.

[3] J. Azéma, M. Yor, Le problème de Skorokhod: compléments à Une solution simple au problème de Skorokhod, in: Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78), in: Lecture Notes in Math., vol. 721, Springer, Berlin, 1979, pp. 625–633.

[4] D. Blackwell, L.E. Dubins, A converse to the dominated convergence theorem, Illinois J. Math. 7 (1963) 508–514.

[5] H. Brown, D. Hobson, L.C.G. Rogers, Robust hedging of barrier options, Math. Finance 11 (3) (2001) 285–314.

[6] L. Carraro, N. El Karoui, J. Obłój, On Azéma–Yor processes, their optimal properties and the Bachelier–Drawdown equation, Ann. Probab. 40 (1) (2012) 372–400.

[7] A.M.G. Cox, D.G. Hobson, An optimal Skorokhod embedding for diffusions, Stochastic Process. Appl. 111 (1) (2004) 17–39.

[8] A.M.G. Cox, D.G. Hobson, Skorokhod embeddings, minimality and non-centred target distributions, Probab. Theory Related Fields 135 (2005) 395–414.

[9] A.M.G. Cox, J. Obłój, Robust hedging of double touch barrier options, SIAM J. Financ. Math. 2 (1) (2011) 141–182.

[10] A.M.G. Cox, J. Obłój, Robust pricing and hedging of double no-touch options, Finance Stoch. 15 (3) (2011) 573–605.

[11] L.E. Dubins, D. Gilat, On the distribution of maxima of martingales, Proc. Amer. Math. Soc. 68 (3) (1978) 337–338.

[12] M. Duembgen, L.C.G. Rogers, The joint law of the extrema, final value and signature of a stopped random walk, 2014. arXiv:1403.0220 [math.PR].

[13] D. Gilat, I. Meilijson, A simple proof of a theorem of Blackwell & Dubins on the maximum of a uniformly integrable martingale, in: Séminaire de Probabilités, XXII, in: Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 214–216.

[14] D.G. Hobson, Robust hedging of the lookback option, Finance Stoch. 2 (4) (1998) 329–347.

[15] D. Hobson, The skorokhod embedding problem and model-independent bounds for option prices, in: R.A. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi (Eds.), Paris-Princeton Lectures on Mathematical Finance 2010, in: Lecture Notes in Math., vol. 2003, Springer, 2010, pp. 267–318.

[16] S.D. Jacka, Doob's inequalities revisited: a maximal H^1-embedding, Stochastic Process. Appl. 29 (2) (1988) 281–290.

[17] R.P. Kertz, U. Rösler, Martingales with given maxima and terminal distributions, Israel J. Math. 69 (2) (1990) 173–192.

[18] R.P. Kertz, U. Rösler, Stochastic and convex orders and lattices of probability measures, with a martingale interpretation, Israel J. Math. 77 (1–2) (1992) 129–164.

[19] R.P. Kertz, U. Rösler, Hyperbolic-concave functions and Hardy–Littlewood maximal functions, in: Stochastic inequalities (Seattle, WA, 1991), in: IMS Lecture Notes Monogr. Ser., vol. 22, Inst. Math. Statist, Hayward, CA, 1993, pp. 196–210.

[20] J. Obłój, The Skorokhod embedding problem and its offspring, Probab. Surv. 1 (2004) 321–390 (electronic).

[21] J. Obłój, The Skorokhod embedding problem, in: Rama Cont (Ed.), Encyclopedia of Quantitative Finance, Wiley, 2010, pp. 1653–1657.

[22] E. Perkins, The Cereteli–Davis solution to the H^1-embedding problem and an optimal embedding in Brownian motion, in: Seminar on Stochastic Processes, 1985 (Gainesville, Fla., 1985), Birkhäuser, Boston, Boston, MA, 1986, pp. 172–223.

[23] L.C.G. Rogers, The joint law of the maximum and terminal value of a martingale, Probab. Theory Related Fields 95 (4) (1993) 451–466.

[24] P. Vallois, On the joint distribution of the supremum and terminal value of a uniformly integrable martingale, in: Stochastic Processes and Optimal Control (Friedrichroda, 1992), in: Stochastics Monogr., vol. 7, Gordon and Breach, Montreux, 1993, pp. 183–199.