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The Brauer Group of Central Separable G-Azumaya Algebras

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Let R be a commutative ring and G a finite abelian group. In [8], Long developed a Brauer group theory for G-dimodule algebras (i.e., algebras with a compatible G-grading and G-action) and constructed $BD(R, G)$, the Brauer group of G-Azumaya algebras. Within $BD(R, G)$ lies $B(R, G)$, the set of classes of algebras which are R-Azumaya (i.e., central separable) as well as G-Azumaya. $B(R, G)$ is not always a group; we show that if every cocycle in $H^2(G, U(R))$ is abelian, then it is. When $B(R, G)$ is a group, we call it the Brauer group of central separable G-Azumaya algebras. If R is connected and $\mathrm{Pic}_m(R) = 0$ where m is the exponent of G, and if every cocycle in $H^2(G, U(R))$ is abelian, then we show that there is a short exact sequence

$$1 \to (BC(R, G)/B(R)) \times (BM(R, G)/B(R)) \to B(R, G)/B(R) \to \mathrm{Aut}(G) \to 1,$$

where $B(R)$ is the usual Brauer group of R, $BM(R, G)$ is the Brauer group of G-module algebras and $BC(R, G)$ is the Brauer group of G-comodule algebras (cf. [8]). If either $BM(R, G)/B(R)$ or $BC(R, G)/B(R)$ is trivial, then the sequence splits. Using the above, we are able to describe $BD(\mathbb{Z}, G)$ for any cyclic G, and $BD(\mathbb{R}, G)$ for any cyclic G of odd order. Our sequence provides a generalization of the results [9, Theorem 5.9] and [11, Theorem 4.4] and should be compared to the sequence in [6, Theorem 5.2] obtained under the assumptions that the order of G is a unit in R and R contains a primitive mth root of unity.

PRELIMINARIES

Let R be a connected commutative ring with unity and G a finite abelian group. We assume throughout that $\mathrm{Pic}_m(R) = 0$ where m is the exponent of G.

All algebras and modules are understood to be R-algebras and R-modules. We write $A \otimes B$ for $A \otimes_R B$, $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_R(A, B)$, etc. The group of units of an algebra A is written $U(A)$. A^0 is the usual opposite algebra of A.

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All formulae defined only for the homogeneous elements of a graded module are to be extended by linearity.

Let RG denote the group ring with basis u_σ, $\sigma \in G$. GR is the dual of RG and has basis v_τ where $v_\tau(u_\sigma) = 1$ if $\tau = \sigma$ and 0 otherwise. If $f \in H^2(G, U(R))$, RG_f denotes the module RG but with multiplication m: $RG_f \otimes RG_f \to RG_f$ defined by $m(u_\sigma, u_\tau) = f(\sigma, \tau)u_{\sigma\tau}$. If Φ is a bilinear map from $G \times G$ to $U(R)$, RG_f^Φ denotes the algebra RG_f with G-action defined by $\sigma(u_\tau) = \Phi(\sigma, \tau)u_\tau$.

A module M is called a G-dimodule if M is a G-graded module and G acts on M as a group of grade-preserving automorphisms. An algebra A is called a G-dimodule algebra if A is a G-dimodule and a G-graded algebra (i.e., $A_\sigma A_\tau \subseteq A_{\sigma\tau}$) and G acts as algebra automorphisms on A. If A is a G-dimodule algebra, $\bar{A}$ is defined to be A as a G-dimodule but with multiplication defined by $\bar{a} \cdot \bar{b} = \overline{\sigma(b)a}$ where $a \in A_\sigma$; $\bar{A}$ is also a G-dimodule algebra. If M is a G-dimodule, then $\mathrm{End}(M)$ is a G-dimodule algebra where, for $f \in \mathrm{End}(M)$, $(\sigma f)(m) = \sigma(f(\sigma^{-1}m))$ and $(f)_\sigma(m) = \sum_\alpha (f(m_\alpha))_{\alpha\sigma}$. If M and N are G-dimodules (or G-dimodule algebras), then so is $M \otimes N$ where $\sigma(m \otimes n) = \sigma(m) \otimes \sigma(n)$ and $(M \otimes N)_\sigma = \sum_{\alpha\beta=\sigma} M_\alpha \otimes N_\beta$.

If A and B are G-dimodule algebras, the smash product $A \# B$ is defined to be $A \otimes B$ as a G-dimodule but with multiplication $(a \# b)(c \# d) = a\sigma(c) \# bd$ for b of grade σ. $A \# B$ is also a G-dimodule algebra. If P is a G-dimodule, $A \# \mathrm{End}(P) \simeq A \otimes \mathrm{End}(P)$.

For a G-dimodule algebra A, the G-dimodule algebra maps F: $A \# \bar{A} \to \mathrm{End}(A)$ and G: $\bar{A} \# A \to \mathrm{End}(A)^0$ are defined by $F(a \# \bar{b})(c) = a\sigma(c)b$ where b has grade σ and $G(\bar{a} \# b)(c) = g^0(c)$ where $g(c) = \sigma(a)\, cb$ for c of grade σ. If A is an R-progenerator and F and G are isomorphisms, A is called G-Azumaya. If A and B are G-Azumaya, so are $\bar{A}$ and $A \# B$; if P is an R-progenerator and G-dimodule, $\mathrm{End}(P)$ is G-Azumaya. Note that G-Azumaya algebras are separable algebras [11, Proposition 2.2]. $BD(R, G)$ is defined to be the group of equivalence classes of G-Azumaya algebras where $A \sim B$ if there exist G-dimodule R-progenerators P and Q such that $A \# \mathrm{End}(P) \simeq B \# \mathrm{End}(Q)$. Multiplication is the smash product.

$B(R, G)$ is defined to be the set of classes of G-Azumaya algebras in which one (and therefore all) algebras are also R-Azumaya. $B(R, G)$ need not be a subgroup of $BD(R, G)$ [11, Example 2.10] but we show that if every cocycle in $H^2(G, U(R))$ is abelian, then it is.

The subgroup $BM(R, G)$ consists of those (classes of) algebras with trivial G-grading; similarly $BC(R, G)$ is the group of (classes of) algebras with trivial G-action. $B(R)$, the usual Brauer group of R, is embedded in all of the above groups as a normal subgroup since elements of $B(R)$ are given trivial G-grading and action and therefore commute with all other elements of $BD(R, G)$. For further discussion of $BD(R, G)$, we refer the reader to [8].

$\mathrm{Gal}(R, GR)$ and $\mathrm{Gal}(R, RG)$ are the groups of isomorphism classes of Galois GR-objects (i.e., the usual Galois extensions of R with group G) and Galois

RG-objects respectively. (See [5] for definitions and details.) By [4, Remark 2.2 and Theorem 1.4] there are split short exact sequences

$$1 \to B(R) \to BC(R, G) \to \mathrm{Gal}(R, GR) \to 1$$

and

$$1 \to B(R) \to BM(R, G) \to \mathrm{Gal}(R, RG) \to 1.$$

From the second, we obtain an isomorphism η from $\mathrm{Gal}(R, RG)$ to $BM(R, G)/B(R)$ defined by

$$\eta(S) = S \# GR \tag{1}$$

where S is a Galois RG-object and $S \# GR$ has G-action induced by that on GR, i.e., $\sigma(s \# v_\tau) = s \# v_\gamma$, $\gamma = \tau\sigma^{-1}$.

Note that (in analogy to the equivalent conditions for an algebra S to be a Galois extension of R with group G), a G-graded algebra S is a Galois RG-object if and only if $S_1 = R$ and for all $\sigma, \tau \in G$, the map $S_\sigma \otimes S_\tau \to S_{\sigma\tau}$ is an R-module isomorphism [3, Proposition 1.3]. Thus, for all $\sigma \in G$, $S_\sigma \in \mathrm{Pic}_m(R)$, where m is the exponent of G. We say that S has normal basis if $S \simeq RG$ as G-graded modules; then if $\mathrm{Pic}_m(R) = 0$, every Galois RG-object has normal basis. The subgroup of $\mathrm{Gal}(R, RG)$ consisting of such algebras is, in fact, isomorphic to $H^2(G, U(R))$, the usual second group cohomology group (cf. [10] or [3, Theorem 1.6]). Thus, since we assume $\mathrm{Pic}_m(R) = 0$, we have an isomorphism φ from $H^2(G, U(R))$ to $\mathrm{Gal}(R, RG)$; φ is defined by

$$\varphi(f) = RG_f \tag{2}$$

for $f \in H^2(G, U(R))$. The isomorphisms η and φ above will be needed in the proof of Theorem 1.2.

1. A Short Exact Sequence Description of $B(R, G)$

The following condition is sufficient for $B(R, G)$ to be a group.

Proposition 1.1. *Let R, G be as above. Suppose all cocycles in $H^2(G, U(R))$ are abelian, i.e., for $f \in H^2(G, U(R))$, $\sigma, \tau \in G$, $f(\sigma, \tau) = f(\tau, \sigma)$. Then $B(R, G)$ is a subgroup of $BD(R, G)$.*

Proof. Let A, $B \in B(R, G)$; we must show that $A \# B \in B(R, G)$. Let $f \in H^2(G, U(R))$ and suppose f is normalized, i.e., $f(\sigma, 1) = f(1, \sigma) = 1$ for all $\sigma \in G$. Let B_f equal B as a G-graded module but with multiplication m defined by

$$m(x \otimes y) = f(\sigma, \tau)\, xy \qquad \text{for} \quad x \in B_\sigma,\ y \in B_\tau.$$

Since f is a normalized cocycle, m defines an algebra structure on B, and, since f is abelian, B is central if and only if B_f is. Further suppose B has separability idempotent $e = \sum_i x_i \otimes y_i^0$. We may assume $e \in (B \otimes B^0)_1$ since $\sum_i x_i y_i = 1$; then using the cocycle identity, it is straight forward to check that

$$e' = \sum_i f(\sigma_i, \sigma_i^{-1})\, x_i \otimes y_i^0$$

is a separability idempotent for B_f where σ_i is the grade of x_i.

Since $\mathrm{Pic}_m(R) = 0$, the elements of G act as inner automorphisms on A [2, Corollary 4.6, p. 108]. For $\sigma \in G$, let x_σ be an element of A such that $\sigma a = x_\sigma a x_\sigma^{-1}$ for every $a \in A$, and choose $x_1 = 1$. Then $x_\sigma x_\tau x_{\sigma\tau}^{-1} a x_{\sigma\tau} x_\tau^{-1} x_\sigma^{-1} = a$ for all $\sigma, \tau \in G$, $a \in A$; thus $x_{\sigma\tau} x_\tau^{-1} x_\sigma^{-1} \in \mathrm{Centre}(A) = R$. Hence we may define a normalized cocycle $f\colon G \times G \to U(R)$ by $x_\sigma x_\tau = f(\sigma, \tau)^{-1} x_{\sigma\tau}$.

Now we imitate the procedure in [11, Lemma 3.2] and define an R-module isomorphism $j\colon A \# B \to A \otimes B_f$ by

$$j(a \# b) = a x_\sigma \otimes b \qquad \text{for} \quad b \in B_\sigma .$$

Since, for $b \in B_\sigma$, $d \in B_\tau$,

$$\begin{aligned} j((a \# b)(c \# d)) &= j(a\sigma(c) \# bd) \\ &= a x_\sigma c x_\sigma^{-1} x_{\sigma\tau} \otimes bd \\ &= f(\sigma, \sigma^{-1}) f(\sigma^{-1}, \sigma\tau)^{-1} a x_\sigma c x_\tau \otimes bd \\ &= f(\sigma, \tau)\, a x_\sigma c x_\tau \otimes bd \\ &= (a x_\sigma \otimes b)(c x_\tau \otimes d) \\ &= j(a \# b) j(c \# d), \end{aligned}$$

j is, in fact, an R-algebra isomorphism. Since A and B_f are R-Azumaya, by [12, Proposition 2.3(d)], $A \otimes B_f$ is R-Azumaya and therefore $A \# B \in B(R, G)$.

Suppose now that every cocycle in $H^2(G, U(R))$ is abelian so that $B(R, G)$ is a subgroup of $BD(R, G)$. In [11, Theorem 4.4], a map $\beta\colon B(R, G) \to \mathrm{Aut}(G)$ was defined as follows. Let $A \in B(R, G)$ and, as in the proof of Proposition 1.1, let x_σ be an element of A such that $\sigma a = x_\sigma a x_\sigma^{-1}$ for $\sigma \in G$, $a \in A$, and choose $x_1 = 1$. By [11, Proposition 4.2], x_σ is homogeneous, say of grade $\alpha_A(\sigma)$. Define $\beta(A) = \beta_A\colon G \to G$ by $\beta_A(\sigma) = \sigma(\alpha_A(\sigma))^{-1}$. In [11, Theorem 4.4], Orzech proved that β_A is independent of the choice of x_σ, is indeed a group automorphism and is well-defined on equivalence classes in $B(R, G)$. We wish to show that β is a group epimorphism. As in the proof of Proposition 1.1, $x_\sigma x_\tau =$

$f(\sigma, \tau)^{-1}x_{\sigma\tau}$ for some normalized $f \in H^2(G, U(R))$. Therefore G acts trivially on the x_σ, since, for $\sigma, \tau \in G$,

$$\begin{aligned}\tau(x_\sigma) &= x_\tau x_\sigma x_\tau^{-1} \\ &= f(\tau, \tau^{-1})f(\tau, \sigma)^{-1}f(\tau\sigma, \tau^{-1})^{-1}x_\sigma \\ &= (f(\sigma, 1)f(\tau, \tau^{-1}))(f(\sigma, \tau)f(\sigma\tau, \tau^{-1}))^{-1}x_\sigma \\ &= x_\sigma .\end{aligned}$$

Given A, $B \in B(R, G)$ and $\sigma \in G$, let the action of σ on A and B be given by conjugation by x_σ and y_σ respectively, where y_σ has grade τ. Let $\rho = \sigma\tau^{-1} = \beta_B(\sigma)$. Then the inner action of σ on $A \# B$ is induced by $x_\rho \# y_\sigma$ since,

$$\begin{aligned}&(x_\rho \# y_\sigma)(a \# b)(x_\rho \# y_\sigma)^{-1} \\ &\quad= (x_\rho \# y_\sigma)(a \# b)(x_\rho^{-1} \# y_\sigma^{-1}) \\ &\quad= x_\rho\tau(a)\, x_\rho^{-1} \# y_\sigma b y_\sigma^{-1} \\ &\quad= x_\rho x_\tau a x_\tau^{-1} x_\rho^{-1} \# \sigma(b) \\ &\quad= f(\rho, \tau)^{-1}f(\tau^{-1}, \rho^{-1})^{-1}f(\tau, \tau^{-1})f(\rho, \rho^{-1})f(\sigma, \sigma^{-1})^{-1}\sigma(a) \# \sigma(b) \\ &\quad= \sigma(a) \# \sigma(b) \text{ by the cocycle identity.}\end{aligned}$$

If x_ρ has grade γ, then $x_\rho \# y_\sigma$ has grade $\gamma\tau$. Then

$$\begin{aligned}\beta_A(\beta_B(\sigma)) &= \beta_A(\rho) \\ &= \rho\gamma^{-1} \\ &= \sigma(\gamma\tau)^{-1} \\ &= \beta_{A\#B}(\sigma),\end{aligned}$$

and β is a group homomorphism.

Furthermore β is onto; we sketch the proof (cf. [11, p. 546]). Let $j \in \operatorname{Aut}(G)$, let $k \in \operatorname{Aut}(G)$ be the automorphism defined by $k(\sigma) = \sigma j(\sigma)^{-1}$, and let $RG(j)$ be the G-graded module RG but with G-action given by $\sigma u_\tau = u_{k(\sigma)\tau}$. Consider the G-dimodule algebra $A = \operatorname{End}(RG(j))$; here, $x_\sigma = \sigma$ viewed as an element of A, and σ has grade $k(\sigma)$. Thus $\beta_A = j$ and it only remains to prove that A is G-Azumaya. To show that F: $A \# \bar{A} \to \operatorname{End}(A)$ is an isomorphism, it suffices, by a dimension argument, to show F is onto. We know that the usual map from $A \otimes A^0$ to $\operatorname{End}(A)$ is an isomorphism. Let $f \in \operatorname{End}(A)$ and let $\sum f_i \otimes g_i{}^0$ be its preimage in $A \otimes A^0$, i.e., for all $h \in A$, $f(h) = \sum f_i \cdot h \cdot g_i$. Now suppose g_i is homogeneous of grade γ_i and let $\rho_i = j^{-1}(\gamma_i)$. Then $\rho_i g_i$ has grade $k(\rho_i)\gamma_i = \rho_i j(j^{-1}(\gamma_i))^{-1}\gamma_i = \rho_i$, and therefore

$$\begin{aligned}F\left(\sum f_i\rho_i^{-1} \# \overline{\rho_i g_i}\right)(h) &= \sum f_i \cdot \rho_i^{-1} \cdot \rho_i \cdot h \cdot \rho_i^{-1} \cdot \rho_i \cdot g_i \\ &= \sum f_i \cdot h \cdot g_i \\ &= f(h).\end{aligned}$$

Similarly G: $\bar{A} \# A \to \operatorname{End}(A)^0$ is onto.

The discussion above yields a group epimorphism β from $B(R, G)$ to $\mathrm{Aut}(G)$. Since $B(R)$ is a normal subgroup of $B(R, G)$ lying in the kernel of β, we have a group epimorphism from $B(R, G)/B(R)$ to $\mathrm{Aut}(G)$. We denote this map also by β, and investigate its kernel in the following theorem.

THEOREM 1.2. *Let G and R be as above. Suppose that every cocycle in $H^2(G, U(R))$ is abelian. Then there is a short exact sequence*

$$1 \to (BC(R, G)/B(R)) \times (BM(R, G)/B(R)) \to B(R, G)/B(R) \to \mathrm{Aut}(G) \to 1.$$

If either $BC(R, G)/B(R)$ or $BM(R, G)/B(R)$ is trivial, the sequence splits.

Proof. Clearly $BM(R, G)/B(R)$ and $BC(R, G)/B(R)$ are subgroups of $\mathrm{Ker}\,\beta$. We show first that $BM(R, G)/B(R)$ is a direct summand of $\mathrm{Ker}\,\beta$.

Let $A \in \mathrm{Ker}\,\beta$ and, for $\sigma \in G$, let $x_\sigma \in U(A)$ be such that $\sigma(a) = x_\sigma a x_\sigma^{-1}$ for all $a \in A$, as above. Then we may define a cocycle $f_A\colon G \times G \to U(R)$ by $f_A(\sigma^{-1}, \tau^{-1}) = x_\sigma x_\tau x_{\sigma\tau}^{-1}$. Suppose that for every $\sigma \in G$, z_σ is another such suitable unit in A, and let h_A be the resulting cocycle. Then $x_\sigma z_\sigma^{-1} \in \mathrm{Centre}(A) = R$, i.e., $x_\sigma = r_\sigma z_\sigma$ for some $r_\sigma \in U(R)$. Define $g\colon G \to U(R)$ by $g(\sigma) = r_\sigma$, and then $f_A = (\delta g)h_A$. Thus the element $f_A \in H^2(G, U(R))$ is independent of the choice of the x_σ.

If $A = \mathrm{End}(P)$, P a G-dimodule R-progenerator, then $x_\sigma = \sigma$ viewed as an element of $\mathrm{End}(P)$, and f_A is trivial. If $A \in B(R)$, then $x_\sigma = 1$ for all $\sigma \in G$, and, again, f_A is trivial. Let A, $B \in \mathrm{Ker}\,\beta$, and for $\sigma \in G$, let the inner action of σ on A and B be conjugation by x_σ and y_σ respectively. Since x_σ and y_σ are homogeneous of grade 1 and are invariant under G-action, then the action of σ on $A \# B$ is given by conjugation by $x_\sigma \# y_\sigma$, and $f_{A\#B} = f_A f_B$. Therefore we now have a well-defined group homomorphism ρ from $\mathrm{Ker}\,\beta$ to $H^2(G, U(R))$ given by $\rho(A) = f_A$.

To show that $BM(R, G)/B(R)$ is a direct summand of $\mathrm{Ker}\,\beta$, we show that the composition

$$H^2(G, U(R)) \xrightarrow{\varphi} \mathrm{Gal}(R, RG) \xrightarrow{\eta} BM(R, G)/B(R) \xrightarrow{\iota} \mathrm{Ker}\,\beta \xrightarrow{\rho} H^2(G, U(R))$$

is the identity on $H^2(G, U(R))$ where η and φ are defined by equations (1) and (2) respectively.

Let $f \in H^2(G, U(R))$. We have that $\iota \cdot \eta \cdot \varphi(f) = RG_f \# GR$, where the G-action on this algebra is induced by that on GR. It is easily checked that the inner action of σ^{-1} on $RG_f \# GR$ is given by conjugation by $u_\sigma \# 1$. Then $x_\sigma x_\tau = f(\sigma^{-1}, \tau^{-1})x_{\sigma\tau}$, and $\rho(RG_f \# GR) = f$. Therefore the inclusion $\iota\colon BM(R, G)/B(R) \to \mathrm{Ker}\,\beta$ is split by $\eta \cdot \varphi \cdot \rho$.

We now have a split short exact sequence

$$1 \to \mathrm{Ker}\,\rho \to \mathrm{Ker}\,\beta \to BM(R, G)/B(R) \to 1,$$

and we investigate $\operatorname{Ker} \rho$. Let $A \in \operatorname{Ker} \rho$ and denote by A_C the G-graded algebra A with trivial G-action. Then

$$\bar{A}_C \# A = (A_C)^0 \otimes A \simeq A \otimes (A_C)^0 \simeq \operatorname{End}(A)$$

as G-graded algebras. Since $A \in \operatorname{Ker} \rho$, the elements x_σ associated to the inner action of G on A may be chosen such that $x_\sigma x_\tau = x_{\sigma\tau}$ for all $\sigma, \tau \in G$. Define A' to be the G-graded module A but with G-action given by $\sigma(a') = (x_\sigma a)'$. It is easily checked that $\bar{A}_C \# A \simeq \operatorname{End}(A')$ as G-dimodule algebras by the isomorphism above. Therefore $\bar{A}_C \# A$ is trivial in $B(R, G)/B(R)$, and, since $A_C \in BC(R, G)/B(R)$, $A \in BC(R, G)/B(R)$. Thus $\operatorname{Ker} \rho = BC(R, G)/B(R)$ and $\operatorname{Ker} \beta \simeq (BM(R, G)/B(R)) \times (BC(R, G)/B(R))$.

Now suppose that $BC(R, G)/B(R)$ is trivial. In the discussion of the map β preceding the theorem, we saw that for $j \in \operatorname{Aut}(G)$, $\beta(\operatorname{End}(RG(j))) = j$. Therefore, if $\operatorname{End}(RG(j)) \# \operatorname{End}(RG(k)) \sim \operatorname{End}(RG(jk))$ for j, $k \in \operatorname{Aut}(G)$ then the map $j \to \operatorname{End}(RG(j))$ is a group homomorphism splitting β. Since $\beta(\operatorname{End}(RG(j)) \# \operatorname{End}(RG(k))) = \beta(\operatorname{End}(RG(jk))) = jk$, then

$$A = \operatorname{End}(RG(j)) \# \operatorname{End}(RG(k)) \# \overline{\operatorname{End}(RG(jk))}$$

is in $\operatorname{Ker} \beta = BM(R, G)/B(R)$, i.e., $A = A_M$, the G-dimodule algebra A with trivial G-grading. Then

$$\begin{aligned} A_M &= (\operatorname{End}(RG(j)) \# \operatorname{End}(RG(k)) \# \overline{\operatorname{End}(RG(jk))})_M \\ &= ((\operatorname{End}(RG(j)) \# \operatorname{End}(RG(k)))_M \# \overline{\operatorname{End}(RG(jk))})_M \\ &= ((\operatorname{End}(RG(j))_M \# \operatorname{End}(RG(k)))_M \# \overline{\operatorname{End}(RG(jk))})_M, \end{aligned}$$

and, since for any $j \in \operatorname{Aut}(G)$, $\operatorname{End}(RG(j))_M = \operatorname{End}(RG)_M$, trivial in $B(R, G)$, A is trivial in $B(R, G)$ and the sequence splits. A similar argument shows that the sequence also splits if $BM(R, G)/B(R)$ is trivial.

Corollary 1.3. *Let R, G be as above. If $H^2(G, U(R)) = 0$, then* $B(R, G) \simeq B(R) \times (BC(R, G)/B(R)) \times \operatorname{Aut}(G) \simeq B(R) \times \operatorname{Gal}(R, GR) \times \operatorname{Aut}(G)$.

Proof. By [11, Lemma 3.2], $B(R)$ is a direct summand of $B(R, G)$; the statement then follows directly from Theorem 1.2.

Corollary 1.4. *Let G, R be as above. Suppose $G \simeq H \times K$ such that*

(i) $H^2(G, U(R)) \simeq H^2(H, U(R)) \times H^2(K, U(R))$

(ii) $\operatorname{Gal}(R, GR) \simeq \operatorname{Gal}(R, HR) \times \operatorname{Gal}(R, KR)$

(iii) $\operatorname{Aut}(G) \simeq \operatorname{Aut}(H) \times \operatorname{Aut}(K)$.

Then $B(R, G)/B(R) \simeq (B(R, H)/B(R)) \times (B(R, K)/B(R))$.

Proof. By [8, Theorem 1.8], $B(R, H)/B(R)$ and $B(R, K)/B(R)$ are subgroups of $B(R, G)/B(R)$, and since $G \simeq H \times K$, elements of these subgroups commute. Also, if $A \in B(R, H)/B(R)$ and $B \in B(R, K)/B(R)$, then $A \# B = A \otimes B$ is nontrivial. For suppose $A \# \text{End}(P) \simeq A \otimes \text{End}(P) \simeq B \# C \# \text{End}(Q) \simeq B \otimes C \otimes \text{End}(Q)$, for some $C \in B(R)$, Q, P G-dimodule R-progenerators. Let P', Q' be the modules P and Q but with trivial K-action and grading, and let B' be the algebra B with trivial K-(and therefore G-)action and grading. Then $A \otimes \text{End}(P') \simeq B' \otimes C \otimes \text{End}(Q') \in B(R)$ and A is trivial in $B(R, G)/B(R)$. Therefore $(B(R, H)/B(R)) \times (B(R, K)/B(R)) \subseteq B(R, G)/B(R)$; an application of the short exact sequence of the theorem then shows equality.

Before applying the above theorem to some computations of $BD(R, G)$, we need the following.

LEMMA 1.5. *Let G be a finite abelian group of order n and let R be such that if p divides n, p is not a unit in R. Then $B(R, G) = BD(R, G)$.*

Proof. Let A be G-Azumaya with centre Z. Let H be the group of gradings of Z and let $\mathcal{M}$ be a maximal ideal in R. Then H is also the group of gradings of $Z/\mathcal{M}Z$, for if $\mathcal{M}Z_\sigma = Z_\sigma$ then Z_σ is annihilated by some element $1 - r$, $r \in \mathcal{M}$ [12, Lemma 1.2], contrary to [11, Corollary 2.5(a)]. Now let p be a prime dividing the order of H and $\mathcal{M}$ a maximal ideal of R containing p; the remainder of the argument follows exactly as in [11, Corollary 2.7].

EXAMPLE 1.6. We now compute $BD(\mathbb{Z}, G)$ for $\mathbb{Z}$ the ring of integers and G any cyclic group. By Lemma 1.5 and the fact that $B(\mathbb{Z}) = 0$, $BD(\mathbb{Z}, G) = B(\mathbb{Z}, G)$, and thus we may apply the above theorem. $\text{Gal}(G, G\mathbb{Z}) = 0$ since any Galois $G\mathbb{Z}$-object would be the ring of integers of an unramified extension of $\mathbb{Q}$ (cf. [1]); thus $BD(\mathbb{Z}, G) \simeq H^2(G, U(\mathbb{Z})) \times \text{Aut}(G)$. Recall that for p an odd prime, $\text{Aut}(C_{p^e}) \simeq C_{p^e - p^{e-1}}$ where C_m denotes the cyclic group of order m, and $\text{Aut}(C_{2^e}) \simeq C_2 \times C_{2^{e-2}}$, $e \geqslant 2$. Also $H^2(G, U(\mathbb{Z})) \simeq C_2$ if the order of G is even and 0 otherwise. Thus $BD(\mathbb{Z}, G)$ has been described explicitly.

Another short lemma will allow us to compute $BD(\mathbb{R}, G)$ for G cyclic of odd order.

LEMMA 1.7. *Let G be a cyclic group of order n and let R be such that* 1 *is the only nth root of unity in R. Then $BD(R, G) = B(R, G)$.*

Proof. Let $A \in BD(R, G)$ with centre Z, and let H be the group of gradings of Z. Then by [11, Proposition 2.11(e)], $Z \simeq RH_f^{\Phi}$ where Φ is a bilinear map from $G \times H$ to $U(R)$. Then if σ generates G and σ^t generates H, $\Phi(\sigma, \sigma^t)$ is an nth root of unity in R and therefore Φ is trivial, i.e., $Z = Z^G$. By [11, Proposition 2.2(a)] $Z^G = R$, and the lemma is proved.

EXAMPLE 1.8. We compute $BD(\mathbb{R}, G)$ for $\mathbb{R}$ the field of real numbers and G

any cyclic group of odd order. By Lemma 1.7, $BD(\mathbb{R}, G) = B(\mathbb{R}, G)$. Since $\text{Gal}(\mathbb{R}, G\mathbb{R})$ and $H^2(G, U(\mathbb{R}))$ are trivial, by Corollary 1.3,

$$BD(\mathbb{R}, G) \simeq C_2 \times \text{Aut}(G).$$

2. A Similar Sequence for $BD(R, G)$

We now suppose that G has order n and R contains a primitive nth root of unity, $\text{Pic}_n(R) = 0$ and n is a unit in R. Then $RG \simeq GR$ as Hopf algebras, so that $\text{Gal}(R, RG) \simeq \text{Gal}(R, GR) \simeq H^2(G, U(R))$. Then, if G is of the form $\prod_i (\prod_{r_i} G_i)$ where G_i is cyclic of order $p_i^{e_i}$, $p_i \neq p_j$, Theorems 5.1 and 5.2 of [6] describe $BD(R, G)/B(R)$ by the short exact sequence:

$$1 \to (H^2(G, U(R))) \times (H^2(G, U(R))) \to BD(R, G)/B(R) \to N \to 1.$$

If $G = \prod_i G_i$, then $N = \prod_i D_i$ where D_i is the dihedral group of order $2(p_i^{e_i} - p_i^{e_i-1})$ for p_i odd. For a further description of N, see [6, Theorem 5.2].

These results of Childs were obtained as a byproduct of his investigation of $B_\Phi(R, G)$, the Brauer group of G-graded Azumaya algebras with Φ a bilinear map from $G \times G$ to $U(R)$ (cf. [7]). If G is cyclic of order $n = p_1 \cdots p_t$, p_i prime, $p_i \neq p_j$, it is also possible to obtain the above short exact sequence by a straightforward investigation of the subgroups of $BD(R, G)$ and an application of the short exact sequence of Theorem 1.2. First suppose that G is cyclic of prime order. Then $B(R, G)$ is a normal subgroup of index 2 in $BD(R, G)$, and $(BM(R, G)/B(R)) \times (BC(R, G)/B(R))$ is a normal subgroup of $BD(R, G)$. The proof makes generous use of the fact that for A G-Azumaya with centre $Z \neq R$, Z is both a Galois RG-object and a Galois GR-object, and thus $A^{RG} \# Z \simeq A \simeq Z \# A^{GR}$, where $\#$ is actually $\otimes$. Note that A^G and A_1 are R-Azumaya algebras, and $Z \simeq RG_f{}^\Phi$ is G-Azumaya [11, Proposition 2.11]. Then a commutative diagram utilizing the short exact sequence of Theorem 1.2 yields the sequence

$$1 \to (BM(R, G)/B(R)) \times (BC(R, G)/B(R)) \to BD(R, G)/B(R) \to N \to 1$$

where N has order $2(p - 1)$. Consideration of the actual elements of N shows that N is the dihedral group of order $2(p - 1)$. The proof that for G cyclic of order $n = p_1 \cdots p_t$, $BD(R, G)/B(R) \supseteq \prod_i BD(R, G_i)/B(R)$, G_i cyclic of order p_i, follows from an easy argument using Corollary 1.4; a further application of the properties of Z yields equality. Details may be found in [3].

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