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On Superheight Conditions for the Affineness of Open Subsets

Holger Brenner

Fakultät für Mathematik, Ruhr-Universität Bochum, Bochum 44780, Germany

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In this paper we consider the open complement U of a hypersurface $Y = V(\mathbf{a})$ in an affine scheme X. We study the relations between the affineness of U, the intersection of Y with closed subschemes, the property that every closed surface in U is affine, the property that every analytic closed surface is Stein, and the superheight of a defining ideal $\mathbf{a}$.

1. INTRODUCTION

Let A be a noetherian ring with an ideal $\mathbf{a} \subseteq A$ and let $X = \operatorname{Spec} A$, $Y = V(\mathbf{a})$. We consider the complement $U = D(\mathbf{a}) = X - Y$. The purpose of this paper is to find geometric conditions for U to be affine. It is well known that if U is affine then Y must be a hypersurface, i.e., $\operatorname{ht} \mathbf{a} \leq 1$; see Proposition 2.4. Note that the converse is by no means true, yet the height condition on Y has a stronger generalization based on the following simple observation.

Let $X' = \operatorname{Spec} A'$ be another affine scheme and $f : X' \longrightarrow X$ be a morphism corresponding to the ring homomorphism $A \longrightarrow A'$. If $U = D(\mathbf{a}) \subseteq X$ is affine then the preimage $U' = f^{-1}(U) = D(\mathbf{a}A')$ is also an affine scheme. Since $Y' = f^{-1}(Y) = V(\mathbf{a}A')$ this means that the height condition must also hold for all minimal primes of the extended ideal $\mathbf{a}' = \mathbf{a}A'$.

It is a general observation, first studied by Neeman [27], that the non-affineness can often be shown by giving a ring homomorphism violating the height condition on the extended ideal of $\mathbf{a}$. In order to illustrate this technique we give the following example.

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EXAMPLE 1.1. Let K be a field, $A = K[R, S, T, Z]/(RS - TZ)$, and $X = \operatorname{Spec} A$. A is a normal three-dimensional domain, the ideal $\mathbf{a} = (R, T)$ is prime of height one. Let $Y = V(R, T)$. Under the reduction $A \longrightarrow A/(Z, S) = K[R, T]$ the extended ideal is (R, T) in $K[R, T]$ which is of height two. Since the complement of a point in the plane is not affine it follows that $U = X - Y$ cannot be affine either.

In this paper we study the connection between the affineness of $U = D(\mathbf{a})$ and the property that the codimension of Y' under every ring homomorphism is restricted by one. This property can be expressed in terms of the superheight of the ideal, namely $\operatorname{supht} \mathbf{a} \leq 1$. This notion was first introduced by Hochster in 1975 [19]. We give an intrinsic definition of superheight depending only on the open set $D(\mathbf{a})$ (not on the ideal) so that the notion of superheight can be extended to arbitrary schemes (2).

In Section 3 we describe situations where the affineness can be obtained from superheight conditions. We show that the affineness of $D(\mathbf{a})$ is equivalent to the property that for all ring homomorphisms $A \longrightarrow A'$ where A' is a Krull domain the height of the extended ideal is ≤ 1. For a noetherian domain we characterize the affineness in terms of finite superheight under the additional condition that the ring of global sections is finitely generated, generalizing a result of Neeman [27]. Furthermore, in the two-dimensional case and in the case of monoid rings the affineness can be read off directly from the behaviour of the height in only one special ring extension.

In Section 4 we consider schemes of finite type over the complex numbers $\mathbf{C}$ and define the notion of analytic superheight and compare it with the algebraic notions of superheight. It will turn out that if U is Stein as a complex space then the analytic and the algebraic superheight is one. We recover the result of Bingener and Storch [3] that, under the condition that the ring of global sections is finitely generated, affineness and Stein are the same.

In Section 5 we consider finitely generated K-Algebras and relate the superheight one condition to the property that every closed subscheme of U of dimension ≤ 2 is affine. We show that in the complex case this property is equivalent to the property that any closed analytic surface in U is Stein. The question whether this last property implies the Stein property for U is the so called hypersurface (or hypersection) problem answered negatively by Coltoiu and Diederich [5, 6].

Finally, in Section 6, we give two classes of examples of non-affine open subsets with superheight one. The first class is constructed from certain curves on smooth projective surfaces, using the intrinsic characterization of superheight and a criterion à la “Riemannscher Fortsetzbarkeitssatz” for superheight one. The other is built from non-torsion divisor classes of a local two-dimensional normal ring, related to a construction of Rees and yielding counterexamples to the hypersurface problem.

2. THE SUPERHEIGHT OF AN IDEAL AND OF A SCHEME

Let $\mathbf{a} \subseteq A$ be an ideal in a commutative ring and $A \longrightarrow A'$ a ring homomorphism. The extended ideal $\mathbf{a}A'$ describes the preimage of the open set $D(\mathbf{a})$ under the mapping $\operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$. The height of an ideal $\mathbf{a} \subset A$ is defined as the minimal height of a minimal prime of $\mathbf{a}$. The maximal height of the minimal primes is called the big height or the altitude of $\mathbf{a}$. We put $\operatorname{ht}(\mathbf{A}) = 1$ in case $A \neq 0$, otherwise $= 0$.

DEFINITION. For an ideal $\mathbf{a} \subseteq A$ in a commutative ring we call

$\operatorname{supht} \mathbf{a} = \max\{\operatorname{ht} \mathbf{a}A' : A \longrightarrow A' \text{ with } A' \text{ noetherian}\}$ the *superheight* of $\mathbf{a}$ or the noetherian superheight.

$\operatorname{supht}^{\text{fin}} \mathbf{a} = \max\{\operatorname{ht} \mathbf{a}A' : A \longrightarrow A' \text{ with } A' \text{ of finite type}\}$ the *finite superheight* of $\mathbf{a}$.

$\operatorname{supht}^{\text{krull}} \mathbf{a} = \max\{\operatorname{ht} \mathbf{a}A' : A \longrightarrow A' \text{ with } A' \text{ Krull domain}\}$ the superheight with respect to Krull domains.

This notion goes back to Hochster and was developed in connection with the direct summand conjecture [19]. This conjecture states that a local regular ring A is a direct summand in any finite extension $A \subseteq B$. The conjecture is known to be true if A contains a field. In general it is equivalent to the monomial conjecture, which can be stated as a proposition about the superheight of an ideal, namely that the ideal $(X_1, \ldots, X_n)$ in

$$\frac{\mathbf{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]}{\left((X_1 \cdots X_n)^k - Y_1 X_1^{k+1} - \cdots - Y_n X_n^{k+1}\right)}$$

has superheight $n-1$ (for every $k \in \mathbf{N}$); see [20, 21] and below for the treatment of the two-dimensional case ($n = 2$) via affineness. There are some important results of Koh [23, 24] on superheight which we will use in the following.

PROPOSITION 2.1. *Let* $\mathbf{a}$ *be an ideal in a noetherian ring* A. *Then the following statements are true* (*in* (2) *and* (3) *suppose* $\mathbf{a} \neq A$):

(1) *Given*

$$\operatorname{ht} \mathbf{a} \leq \operatorname{bight} \mathbf{a} \leq \operatorname{supht}^{\text{fin}} \mathbf{a} \leq \operatorname{supht} \mathbf{a} \leq \operatorname{ara} \mathbf{a}.$$

(2) *The finite superheight equals*

$$\operatorname{supht}^{\mathrm{fin}} \mathbf{a} = \max\{\operatorname{ht} \mathbf{m} : \mathbf{m} \textit{ is a maximal ideal of } A', A \longrightarrow A' \textit{ is of finite type and } V(\mathbf{a}A') = V(\mathbf{m})\}.$$

(3) *The superheight equals*

$$\operatorname{supht} \mathbf{a} = \max\{\dim A' : A' \textit{ is a noetherian local complete normal domain}, A \longrightarrow A' \textit{ is a ring homomorphism with } V(\mathbf{a}A') = V(\mathbf{m})\}.$$

Proof. (1) The first and third inequalities are clear; the second is proved below. ara $\mathbf{a}$ denotes the minimal number of functions $f_1, \ldots, f_k$ with $V(\mathbf{a}) = V(f_1, \ldots, f_k)$, so the fourth inequality follows from the general Krull Hauptidealsatz [7, Theorem 10.2].

(2) and (3) Let $\mathbf{a}A' \subseteq \mathbf{p}$ be a minimal prime ideal in A', and $\mathbf{p}_1, \ldots, \mathbf{p}_r$ the others. Prime avoidance shows that there exists $f \notin \mathbf{p}$ and $f \in \mathbf{p}_i$ for $i = 1, \ldots, r$. After the change $A' \longrightarrow A'_f$ the prime ideal $\mathbf{p}A'_f$ is the only minimal prime over $\mathbf{a}A'_f$, and the height is the height of $\mathbf{p}$. This proves the second inequality of (1). If the height of $\mathbf{p}$ is taken over the prime $\mathbf{q}$ of height zero, we get modulo $\mathbf{q}$ a domain. So in both cases we can restrict to morphisms where A' is a domain and $V(\mathbf{a}A') = V(\mathbf{p})$ irreducible.

We show that for a prime ideal $\mathbf{p}$ of height n in a noetherian ring there exists a residue class domain where $\mathbf{p}$ extends geometrically (as a radical) to a maximal ideal of height n. If $\mathbf{p}$ is maximal we are done, so let $\mathbf{q}$ be a direct prime over $\mathbf{p}$. Let $x \notin \mathbf{p}$, $x \in \mathbf{q}$. Then $\mathbf{q}$ is modulo x a minimal prime over $\mathbf{p}A'/x$ (and over $\mathbf{a}A'/x$), and we have in A'/x the relations (with $\mathbf{q}' = \mathbf{q}(A'/x)$)

$$\operatorname{ht} \mathbf{q}' = \dim(A'/x)_{\mathbf{q}'} = \dim(A'_{\mathbf{q}}/x) \geq \dim(A'_{\mathbf{q}}) - 1 \geq \operatorname{ht} \mathbf{p},$$

cf. [7, Corollary 10.9]. Sucessively we arrive at a maximal ideal. This proves (2).

(3) Localization at $\mathbf{p}$ yields a local ring; there the extended ideal describes geometrically exactly the maximal ideal, and the dimension is the superheight of $\mathbf{a}$. Under completion the dimension does not change and in considering a component of maximal dimension we have a complete local domain R. R is excellent, and therefore [15, 7.6.2] its normalization is again noetherian complete and local of the same dimension. ∎

THEOREM 2.2. *Let A be a noetherian ring and $\mathbf{a} \subseteq A$ an ideal. Then*

$$\operatorname{supht}^{\mathrm{fin}} \mathbf{a} = \sup\{\operatorname{bight} \mathbf{a}A' : A' \textit{ is the normalization of a residue class domain}\}.$$

Proof. See [23]. ∎

THEOREM 2.3. *Let K be a field and A a finitely generated K-Algebra, $\mathbf{a} \subseteq A$ an ideal. Then* $\operatorname{supht}^{\mathrm{fin}} \mathbf{a} = \operatorname{supht} \mathbf{a}$.

Proof. See [24, Theorem 1]. ∎

We extend the notion of superheight to an arbitrary scheme.

DEFINITION. Let X be a scheme. The *superheight* of X is the biggest number d such that there exists

(i) a noetherian affine scheme T with a closed point $P \in T$ of height d.

(ii) an affine morphism $f : T - \{P\} \longrightarrow X$.

If X is a variety over a field K, we call the same number, under the restriction that T be an affine variety, the finite superheight of X.

Remark. In determining the superheight of a scheme one may only look at local complete normal noetherian domains $T = \operatorname{Spec} A$. For this, first localize at P and then do the same steps as in the proof of 2.1 (3).

If X is empty we have $\operatorname{supht} X = 0$, because then $T - \{P\}$ has to be empty, hence $\dim T = 0$. On the other hand, a non-empty scheme X has superheight ≥ 1. For a point $\operatorname{Spec} K \longrightarrow X$ (K a field) the morphism

$$\operatorname{Spec} K[Y]_{(Y)} \supseteq D(Y) = \operatorname{Spec} K(Y) \longrightarrow \operatorname{Spec} K \longrightarrow X$$

is affine and $T = \operatorname{Spec} K[Y]_{(Y)}$ is one-dimensional.

If X is affine, we have $\operatorname{supht} X \leq 1$, because in this case the affineness of $T - \{P\} \longrightarrow X$ implies the affineness of $T - \{P\}$. The following proposition, which is the starting point of this whole subject, shows that $\dim T \leq 1$.

PROPOSITION 2.4. *Let X be a noetherian separated scheme and $U \subseteq X$ an affine open subscheme. Then every component of $Y = X - U$ has codimension ≤ 1. The same is true for $X = \operatorname{Spec} A$ where A is a Krull domain.*

Proof. Let η be the generic point of a component of Y, $A = \mathcal{O}_\eta$. Since X is separated, $i : \operatorname{Spec} A \hookrightarrow X$ is an affine morphism and thus $D(\eta) = i^{-1}(U)$ is again affine. So we have to show that in a local noetherian ring A the complement of the closed point is affine only in case $\dim A \leq 1$. We may assume that A is a domain. The normalization A_{nor} of A is a semilocal Krull domain (see [25]), so we are led to a local Krull domain A. But for a Krull domain with $\dim A \geq 2$ we have $\Gamma(D(\mathbf{m}), \mathcal{O}_X) = A$, hence $D(\mathbf{m})$ is not affine. ∎

The assumption in the following criterion for $\operatorname{supht}(X) \leq 1$ says that X satisfies as target the “Riemannscher Fortsetzbarkeitssatz.”

LEMMA 2.5. *Let X be a noetherian separated scheme satisfying the following property: For any normal noetherian scheme T and any closed point $P \in T$ with* $\operatorname{ht} P \geq 2$, *every morphism $T - \{P\} \longrightarrow X$ is extendible to T.*

Then $\operatorname{supht}(X) \leq 1$.

If X is quasi-affine, the converse is also true.

Proof. Let T be affine. If X is separated, an affine morphism $f : T - \{P\} \longrightarrow X$ with $\operatorname{ht} P \geq 2$ cannot be extended to the whole of T. An extension $\bar{f} : T \longrightarrow X$ would be an affine morphism, and for an affine open neighbourhood $\bar{f}(P) \in V$ the sets $\tilde{f}^{-1}(V)$ and $f^{-1}(V)$ must both be affine. But $\bar{f}^{-1}(V) = f^{-1}(V) \cup \{P\}$ and P is a point of height ≥ 2, so this is not possible. Therefore an affine morphism $T - \{P\} \longrightarrow X$ with $\operatorname{ht} P \geq 2$ and T normal contradicts the assumption.

Let $X \subseteq \operatorname{Spec} A$ quasi-affine with superheight ≤ 1 and $f : T - \{P\} \longrightarrow X$ a morphism with T normal and affine, $\operatorname{ht}(P) \geq 2$. f is not affine, but there is an affine extension $\bar{f} : T \longrightarrow \operatorname{Spec} A$, corresponding to the ring homomorphism $A \to \Gamma(T - \{P\}, \mathcal{O}_T) = \Gamma(T, \mathcal{O}_T)$. If $\tilde{f}(P) \notin X$, f would be the restriction of $\bar{f}$ on X, hence affine. So $\bar{f}(P) \in X$ and f is extendible as a mapping to X. ∎

As the following proposition shows, the superheight of an ideal $\mathbf{a}$ and the superheight of the open set $D(\mathbf{a})$ coincide.

PROPOSITION 2.6. *For an ideal $\mathbf{a} \subseteq A$ the equality* $\operatorname{supht} D(\mathbf{a}) = \operatorname{supht} \mathbf{a}$ *holds.*

Proof. Let $A \longrightarrow R$ be a ring homomorphism in a local normal noetherian domain of dimension $m = \operatorname{supht} \mathbf{a}$ with $V(\mathbf{a}R) = V(\mathbf{m}_R) = \{P\}$. Let $U = D(\mathbf{a})$. Then the mapping $f^{-1}(U) = \operatorname{Spec} R - \{P\} \longrightarrow U$ is affine and therefore $\operatorname{supht} U \geq \operatorname{supht} \mathbf{a}$.

For the converse inequality let $f : T - \{P\} \longrightarrow U$ be affine morphism where T is local normal noetherian and $d = \dim T = \operatorname{supht} U$. If $d = 0$, there is nothing to show. If $d = 1$, it follows that U is not empty. Let $\mathbf{q} \in U$ be a prime ideal and consider $A \longrightarrow A_{\mathbf{q}} = A'$. Then we have $\mathbf{a}A_{\mathbf{q}} = A_{\mathbf{q}}$ and therefore $\operatorname{supht} \mathbf{a} \geq 1$ follows from the definition.

So let $d \geq 2$. Since $T = \operatorname{Spec} R$ is normal, we have $\Gamma(T - \{P\}, \mathcal{O}_T) = \Gamma(T, \mathcal{O}_T) = R$ and f corresponds to the global ring homomorphism $A \to R$, so the mapping is extendible to a mapping $\bar{f} : \operatorname{Spec} R \longrightarrow \operatorname{Spec} A$. $\bar{f}(P) \in U$ is not possible, for otherwise the mapping would be extendible as a mapping into U, but this is excluded by the proof of the previous lemma. So under $A \longrightarrow R$ the extended ideal describes $V(\mathbf{a}R) = \{P\}$ and therefore $\operatorname{supht} \mathbf{a} \geq d = \operatorname{supht} U$. ∎

We gather together some properties of the superheight of a scheme.

PROPOSITION 2.7. (1) *For an affine morphism* $X' \longrightarrow X$ *we have* $\operatorname{supht} X' \leq \operatorname{supht} X$.

(2) *The superheight of* X *equals the maximum of the superheights of the irreducible components of* X.

(3) *Suppose* X *is noetherian. For* $Y \subseteq X$ *closed and* $U = X - Y$ *we have*

$$\operatorname{supht} X \leq \operatorname{supht} Y + \operatorname{supht} U.$$

(4) *If* $X = U \cup V$ *with* U, V *open, we have* $\operatorname{supht} X \leq \operatorname{supht} V + \operatorname{supht} U$.

(5) $\operatorname{supht} X \leq \dim X + 1$.

(6) *For a noetherian separated scheme* X *we have* $\operatorname{supht} X \leq \operatorname{cd} X + 1$ (cd *denotes the cohomological dimension of* X *in the sense of R. Hartshorne, meaning the maximal number* $n \in \mathbf{N}$ *such that there is a quasicoherent sheaf* $\mathcal{F}$ *on* X *with* $H^n(X, \mathcal{F}) \neq 0$).

Proof. Part (1) is clear. For (2), let $f : T - \{P\} \longrightarrow X$ be affine with T irreducible. The image of T lies in a component X_i of X and this component must have the superheight of X.

(3) Let $f : T \supseteq T - \{P\} \longrightarrow X$ be an affine morphism with $T = \operatorname{Spec} R$, R being a noetherian local complete domain, and with $\dim T = \operatorname{supht} X$. Set $f^{-1}(Y) = V(\mathbf{a}) - \{P\}$ with an ideal $\mathbf{a} \subseteq R$. On one hand, we have $\dim V(\mathbf{a}) \leq \operatorname{supht} Y$ as is shown by the restriction $V(\mathbf{a}) - \{P\} = f^{-1}(Y) \longrightarrow Y$. On the other hand, the restriction $D(\mathbf{a}) = f^{-1}(U) \longrightarrow U$ is also affine and so $\operatorname{bight} \mathbf{a} \leq \operatorname{supht} D(\mathbf{a}) \leq \operatorname{supht} U$. Since R is complete, R is catenary (see [7, Corollary, 18.10]) and so for a minimal prime $\mathbf{p}$ of $\mathbf{a}$ we have the inequalities

$$\begin{aligned} \operatorname{supht} X = \dim R &= \dim R/\mathbf{p} + \operatorname{ht} \mathbf{p} \\ &\leq \dim R/\mathbf{a} + \operatorname{bight} \mathbf{a} \\ &\leq \operatorname{supht} Y + \operatorname{supht}(X - Y). \end{aligned}$$

(4) Let $X = U \cup V$. Then $Y = X - U$ is a closed subset of V leading to $\operatorname{supht} Y \leq \operatorname{supht} V$ and the statement follows from (3).

(5) We do induction on the dimension; the beginning is clear. Because of (2) we may assume that X is irreducible of dimension d. For a non-empty open affine subset U, (3) yields $\operatorname{supht} X \leq \operatorname{supht} U + \operatorname{supht}(X - U) \leq 1 + d$.

(6) $\operatorname{supht} X = 0$ if and only if $X = \emptyset$. In this case $\operatorname{cd} X = -1$. So suppose $\operatorname{supht} X \geq 1$. If T is a local noetherian affine scheme of dimension $d \geq 1$ a theorem of Grothendieck says that $H^d_{\mathbf{m}}(T, \mathcal{O}) \neq 0$. The natural map

of local cohomology $H^{i-1}(T-\{P\},\mathscr{O}) \longrightarrow H^i_{\mathbf{m}}(T,\mathscr{O})$ is bijective for $i \geq 2$ and surjective for $i=1$. Thus $H^{d-1}(T-\{P\},\mathscr{O}) \neq 0$. If $f: T-\{P\} \longrightarrow X$ is affine and $d = \operatorname{supht} X$ it follows that $H^{d-1}(X, f_*\mathscr{O}) \neq 0$ and cd $X \geq d-1 = \operatorname{supht} X - 1$. This gives also another proof of (5) and of 2.4. ∎

EXAMPLE 2.1. Let Y be a projective variety of dimension d. The mapping of a punctured affine cone $X - \{P\} \longrightarrow Y$ is affine, hence the superheight of Y is $\geq d+1$ and equality must hold because of (5). It is reasonable to ask whether maximal possible superheight—the existence of such an affine cone—ensures for a normal separated variety projectivity. A result of Kleiman states that a normal separated variety is proper if and only if the cohomological dimension is maximal [22].

COROLLARY 2.8. *Let X be a scheme with* $\operatorname{supht} X \leq d$. *Then the complement of X in any open embedding $X \subseteq X'$ with X' noetherian and separated has codimension $\leq d$.*

Proof. For an affine subset U of X' the morphism $U \cap X \hookrightarrow X$ is affine, so $U \cap X$ fulfills the assumption as well. Since the conclusion is local, we may assume X' to be affine. Thus the statement follows from 2.6. ∎

3. AFFINENESS AND SUPERHEIGHT ONE

Let $a = (f_1, \ldots, f_n) \subseteq A$ be an ideal in a commutative ring, $U = D(\mathbf{a}) \subseteq \operatorname{spec} A = X$, and $B = \Gamma(U, \mathscr{O}_X)$ the ring of global sections on U. In this situation we have an open embedding $U = D(\mathbf{a}B) \hookrightarrow \operatorname{Spec} B$, and U is affine if and only if $\mathbf{a}B$ is the unit ideal. In this case we have $1 = q_1 f_1 + \cdots + q_n f_n$ with $q_i \in \Gamma(U, \mathscr{O}_X)$, and the functions yield a closed embedding $(q_1, \ldots, q_n) : U \hookrightarrow \operatorname{Spec} A[T_1, \ldots, T_n]$, showing by the way that in the affine case B is an A-algebra of finite type. If this is not the case, the height of this extended ideal is larger than one.

THEOREM 3.1. *Let A be a noetherian ring and* $\mathbf{a}$ *an ideal*, $U = D(\mathbf{a})$. *Then U is affine if and only if* $\operatorname{supht}^{\mathrm{krull}} \mathbf{a} \leq 1$.

Proof. If U is affine and $A \longrightarrow A'$ is a ring homomorphism, where A' is a Krull domain, then the preimage $U' = D(\mathbf{a}A')$ is affine and $V(\mathbf{a}A')$ has codimension ≤ 1; see 2.4.

So suppose U is not affine. Since a noetherian scheme is affine if and only if all its (reduced) components are affine (see [17, II.1.4]) we find $A \longrightarrow A'$, where A' is a noetherian domain and where $D(\mathbf{a}A')$ is not affine. So we may assume that A is a domain. Consider the normalization $\varphi : \operatorname{Spec} A_{\mathrm{nor}} \longrightarrow \operatorname{Spec} A$. If $\mathbf{a} = (f_1, \ldots, f_n)$ and $\varphi^{-1}(U)$ were affine, there would exist $q_i \in \Gamma(\varphi^{-1}(U), A_{\mathrm{nor}})$ with $q_1 f_1 + \cdots + q_n f_n = 1$. But these functions are already defined on the corresponding open set in a finite extension $A \subset B \subset A_{\mathrm{nor}}$, and the theorem of Chevalley [15, 17, II.1.5] shows that U itself would be affine.

So we may assume that A is a Krull domain. For an open subset W in $\operatorname{Spec} A$ of a Krull domain the ring of global sections is given by the intersection of discrete valuation domains,

$$\Gamma(W, \mathscr{O}_X) = \bigcap_{\operatorname{ht}(\mathbf{p})=1,\, \mathbf{p}\in W} A_{\mathbf{p}}.$$

From this we see that the ring of global sections $B = \Gamma(U, \mathscr{O}_X)$ is again a Krull domain. We have $U = D(\mathbf{a}) \cong D(\mathbf{a}B) \subseteq \operatorname{Spec} B$, and $\mathbf{a}B$ is not the unit ideal. On the other hand, we have $B = \Gamma(U, \mathscr{B})$, and this can only hold if U contains all prime ideals of height one of the Krull domain B. For if $\mathbf{p}$ of height one is not in U, let p be a generator of the maximal ideal in the discrete valuation ring $B_{\mathbf{p}}$ and let $q = 1/p$. Let $\mathbf{q}_1, \ldots, \mathbf{q}_m$ be the other poles of q. We find $f \in B$ with $f \notin \mathbf{p}$, $f \in \mathbf{q}_i$ for $i = 1, \ldots, m$. Then for all n big enough the function $f^n q$ has its only pole in $\mathbf{p}$ and is defined on $\cup \subseteq D(\mathbf{p})$. So we conclude that $\mathbf{a}B$ has height ≥ 2. ∎

Under additional conditions on the ring of global sections the superheight condition for smaller classes of rings guarantees affineness. The following result can also be found in [27] in the case that A is normal and of finite type over a field.

THEOREM 3.2. *Let A be a noetherian domain and $\mathbf{a}$ an ideal, $U = D(\mathbf{a})$. Then U is affine if and only if the ring of global sections $\Gamma(U, \mathscr{O}_X)$ is of finite type over A and* $\operatorname{supht}^{\mathrm{fin}} \mathbf{a} \leq 1$.

Proof. If U is affine, it is known that $B = \Gamma(U, \mathscr{O}_X)$ is finitely generated over A, so suppose U is not affine with a finitely generated ring B of global sections. B is a noetherian domain and the extended ideal is not the unit ideal, but $U \cong D(\mathbf{a}B)$ contains all prime ideals of height one of B. For if $\mathbf{p} = (f_1, \ldots, f_m)$ is a prime ideal in B of height one there is a function $f \in \mathbf{p}$ with $\operatorname{Rad}(f) = \mathbf{p}B_{\mathbf{p}}$. This yields equations $f_i^n = (a_i/r_i)f$ with $r_i \notin \mathbf{p}$. With $r = r_1 \cdot \cdots \cdot r_m$ we may write $f_i^n = (b_i/r)f$ or $r/f = b_i/f_i^n$, showing that this is a function defined on $D(\mathbf{p})$ not belonging to B, since otherwise $f(r/f) = r \in \mathbf{p}$. So we have height $\mathbf{a}B \geq 2$ and $\operatorname{supht}^{\mathrm{fin}} \mathbf{a} \geq 2$. ∎

In deciding whether an open subset of an affine scheme is again affine, one may look at the ring of global sections and the height of the extended ideal in it. If this ideal is the unit ideal, U is affine and the superheight is one. If this is not the case, the ring of global sections is just one candidate among others to show that the superheight is ≥ 2.

EXAMPLE 3.1. Let K be a domain and consider in the domain

$$A = \frac{K[X_1, X_2, Y_1, Y_2]}{(X_1^k X_2^k + Y_1 X_1^{k+1} + Y_2 X_2^{k+1})}$$

the ideal $\mathbf{a} = (X_1, X_2)$, $U = D(\mathbf{a})$. The functions

$$Z = \frac{-Y_2}{X_1^k} = \frac{(X_2^k + Y_1 X_1)}{X_2^{k+1}} \qquad \text{and} \qquad W = \frac{-Y_1}{X_2^k} = \frac{(X_1^k + Y_2 X_2)}{X_1^{k+1}}$$

are defined on U and one has $X_2 Z + X_1 W = 1$, hence U is affine.

This example is for $K = \mathbf{Z}$ the two-dimensional case of the superheight version of the monomial conjecture, and the easiest way to settle this instance is by showing the affineness. For another proof see [21].

EXAMPLE 3.2. Now we look at the prime ideal $\mathbf{a} = (X_1, X_2)$ in the domain

$$A = \frac{K[X_1, X_2, Y_1, Y_2]}{(X_1^k X_2^k + Y_1 X_1^k + Y_2 X_2^{k+1})}.$$

Consider the morphism $A \longrightarrow A' = K[X_1, X_2]$ given by the substitution $Y_1 \longmapsto -X_2^k$, $Y_2 \longmapsto 0$. Then $\mathbf{a}A' = (X_1, X_2)$ has height two, and $D(\mathbf{a})$ is not affine.

Two-Dimensional Rings

A theorem of Nagata states that on a normal affine surface the complement of any (pure one-dimensional) curve is affine; see [26]. From the proof of this theorem one can get the following theorem.

THEOREM 3.3. *Let A be a two-dimensional noetherian ring, $\mathbf{a} \subseteq A$. Then $D(\mathbf{a})$ is affine if and only if the noetherian superheight of $\mathbf{a}$ is ≤ 1.*

Proof. Suppose $U = D(\mathbf{a})$ is not affine. We may assume that A is a two-dimensional noetherian normal and local domain, since the normalization of a noetherian two-dimensional domain is again noetherian. $B = \Gamma(U, \mathcal{O}_X)$ is a Krull domain, and, since U is not affine, the height of the extended ideal $\mathbf{b} = \mathbf{a}B$ is at least two. By a faithfully flat extension as in [26] one may assume that there exist infinitely many prime elements in A. Then one can show for a minimal prime $\mathbf{m}'$ of $\mathbf{b}$ that $R = B_{\mathbf{m}'}$ is the desired two-dimensional and noetherian ring. ∎

THEOREM 3.4. *Let A be an excellent two-dimensional domain. The complement of a curve $Y \subseteq X = \operatorname{Spec} A$ is affine if and only if every component of the preimage of Y in the normalization $\widetilde{X}$ has codimension one. This means that the preimage does not have isolated points.*

Proof. If the preimage $\widetilde{Y}$ has pure codimension one, the theorem of Nagata (which is valid for affine excellent surfaces) says that $\widetilde{Y}$ has an affine complement, and the theorem of Chevalley says that this holds for Y itself. ∎

Remark. Of course, if the normalization is a bijection any complement of a curve is affine. If this is not the case it is quite easy to find curves with nonaffine complement. If $Q, R \in \widetilde{X}$ are different points mapping to $P \in X$, look for curves Y' on $\widetilde{X}$ lying generically inside the open set where the normalization is an open embedding (say X excellent) and with $Q \in Y', R \notin Y'$. Then the image Y of Y' cannot have an affine complement, because the preimage $\widetilde{Y} = Y' \cup \{R\}$ and R is an isolated point in it. On X itself we have to look for regular (or at least cuspidal) curves C through P not totally lying on $\operatorname{Sing} X$.

Monoid Rings

Let M be a normal torsion-free finitely generated monoid with quotient lattice $\Gamma = \mathbf{Z}M \cong \mathbf{Z}^d$. Let M be positive, meaning that 0 is the only unit of M. Then there exists an embedding with the intersection property, namely $M \hookrightarrow \mathbf{Z}^k$ with $M = \Gamma \cap \mathbf{N}^k$, see [4, exc. 6.1.10], or take the natural embedding given by the divisor class representation. Such an embedding yields an inclusion of rings

$$K[M] \hookrightarrow K[\mathbf{N}^k] = K[T_1, \dots, T_k],$$

and $K[M]$ is the ring of degree zero under the D-graduation of the polynomial ring given by $\mathbf{Z}^k \longrightarrow \mathbf{Z}^k/\Gamma =: D$. In particular $K[M]$ is a direct summand of $K[T_1, \dots, T_k]$.

THEOREM 3.5. *Let M be a finitely generated torsion free monoid and K a noetherian factorial domain. Then there exists a ring extension of finite type $K[M] \hookrightarrow B$ such that an open subset $U = D(\mathbf{a}) \subseteq \operatorname{Spec} K[M]$ is affine if and only if* $\operatorname{bight} \mathbf{a}B \leq 1$. *In particular U is affine if and only if* $\operatorname{supht}^{\text{fin}} \mathbf{a} \leq 1$.

Proof. Let $\widetilde{M}$ be the normalization of M and $\widetilde{M} = \mathbf{Z}^s \times M'$ with M' positive; see [4, Theorem 6.1.4 and Proposition 6.1.3]. Let $M' \hookrightarrow \mathbf{N}^k$ be a representation with the intersection property. Then the mapping

$$K[M] \longrightarrow K[\widetilde{M}] \cong K[\mathbf{Z}^s][M'] \longrightarrow K[V_1, \dots, V_s, V_1^{-1}, \dots, V_s^{-1}][T_1, \dots, T_k] = B$$

is of finite type. Let $\mathbf{a}K[M]$ be an ideal with $\operatorname{bight} \mathbf{a}B \leq 1$. Since B is factorial, we know that $D(\mathbf{a}B)$ is affine and we have to show that this property holds already for $D(\mathbf{a})$. For a finite extension this is the theorem of Chevalley, and for a direct summand $A \subseteq B = A \oplus V$ this is true, since $\Gamma(D(\mathbf{a}B), \mathcal{O}_B) = \Gamma(D(\mathbf{a}), \mathcal{O}_A) \oplus \Gamma(D(\mathbf{a}), \overline{V})$, and, if $\mathbf{a}$ generates the unit ideal in $\Gamma(D(\mathbf{a}B), \mathcal{O}_B)$, this is also true in the first component. ∎

4. AFFINENESS, THE STEIN PROPERTY, AND SUPERHEIGHT ONE

In the case $K = \mathbf{C}$, we can associate to an algebraic variety X the corresponding complex space X^{an}. If X is an affine variety, then X^{an} is a Stein space; see [13, V, Sect. 1 Satz 1]. We will show that the analytic property of being Stein is strong enough to guarantee that the noetherian superheight is one. Thus the existence of Stein but non-affine quasi-affine schemes yields directly to non-affine quasi-affine varieties with superheight one. We consider only separated varieties and complex Hausdorff spaces. Some results and ideas of this section can also be found in Neeman [27] and in Bingener and Storch [3].

Definition. Let X be a complex space and $Y \subseteq X$ a closed analytic subset. We define the analytic superheight of Y by

$$\operatorname{supht}^{\mathrm{an}}(Y, X) = \sup\{\operatorname{codim}_{x'}(f^{-1}(Y), X') : x' \in X', f : X' \longrightarrow X\}.$$

Here we put $\operatorname{codim}_x(Y, X) = \dim_x X - \dim_x Y$ with $\dim_x X = \dim \mathcal{O}_x = \dim \widehat{\mathcal{O}}_x$; see [12, Kap. II, Sect. 4ff]. If the analytic set Y_x is described at the point $x \in X$ by the ideal $\mathbf{a}$, we have $\operatorname{codim}_x(Y, X) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{X,x}/\mathbf{a})$. If X is irreducible this equals the height of the ideal, since the analytic rings are catenary.

Lemma 4.1. *Let $Y \subseteq X$ be a closed analytic subset in a complex space with $U = X - Y$ Stein. Then* $\operatorname{supht}^{\mathrm{an}}(Y, X) \leq 1$.

Proof. Let $f : X' \longrightarrow X$ be a morphism of complex spaces and x' a point of X'. Since the codimension is local, we can assume that X' is Stein. f factors through the closed graph $X' \overset{\Gamma_f}{\hookrightarrow} X' \times X \xrightarrow{p_2} X$ and therefore $f^{-1}(X - Y)$ is isomorphic to a closed subset of $X' \times (X - Y)$. Since X' and $X - Y$ are Stein, the product $X' \times (X - Y)$ is also Stein and so is $X' - f^{-1}(Y) \subseteq X$; see [13, V, Sect. 1, Satz 1]. But the complement of an open Stein subset in a Stein space has codimension ≤ 1; see [13, V, Sect. 3, Satz 4]. ∎

Theorem 4.2. *Let $X = \operatorname{Spec} A$ be an affine algebraic $\mathbf{C}$-variety and $V(\mathbf{a}) = Y \subseteq X$. Then the algebraic and the analytic superheight coincide.*

$$\operatorname{supht} \mathbf{a} = \operatorname{supht}^{\mathrm{fin}} \mathbf{a} = \operatorname{supht}^{\mathrm{an}}(Y^{\mathrm{an}}, X^{\mathrm{an}}).$$

Proof. The first equality follows from the theorem of Koh (Theorem 2.3). Of course, the analytic superheight is not lower than the finite algebraic superheight, since we can interpret every algebraic test variety as an analytic variety, and the algebraic and analytic dimension coincide.

For the converse, let $f : X' \longrightarrow X^{\mathrm{an}}$ be a morphism of complex spaces, $x' \in X'$, $f(x') = x$. We may suppose that X' is irreducible. The extended ideal $\mathbf{a}\mathcal{O}_{X^{\mathrm{an}},x}$ under $A \longrightarrow \mathcal{O}_{X^{\mathrm{an}},x}$ describes the zero set Y^{an} in x, and the preimage Y' in x' is described by $\mathbf{a}\mathcal{O}_{X',x'}$. Since the local rings in a complex space are noetherian, see [12, Kap. I, 35.2, Satz 3, and Kap. II, 30; Satz 1], we have $\operatorname{codim}_{x'}(Y', X') = \operatorname{ht}(\mathbf{a}\mathcal{O}_{X',x'}) \leq \operatorname{supht} \mathbf{a}$. ∎

COROLLARY 4.3. *Let A be a $\mathbf{C}$-algebra of finite type and $U = D(\mathbf{a}) \subseteq X$ an open subset with U^{an} Stein. Then* $\operatorname{supht} \mathbf{a} \leq 1$.

Proof. This follows from the theorem and the lemma. ∎

COROLLARY 4.4. *Let A be a domain of finite type over $\mathbf{C}$ and $U \subseteq \operatorname{Spec} A$ an open subset with $\Gamma(U, \mathcal{O}_X)$ finitely generated. Then U is affine if and only if U is Stein.*

Proof. The previous corollary shows that the finite superheight is one. This together with the finiteness of the global ring shows that U is affine. (For another proof see [3, 5.1].) ∎

5. SUPERHEIGHT ONE AND AFFINENESS OF TWO-DIMENSIONAL SUBSCHEMES

Let U be a separated scheme of finite type over a field K. In this section we study the property that every closed surface in U is affine.

THEOREM 5.1. *Let A be a domain of finite type over a field K, $D(\mathbf{a}) = U \subseteq X = \operatorname{Spec} A$ an open subset. Then the following are equivalent.*

(1) $\operatorname{supht} \mathbf{a} \leq 1$.

(2) *Every closed subvariety of dimension ≤ 2 of U is affine.*

If $K = \mathbf{C}$, this is also equivalent to the following.

(3) *For every closed analytic surface $S \subseteq X^{\mathrm{an}}$ the intersection $S \cap U^{\mathrm{an}}$ is Stein.*

Proof. Suppose (1) holds. For points and curves the statement (2) is always true, so let $S \hookrightarrow U$ be a closed reduced irreducible surface in U, and let S' be the closure of S in X. Then $S' \hookrightarrow X$ is again a surface, because the dimension of an irreducible variety does not change in passing to a non-empty open subset. Let $\overline{S} \longrightarrow S'$ be the normalization. The preimage of $Y = V(\mathbf{a})$ under $\overline{S} \longrightarrow X$ is due to the superheight property of pure codimension one and hence due to the theorem of Nagata it has an affine complement. The theorem of Chevalley shows that the complement of $S' \cap Y$ is again affine, so $S = S' \cap U = S' - S' \cap Y$ is affine.

For the converse let $\operatorname{supht}^{\mathrm{fin}} \mathbf{a} = \operatorname{supht} \mathbf{a} \geq 2$. Then there exists (Theorem 2.2) an irreducible surface $\operatorname{Spec} R = S \subseteq X$ with normalization $S' = \operatorname{Spec} R_{\mathrm{nor}}$ such that $\mathbf{a}R_{\mathrm{nor}}$ has big height 2. Thus $D(\mathbf{a}R_{\mathrm{nor}})$ and $D(\mathbf{a}R)$ cannot be affine, and $D(\mathbf{a}R) = U \cap S$.

Now let $K = \mathbf{C}$ and suppose (1) holds. Let $S \subseteq X^{\mathrm{an}}$ be a closed analytic surface with normalization $f : \overline{S} \longrightarrow S \hookrightarrow X^{\mathrm{an}}$. Then the codimension of $f^{-1}(Y)$ on the normal surface $\overline{S}$ is ≤ 1, because the algebraic superheight equals the analytic superheight. The theorem of Simha (this is the analytic analogue to the theorem of Nagata, see [29]) says that $\overline{S} - f^{-1}(Y)$ is a Stein space. This means that the normalization of $U \cap S$ is Stein, and so due to the analytic version of the theorem of Chevalley $U \cap S$ itself is Stein.

Now suppose (3) holds and let an algebraic surface $S' \subseteq U$ be given. We can write $S' = S \cap U$ with a closed algebraic surface S in X. By (3) we know that $S' = S \cap U$ is Stein, so by 4.3 and 3.3 it is affine. ∎

COROLLARY 5.2. *Let U be a quasi-affine variety over K such that the ring of global sections is finitely generated. If all irreducible closed surfaces of U are affine, U itself is affine.*

Proof. This follows directly from the theorem and Theorem 3.2. ∎

Remark. In case $K = \mathbf{C}$, the last statement of the theorem is fulfilled if U itself is Stein. The hypersection (or hypersurface) problem in complex analysis asks the following: Given a Stein space X of dimension ≥ 3 and an open subset $U \subseteq X$ with the property that for any analytic hypersurface $S \subseteq X$ the intersection $S \cap U$ is Stein, is then U itself Stein? If $U \subseteq X$ is algebraic and $\dim X = 3$, statement (3) of the above theorem is exactly the condition of the hypersurface problem.

However, the hypersurface problem is now known not to be true in general, as first shown by an example of Coltoiu and Diederich; see [5, 6]. In Section 6 we will give a class of examples of non-Stein open subsets with superheight one, and 5.1 shows that the assumptions of the hypersection problem are fulfilled.

EXAMPLE 5.1. The affineness of an open subset cannot be tested (even if the ring of global sections is finitely generated) with more restrictive classes of surfaces. The following example shows that homogeneous surfaces do not suffice.

Let S be the projective plane, blown up in one point P. Let E be the exceptional divisor and C a projective line not passing through the point. $W = S - (E \cup C)$ is then a punctured affine plane, so W is quasi-affine and contains no projective lines. Let A be a homogeneous coordinate ring for S, $W = D_+(\mathbf{a})$, $U = D(\mathbf{a}) \subseteq X = \operatorname{Spec} A$. For an irreducible homogeneous surface $V(\mathbf{p})$ in the affine cone X the corresponding projective curve $V_+(\mathbf{p})$ intersects $V_+(\mathbf{a})$, and therefore $V_+(\mathbf{p}) \cap W$ is affine, hence also the preimage $V(\mathbf{p}) \cap U$. This means that all homogeneous surfaces inside U are affine. But not all surfaces in U are affine. U is the cone over a punctured affine plane and thus isomorphic to $\mathbf{A}^{\times} \times (\mathbf{A}^2 - \{P\})$. As a subset of $\mathbf{A}^{\times} \times \mathbf{A}^2$ it has height two, and this gives a lot of non-affine surfaces.

6. NON-AFFINE SUBSETS WITH SUPERHEIGHT ONE

A theorem of Goodman states that on a smooth projective surface S an open subset $U = S - Y$ is affine if and only if there exists an ample effective divisor H with $\operatorname{supp} H = Y$ [9; 17, II.4.2]. A weaker condition on Y and H still implies that U has superheight one.

THEOREM 6.1. *Let S be a smooth projective surface over an algebraically closed field K, $Y \subset S$ a curve, and $U = S - Y$. Suppose there exists an effective divisor H with* $\operatorname{supp} H = Y$ *and with $H.Y_i \geq 0$ for all irreducible components Y_i of Y and with $H.C > 0$ for all curves $C \not\subseteq Y$. Then every morphism $T \supseteq T - \{P\} \longrightarrow U$, where T is a two-dimensional normal irreducible affine variety, is extendible to T.*

If $S =$ Proj A with a finitely generated graded K-Algebra A and $U = D_+(\mathbf{a})$, then $\operatorname{supht} \mathbf{a} = 1$.

Proof. We have already seen in 2.5 that $\operatorname{supht}^{\text{fin}}(U) = 1$ follows from the described extendibility property. Since the cone mapping is affine, it follows that $\operatorname{supht}^{\text{fin}} \mathbf{a} = 1$ and, due to the theorem of Koh, $\operatorname{supht} \mathbf{a} = 1$.

So let $f : T - \{P\} \longrightarrow U$ be a morphism of a reduced irreducible normal affine surface T. We may assume that $T - \{P\}$ is regular. If $f(T - \{P\})$ is a point, f is of course extendible. If $f(T - \{P\}) \subseteq U$ lies inside an irreducible curve $C \subseteq U$, this curve C is due to the assumption not projective, hence affine. Then f corresponds to a ring-homomorphism and is thus extendible to $C \subseteq U$ as in the proof of Theorem 2.6. So suppose that the image of f is two-dimensional and f dominates U.

Let $T \hookrightarrow T'$ be an open embedding in a projective surface with complement D'. Let $p : \overline{T} \longrightarrow T'$ be a resolution of singularities of T' and a resolution of the undefined points of $f : T' \supseteq T - \{P\} \longrightarrow S$; see [18, V.3.8.1 and Theorem V.5.5]. So we have an extension $\bar{f} : \overline{T} \longrightarrow S$ of f on $T - \{P\} \cong \overline{T} - p^{-1}(P) - p^{-1}(D')$. Let $C_1, \dots, C_n$ be the irreducible one-dimensional components of $p^{-1}(P)$ and let $D_1, \dots, D_m$ be the components of $p^{-1}(D')$.

Since $\bar{f}$ is surjective, it induces a mapping $\bar{f}^*$ of the divisors (= Cartier-divisors). Let $\bar{f}^*(H) = C + D$ with $C = k_1 C_1 + \cdots + k_n C_n$ and $D = l_1 D_1 + \cdots + l_m D_m$ be the pull-back of the divisor H; there cannot be other components. $\bar{f}_*(C)$ is a non-negative combination of the Y_j, so the assumptions concerning the intersections of H with its components yield

$$0 \leq \bar{f}_*(C).H = C.\bar{f}^*(H) = C.C + C.D = C.C.$$

But due to [1, theorem 2.3], the self intersection number of an effective divisor $\neq 0$ is negative, if all its components are (possibly singular) contractible. Since the components of C are contracted by p to P, we must have $C = 0$. So for all components C_i we have $\bar{f}(C_i) \not\subseteq Y$.

So the preimage of Y under $\bar{f}$ contains only some points on the C_i. If $Q \in C_i$ with $\bar{f}(Q) = R \in Y$, we find—since S is regular and hence locally factorial—an affine neighbourhood W of R where Y is described by one function, so the preimage of Y must be a curve, which is already excluded.

So we conclude that $\bar{f}(C_i) \subseteq U = S - Y$ for all curves C_i. Since on U there exist no projective curves, all these curves are contracted by $\bar{f}$ to a point of U, and to exactly one point, because the C_i are connected. So f itself is extendible in P as a function to U. ∎

Remark. We cannot weaken the assumptions in this theorem. If Y is irreducible with $Y^2 = 0$ and the complement contains no projective curves, this has no consequence on the superheight as shown by the example at the end of Section 5. If $Y \cap C = \emptyset$ for some curve C, then C is a projective curve lying inside U, and the cone mapping of this curve is not extendible to the vertex of the cone.

If Y is irreducible, the condition of the theorem says $Y^2 \geq 0$ and $S - Y$ contains no projective curve. In this case we cannot avoid the assumption $Y^2 \geq 0$. If $K = \mathbf{C}$ and $Y^2 < 0$, one can contract Y onto a (possibly non-algebraic) complex space; see [11]. This contraction yields a mapping back on this complex space defined outside the contraction point, and this mapping is not extendible.

COROLLARY 6.2. *Suppose the situation of the theorem holds, but there exists no ample effective divisor H with support Y. Then $U = D_+(\mathbf{a})$ and the preimage $D(\mathbf{a})$ in the affine cone are not affine, yet their superheight is one. This is in particular the case if Y is irreducible with $Y^2 = 0$ or if Y is not connected.*

Proof. If U is affine then there exists an ample divisor H with support Y; see [9, 17]. If U is not affine then also the preimage in an affine cone cannot be affine. This can be seen for example by considering the cohomology of quasi-coherent sheaves coming from graded modules. An effective ample divisor has a positive self intersection number and is connected; see [18, II.6.2]. ∎

Remark. A problem of Hartshorne [18, VI. 3.4; 30; 31] asks the following: Suppose we are given a smooth complete algebraic surface S over $\mathbf{C}$ and an irreducible curve Y intersecting every other curve positively and with self intersection zero. Is then $S - Y$ Stein?

Our theorem states that in this situation $S - Y$ fulfills all geometric conditions which would follow from the Stein property, so at least it is not possible to refute this conjecture by geometrical means. Furthermore, our theorem relates this problem to the hypersurface problem: from the assumptions on $Y \subset S$ in the problem of Hartshorne it follows via 6.1 and 5.1 that the corresponding open subset in an affine cone over S fulfills the assumptions of the hypersurface problem (the conclusion of both problems being the same). The original problem of Hartshorne is still open; Vo Van proves it in [31] in the case where S is a ruled surface.

We will give some examples of curves on smooth surfaces where the situation of the corollary (and of the theorem) occurs.

EXAMPLE 6.1 (see [10, 6.10, 18, V. 5.7.3]). Let K be an algebraically closed field, $Y_0 \subseteq \mathbf{P}^2_K$ be a smooth curve of degree three, hence an elliptic curve, and let $P_1, \dots, P_9$ be nine points on Y_0. Let S be the blown-up surface of these 9 points and let Y be the proper transform of Y_0. The self-intersection is 0. Y intersects all exceptional divisors and the intersection with the other curves is also positive if the points are chosen in such a way that there does not exist a relation between them in the group structure on Y_0.

EXAMPLE 6.2 (see [2, 17, 30] with the corrections due to [27]). This is the classical example of a non-affine but Stein surface. Let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

be a non-split exact sequence of sheaves on an elliptic curve C over $\mathbf{C}$, where $\mathcal{E}$ is locally free of rank two. Let $s : C \longrightarrow S$ be the section in $S = \mathbf{P}(\mathcal{E})$ corresponding to the epimorphism and put $Y = s(C)$ and $U = S - Y$. Then Y fulfills the conditions in 6.2, and it is also Stein, the same being true in the affine cone.

We construct a second class of non-affine, quasiaffine schemes with superheight one. For this, let R be a noetherian normal domain and let M be a reflexive (finitely generated) R-module of rank one, corresponding to a Weil divisor. Let $S(M)$ be the symmetric algebra of M and put $X = \operatorname{Spec} S(M)$ with restriction map $p : X \longrightarrow \operatorname{Spec} R = Y$.

Let $V \subseteq Y$ be an open subset containing the points of codimension one such that M defines an invertible sheaf $\mathcal{L}$ on V. Then $X|_V = p^{-1}(V) \longrightarrow V$ is a line bundle. Its ring of global sections is given by

$$\Gamma(p^{-1}(V), \mathcal{O}_X) = \oplus_{k \geq 0} \Gamma(V, \mathcal{L}^k) = \oplus_{k \geq 0} (M^{\otimes k})^{**}.$$

If $M = \mathbf{p}$ is a prime ideal of height one, this ring equals also $A + \mathbf{p} + \mathbf{p}^{(2)} + \dots$. The zero-section in X defines the closed subscheme $Z = V(S(M)_+)$. Above V the open subset $U = p^{-1}(V) \cap (X - Z)$ is a G_m-fiber bundle; its ring of global sections is given by

$$\Gamma(U, \mathcal{O}_X) = \oplus_{k \in \mathbf{Z}} \Gamma(V, \mathcal{L}^k).$$

A line bundle is trivial if and only if there exists a section without zero and a G_m-fiber bundle is trivial if and only if it has a section.

THEOREM 6.3. *Let R be a noetherian normal domain with a closed point $P \in \operatorname{Spec} R = Y$ of height $d \geq 2$ such that $V = Y - \{P\}$ is locally factorial. Let $\mathcal{L} \in \operatorname{Pic} V \cong \operatorname{Cl} R$ be an non-torsion element in $\operatorname{Cl} R_P$. Let U be the corresponding G_m-fiber bundle over V. Then the cohomological dimension of U is $d-1$ and its finite superheight is $\leq d-1$.*

If P is a closed point on a normal affine surface, then U has finite superheight one, but is not affine.

Proof. For a finitely generated positively graded algebra S over R and a homogeneous ideal $\mathbf{a}$ the cohomological dimension of $D(\mathbf{a})$ and $D_+(\mathbf{a}) \subseteq \operatorname{Proj} S$ is the same. This follows from the fact that any coherent sheaf on $D_+(\mathbf{a})$ comes from a graded module. We may apply this to $U \longrightarrow V$ and therefore $\operatorname{cd} U = \operatorname{cd} V = d-1$.

Let now R' be a normal noetherian domain of dimension d and let $f : Y' = \operatorname{Spec} R' \longrightarrow X = \operatorname{Spec} S$ be morphism of finite type. We have to show that $f^{-1}(U) \neq Y' - \{P'\}$, where P' is a closed point of height d. First observe that $p(f(P')) = P$, for otherwise $p(f(P')) \in W$, where W is an affine neighbourhood with $X|_W$ trivial, and $f(P') \in Z \cap p^{-1}(W)$; but this is not possible since $\operatorname{ht} P' \geq 2$. Therefore $g = p \circ f : Y' \longrightarrow Y$ is a morphism of finite type with $g^{-1}(P) = \{P'\}$, and we have to exclude that $f : Y' - \{P'\} \longrightarrow X|_V$ does not meet Z at all. But such a mapping would yield a zero-free section $f' : Y' - \{P'\} \longrightarrow g^*(X|_V)$ on the pull back of the line bundle $X|_V$ and this would be trivial, but this is not possible as the following lemma shows. ∎

LEMMA 6.4. *Let R and R' be normal excellent domains with maximal ideals $\mathbf{m}$ and $\mathbf{m}'$ of same height $d \geq 2$. Let $R \longrightarrow R'$ be a ring homomorphism of finite type with $V(\mathbf{m}R') = V(\mathbf{m}')$. Then the kernel of $ClR_P \longrightarrow ClR'_{P'}$ consists of torsion elements.*

Proof. We may assume that R and R' are local, and from $V(\mathbf{m}R') = V(\mathbf{m}')$ we see that also $\widehat{R} \longrightarrow \widehat{R'}$ is of finite type. Since we assume excellence, normality is preserved by completion, and $\operatorname{Cl} R \longrightarrow \operatorname{Cl} \widehat{R}$ is injective; see [8, Corollary 6.12]. Thus we may assume that both rings are complete. Since R and R' have the same dimension and the closed fiber is zero-dimensional it follows that $R \longrightarrow R'$ is quasifinite. Due to [14, 6.2.6], it is already finite and the result follows by taking the norm. ∎

EXAMPLE 6.3. To construct examples of the desired type we have to look for affine normal surfaces $Y = \operatorname{Spec} R$ with prime ideals $\mathbf{p}$ of height one which are not torsion at a point $P \in Y$. One can take for instance the homogeneous coordinate ring of a smooth projective curve of genus ≥ 1. If the curve is elliptic, such divisors are given by points which are not torsion in the group structure. Another example is given in [2, 2.10, (3)].

Examples of such prime ideals were first used by Rees to construct examples of non-finitely generated rings of global sections. From the properties established in the theorem it follows by 3.2 that the global ring of U is not finitely generated.

Remark. Take an example as above where R is a finitely generated normal $\mathbf{C}$-Algebra of dimension two. Then $U^{\text{an}} \subseteq X^{\text{an}}$ is an example of a complement of a hypersurface in a Stein space, fulfilling the assumptions in the hypersection problem but not the conclusion. For in that case it follows from superheight one via 5.1 that for every closed analytic surface (=hypersurface) $T \hookrightarrow X^{\text{an}}$ the intersection $T \cap U$ is Stein. However, on a complex manifold V the complement of the zero-section in a line bundle L can only be Stein in case V itself is Stein; see [5, Lemma 3.21]. But here $V = Y - \{P\}$ is not Stein. The example of Coltiou and Diederich can be interpreted in this context as in the context of 6.2 as well.

We will discuss a third class of non-affine schemes with superheight one arising from tight closure in characteristic 0 and related to Example 6.2 in another paper.

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