LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 283 (1998) 61–73

Ergodic decomposition of Markov chains [1]

César E. Villarreal [2]

Departamento de Matemáticas, Cent. de Investigacion y de Estudios, Avanzados del I.P.N., Apartado Postal 14-740, 07000 México D.F., Mexico

Received 14 October 1997; received in revised form 20 April 1998; accepted 23 April 1998

Submitted by R.A. Brualdi

Abstract

We explicitly find the spectral decomposition, when it exists, of a Markov operator $P^*: \ell^1 \to \ell^1$ using the asymptotic periodicity of the associated infinite Markov matrix. We give a simple condition under which an infinite Markov matrix is asymptotically periodic. We also determine the set of P^*-invariant distributions in ℓ^1 and the set of P^*-ergodic distributions.

1. Introduction

Lasota et al. [1] prove a spectral decomposition theorem for a class of Markov operators T, called *strongly constrictive*, acting on an arbitrary space $L^1(X, \Sigma, \mu)$ with a σ-finite measure μ. For these operators all the sequences $(T^n f)$, with $f \in L^1$, are asymptotically periodic. The result by Lasota et al. was extended by Komorník [2] to the case of a *weakly constrictive* Markov operator.

In this paper we give a method to explicitly find the spectral decomposition of a Markov operator $P^*: \ell^1 \to \ell^1$. The method is similar to the one given in [3] for finite Markov matrices, which is based on results by Chi [4].

In Section 2 we state the basic definitions. In Section 3 we prove some results on idempotent infinite Markov matrices, which are needed to explicitly

[1] This work was partially supported by CONACyT grant 3115P-E9608.

[2] E-mail: cevilla@math.cinvestav.mx.

PII: S0024-3795(98)10080-0

find the spectral decomposition given by the Spectral Decomposition Theorem (SDT) of Section 4 for a Markov operator $P^*: \ell^1 \to \ell^1$. We provide a condition for P^* to be constrictive and a method is given to determine the number of orthogonal vectors in the SDT. In Section 5 we characterize the P^*-invariant distributions and the P^*-ergodic distributions.

2. Preliminaries

Definition 1. An *infinite Markov* (or *stochastic*) *matrix* is an infinite matrix $P = (p_{ij})_{i,j=1}^{\infty}$ with nonnegative components p_{ij} such that the sum of the entries of each row is 1, that is, $\sum_{j=1}^{\infty} p_{ij} = 1$ for all $i = 1, 2, \ldots$

Let ℓ^1 be the Banach space of sequences $\mu = (x_1, x_2, \ldots)$ in $\mathbb{R}$, seen as row vectors, such that $\|\mu\|_1 := \sum_{k=1}^{\infty} |x_k| < \infty$. The convex set $D := \{\mu \in \ell^1 : \|\mu\|_1 = 1, \text{ and } \mu \geqslant 0\}$ is referred to as the set of *distributions* in ℓ^1.

Definition 2. A positive linear operator $P^*: \ell^1 \to \ell^1$ is called a *Markov operator* on ℓ^1 if it maps D into itself, that is,

$$P^*(D) \subset D.$$

In particular, an infinite Markov matrix P defines a Markov operator P^* on ℓ^1 as

$$P^*(\mu) := \mu P, \tag{1}$$

so that the jth component of $P^*(\mu)$ is $(\mu P)_j = \sum_{k=1}^{\infty} x_k p_{kj}$.

Also note that P^* is a contraction map, that is,

$$\|P^*(\mu)\|_1 \leqslant \|\mu\|_1 \quad \forall \mu \in \ell^1 \tag{2}$$

and, moreover, P^* preserves the norm of $\mu \in \ell^1$ if μ is nonnegative, i.e.,

$$\|P^*(\mu)\|_1 = \|\mu\|_1 \quad \forall \mu \in \ell^{1+}. \tag{3}$$

Throughout the following, $P = (p_{ij})_{i,j=1}^{\infty}$ denotes a given (infinite) Markov matrix, and P^* stands for the corresponding Markov operator defined by Eq. (1). Further, $\mathbb{N}$ denotes the set of positive integers, and if B is a subset of $\mathbb{N}$, we define

$$P(i, B) := \sum_{j \in B} p_{ij} \quad \text{for } i \in \mathbb{N}.$$

As in Markov chains theory we interpret $P(i, B)$ as the "probability" of going from i to the set B in one time unit.

We will identify a sequence $\mu = (x_i)_{i=1}^{\infty} \in \ell^1$ with the finite signed measure (also denoted by μ) on the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$ such that $\mu(B) := \sum_{i \in B} x_i$ for $B \subset \mathbb{N}$.

Let us recall some definitions (see, for instance, [5]).

Definition 3. (a) A set $B \subset \mathbb{N}$ is called *P-invariant* if $P(i, B) = 1$ for all $i \in B$.
(b) A distribution $\mu \in D$ is called P^*-invariant if $P^*(\mu) = \mu$.
(c) A distribution $\mu \in D$ is said to be *P^*-ergodic* if it is P^*-invariant and $\mu(B) = 0$ or 1 for any P-invariant set B.

In Section 5 we give conditions to identify the set D^I_∞ of P^*-invariant distributions and the subset $D^E_\infty \subset D^I_\infty$ of P^*-ergodic distributions.

Definition 4. Two sequences $\mu = (x_i)_{i=1}^\infty$ and $\nu = (y_i)_{i=1}^\infty$ in ℓ^1 are said to be *orthogonal* if $\mu \cdot \nu := \sum_{i=1}^\infty x_i y_i = 0$. In this case, we write $\mu \perp \nu$.

3. Idempotent infinite Markov matrices

The main result in this section is Theorem 10, which requires some properties of idempotent infinite Markov matrices. These properties, stated in the following lemmas, are also used in the next section. (Recall that a matrix A is said to be *idempotent* if $A^2 = A$.)

Lemma 5. *Let $A = (a_{ij})_{i,j=1}^\infty$ be an idempotent infinite Markov matrix.*
(a) *If $a_{kk} = 0$ for some $k \in \mathbb{N}$, then $a_{ik} = 0$ for all $i \in \mathbb{N}$.*
(b) *For all $k, i \in \mathbb{N}$, we have $a_{ik} \leqslant a_{kk}$.*

Proof. (a) Arguing by contradiction, suppose that $a_{kk} = 0$ and $a_{ik} > 0$ for some $i \in \mathbb{N}$. Let $\bar{a} = \sup\{a_{ik}: i \in \mathbb{N}\}$. For each $\varepsilon > 0$ we take $i_\varepsilon \in \mathbb{N}$ such that $\bar{a} - a_{i_\varepsilon k} < \varepsilon$ and $a_{i_\varepsilon k} > 0$. Then by the idempotency hypothesis,

$$a_{i_\varepsilon k} = \sum_{j=1}^\infty a_{i_\varepsilon j} a_{jk} = \sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j} a_{jk} \leqslant (a_{i_\varepsilon k} + \varepsilon) \sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j}$$
$$\leqslant a_{i_\varepsilon k} \left(\sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j} + \frac{\varepsilon}{a_{i_\varepsilon k}} \right)$$

and, therefore,

$$\sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j} \geqslant 1 - \frac{\varepsilon}{a_{i_\varepsilon k}}. \tag{4}$$

Now, as $a_{kk} = 0$, we have that $k \notin \{j: a_{jk} > 0\}$. On the other hand, $a_{i_\varepsilon k} > 0$ yields

$$1 \geqslant a_{i_\varepsilon k} + \sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j} > \sum_{\{j: a_{jk} > 0\}} a_{i_\varepsilon j}. \tag{5}$$

Let us now consider a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$ and, furthermore, the limit $\lim_{n\to\infty}\sum_{\{j:a_{jk}>0\}} a_{i_{\varepsilon_n}j} =: Q$ exists. Then, from Eqs. (4) and (5),

$$1 \geqslant a_{i_{\varepsilon_n}k} + \sum_{\{j:\,a_{jk}>0\}} a_{i_{\varepsilon_n}j} > \sum_{\{j:\,a_{jk}>0\}} a_{i_{\varepsilon_n}j} \geqslant 1 - \frac{\varepsilon_n}{a_{i_{\varepsilon_n}k}}. \tag{6}$$

Hence, since $\bar{a} - \varepsilon_n < a_{i_{\varepsilon_n}k} \leqslant \bar{a}$, letting $n \to \infty$ in Eq. (6) we obtain

$$1 \geqslant \bar{a} + Q \geqslant Q \geqslant 1,$$

which is impossible because $\bar{a} > 0$. This contradiction yields that we must have $a_{ik} = 0$ for all i.

(b) If $a_{ik} = 0$, then $a_{ik} \leqslant a_{kk}$. Now, if a_{ik} is positive, then, by (a), so is a_{kk}. Let $\bar{a}$ be as in the previous proof, that is, $\bar{a} := \sup\{a_{ik}: i \in \mathbb{N}\}$, and for $0 < \varepsilon \leqslant \bar{a}$ let $i_\varepsilon \in \mathbb{N}$ be such that $\bar{a} - a_{i_\varepsilon k} < \varepsilon$. As A is idempotent, we have

$$a_{i_\varepsilon k} = \sum_{j=1}^{\infty} a_{i_\varepsilon j}a_{jk} \leqslant \sum_{j=1}^{\infty} a_{i_\varepsilon j}(a_{i_\varepsilon k} + \varepsilon) = a_{i_\varepsilon k} + \varepsilon. \tag{7}$$

Now, in Eq. (7), the jth term of the first sum is less than or equal to the jth term of the second and, further, the difference between these terms is at most ε. Thus if $a_{i_\varepsilon j} > 0$, then $a_{i_\varepsilon j}a_{jk} + \varepsilon > a_{i_\varepsilon j}a_{i_\varepsilon k}$, so that $a_{jk} + \varepsilon/a_{i_\varepsilon j} > a_{i_\varepsilon k}$. In particular,

$$a_{kk} + \frac{\varepsilon}{a_{i_\varepsilon k}} > a_{i_\varepsilon k} > \bar{a} - \varepsilon. \tag{8}$$

Finally, in Eq. (8) we take the limit as $\varepsilon \to 0$ and we get $a_{kk} \geqslant \bar{a}$, so that, by definition of $\bar{a}$, we have $a_{ik} \leqslant a_{kk}$ for all $k, i \in \mathbb{N}$. □

Lemma 6. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an idempotent infinite Markov matrix. If $a_{kk} > 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_{kk} < \infty$, then:*

(a) $a_{ji} > 0 \Rightarrow a_{ij} > 0$ (*or, equivalently*, $a_{ij} = 0 \Rightarrow a_{ji} = 0$).

(b) $a_{ik} > 0 \Rightarrow a_{ik} = a_{kk}$.

Proof. (a) By Lemma 5(b),

$$\sum_{i=1}^{\infty} a_{ii} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}a_{ji} \leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{jj}a_{ji} = \sum_{j=1}^{\infty} a_{jj}\sum_{i=1}^{\infty} a_{ji} = \sum_{j=1}^{\infty} a_{jj} = \sum_{i=1}^{\infty} a_{ii}.$$

Thus, as $\sum_{i=1}^{\infty} a_{ii} < \infty$, we obtain $a_{ij}a_{ji} = a_{jj}a_{ji}$. Therefore, $a_{ji} > 0$ implies $a_{ij} = a_{jj}$. Hence, as $a_{jj} > 0$ (by hypothesis), we conclude that $a_{ji} > 0$ implies $a_{ij} > 0$.

(b) By Lemma 5(b),

$$a_{kk} = \sum_{i=1}^{\infty} a_{ki}a_{ik} \leqslant \sum_{i=1}^{\infty} a_{ki}a_{kk} = a_{kk}.$$

Therefore, $a_{ki} > 0$ implies $a_{ik} = a_{kk}$, so that, by (a), $a_{ik} > 0$ implies $a_{ik} = a_{kk}$. □

Remark 7. If $A = (a_{ij})_{i,j=1}^{\infty}$ is an idempotent infinite Markov matrix such that all the entries of the columns $k_1, k_2, \ldots$ are zero, then the matrix obtained by cancelling all the rows $k_1, k_2, \ldots$ and all the columns $k_1, k_2, \ldots$ is also an (infinite or possibly finite) idempotent Markov matrix.

Remark 8. Two sequences $\mu = (x_i)_{i=1}^{\infty}$ and $\nu = (y_i)_{i=1}^{\infty}$ in D are orthogonal if and only if their supports are disjoint, that is, $x_i > 0$ implies $y_i = 0$, and $y_i > 0$ implies $x_i = 0$.

We will denote by $a_{i\cdot}$ the ith row of A, that is, $a_{i\cdot} = (a_{ij})_{j=1}^{\infty}$.

Lemma 9. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an idempotent infinite Markov matrix with $a_{ii} > 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} a_{ii} < \infty$. Then any two rows of A are either equal or orthogonal; that is, for all $i, k \in \mathbb{N}$ we have $a_{i\cdot} = a_{k\cdot}$ or $a_{i\cdot} \perp a_{k\cdot}$.*

Proof. Suppose that the rows $a_{i\cdot}$ and $a_{k\cdot}$ of A are neither equal nor orthogonal. Let: $B = \{j_1, j_2, \ldots\}$ be the set of all indices such that $a_{ij_p} = a_{kj_p} > 0$ for all $j_p \in B$; $B^* = \{j_1^*, j_2^*, \ldots\}$ be the set of all indices such that $a_{ij_p^*} > 0 = a_{kj_p^*}$ for all $j_p^* \in B^*$; $B^{**} = \{j_1^{**}, j_2^{**}, \ldots\}$ be the set of all indices such that $a_{ij_p^{**}} = 0 < a_{kj_p^{**}}$ for all $j_p^{**} \in B^{**}$; and $\tilde{B} = \{\tilde{j}_1, \tilde{j}_2, \ldots\}$ be the set of all indices such that $a_{i\tilde{j}_p} = a_{k\tilde{j}_p} = 0$ for all $\tilde{j}_p \in \tilde{B}$.

As A is a Markov matrix and the vectors $a_{i\cdot}$ and $a_{k\cdot}$ are neither equal nor orthogonal, from Lemma 6(b) and Remark 8 we have $B \neq \emptyset$, $B^* \neq \emptyset$ and $B^{**} \neq \emptyset$. By definition,

$$0 = a_{ij_q^{**}} = \sum_{p=1}^{\infty} a_{ip} a_{pj_q^{**}} = \sum_{j_p \in B} a_{ij_p} a_{j_p j_q^{**}}.$$

Thus

$$a_{j_p j_q^{**}} = 0 \quad \text{for all } j_p \in B \text{ and for all } j_q^{**} \in B^{**}. \tag{9}$$

Now, if $a_{rs} > 0$, by Lemma 6(b) we have

$$a_{rs} = \sum_{p=1}^{\infty} a_{rp} a_{ps} = a_{rs} \sum_{\{p: a_{ps} \neq 0\}} a_{rp}.$$

Therefore, $\sum_{\{p: a_{ps} \neq 0\}} a_{rp} = 1$ and, as A is a Markov matrix,

$$(a_{rs} > 0 \text{ and } a_{ps} = 0) \;\Rightarrow\; a_{rp} = 0. \tag{10}$$

By Eq. (10), and because $a_{ij_1^*} > 0 = a_{kj_1^*}$, we have $a_{ik} = 0$; that is, k is in the union $B^{**} \cup \tilde{B}$. We will next show that this leads to a contradiction. Indeed, suppose that $k \in \tilde{B}$. In this case, $a_{kk} = 0$, which contradicts our hypothesis; therefore, $k \in B^{**}$. But, from Eq. (9), $a_{j_1 k} = 0$; thus, by Lemma 6(a), $a_{kj_1} = 0$, which contradicts that $j_1 \in B$. Therefore, the rows $a_i.$ and $a_k.$ are either equal or orthogonal. □

Theorem 10. *Let $A = (a_{ij})_{i,j=1}^{\infty}$ be an idempotent infinite Markov matrix such that $\sum_{k=1}^{\infty} a_{kk} < \infty$. Then:*

(a) *Any two rows with positive diagonal entries are either equal or orthogonal; that is, for all $i, k \in \mathbb{N}$ such that $a_{ii} > 0$ and $a_{kk} > 0$ we have $a_i. = a_k.$ or $a_i. \perp a_k.$.*

(b) *Each row of A is a convex combination of the rows which have a positive diagonal entry.*

Proof. (a) This part follows from Lemma 9 and Remark 7. (b) Let $a_k.$ be the kth row of A for a fixed positive integer k. To prove (b) we wish to write $a_k.$ as a convex combination

$$a_k. = \alpha(m_1, k)a_{m_1}. + \alpha(m_2, k)a_{m_2}. + \cdots,$$

where $a_{m_1}., a_{m_2}., \ldots$ are rows with positive diagonal entry. We will first show that if $a_m.$ is a row such that a_{mj}, a_{ml} and a_{mm} are all positive; then:

(i) The rows $a_j., a_l.$ and $a_m.$ are equal.

(ii) If $a_{ij} = a_{jj}$ for some i, then $a_{il} > 0$; and similarly, if $a_{il} = a_{ll}$ for some i, then $a_{ij} > 0$.

(iii)

$$\sum_{\{i:\, a_{ij} = a_{jj}\}} a_{ki} = \sum_{\{i:\, a_{il} = a_{ll}\}} a_{ki} =: \alpha(m, k).$$

(iv)

$$a_{kj} = \alpha(m, k)a_{mj} \quad \text{and} \quad a_{kl} = \alpha(m, k)a_{ml}.$$

Proof of (i). By Lemma 5 and part (a), the rows $a_j., a_l.$ and $a_m.$ are equal.

Proof of (ii). It is impossible to have $a_{ij} = a_{jj}$ and $a_{il} = 0$ because this would imply, by idempotency, that $a_{ij}a_{jl} = 0$ with $a_{ij} > 0$. Therefore $a_{jl} = 0$. However, by Theorem 5(b), $a_{ll}, a_{jj} > 0$, so, by (a), the rows $a_l.$ and $a_j.$ would be orthogonal, which contradicts that they are equal. Hence $a_{ij} = a_{jj}$ for some i implies $a_{il} > 0$. Similarly, $a_{il} = a_{ll}$ for some i implies $a_{ij} > 0$.

Proof of (iii). By (a) and Lemma 5(a), $(0 < a_{il} < a_{ll}$ or $0 < a_{ij} < a_{jj}) \Rightarrow a_{ii} = 0 \Rightarrow a_{ki} = 0$. From this fact and (ii) we have that if $a_{ij} = a_{jj}$ and $a_{il} < a_{ll}$, or $a_{ij} < a_{jj}$ and $a_{il} = a_{ll}$; then $a_{ki} = 0$. Hence $\{i\colon a_{ij} = a_{jj}$ and $a_{ki} > 0\} = \{i\colon a_{il} = a_{ll}$ and $a_{ki} > 0\}$, and, therefore,

$$\sum_{\{i:\, a_{ij} = a_{jj}\}} a_{ki} = \sum_{\{i:\, a_{il} = a_{ll}\}} a_{ki},$$

which proves (iii).

Proof of (iv). In the proof of (iii) we have that $0 < a_{ij} < a_{jj} \Rightarrow a_{ki} = 0$. Thus, by (i),

$$a_{kj} = \sum_{i=1}^{\infty} a_{ki}a_{ij} = \sum_{\{i:a_{ij}=a_{jj}\}} a_{ki}a_{ij} = a_{jj} \sum_{\{i:a_{ij}=a_{jj}\}} a_{ki}$$
$$= a_{mj} \sum_{\{i:a_{ij}=a_{jj}\}} a_{ki} = \alpha(m,k)a_{mj},$$

and similarly

$$a_{kl} = \alpha(m,k)a_{ml},$$

which proves (iv).

Note that if two rows $a_{m\cdot}$, and $a_{n\cdot}$ with positive diagonal entry are equal, then the coefficients $\alpha(m,k)$ and $\alpha(n,k)$ are equal.

Now, having (i) to (iv), we can easily complete the proof of part (b). Let us define $m_1, m_2, \ldots \in \mathbb{N}$ recursively as follows: the integer m_1 is such that $a_{m_1\cdot}$ is the first row of A with positive diagonal entry. Next, m_2 is the smallest integer greater than m_1 such that the row $a_{m_2\cdot}$ has a positive diagonal entry and $a_{m_2\cdot} \neq a_{m_1\cdot}$. Continuing this process we obtain a (possibly finite) sequence $(m_1, m_2, \ldots)$ in which, given m_{q-1}, we choose m_q as the smallest integer greater than m_{q-1} such that the row $a_{m_q\cdot}$ has positive diagonal entry and $a_{m_q\cdot} \neq a_{m_i\cdot}$ for all $i = 1, 2, \ldots, q-1$. Let $r = \#\{m_1, m_2, \ldots\}$ be the number of elements in the set $\{m_1, m_2, \ldots\}$, so that $r \in \mathbb{N} \cup \{\infty\}$. (In Theorem 19, we give an explicit value for r.) By construction, for each row $a_{i\cdot}$ such that $a_{ii} > 0$ there exists a unique $m_q \in \{m_1, m_2, \ldots\}$ such that $a_{i\cdot} = a_{m_q\cdot}$. If $a_{kk} = 0$, then from (iv), the definition of $m_1, m_2, \ldots$, part (a) and Remark 8, we conclude that

$$a_{k\cdot} = \sum_{q=1}^{r} \alpha(m_q, k)a_{m_q\cdot}.$$

Thus, as A is a Markov matrix, we have $\sum_{q=1}^{r} \alpha(m_q, k) = 1$ with $\alpha(m_q, k) \geqslant 0$ and the proof of the theorem is complete. □

4. The spectral decomposition theorem

In this section we first state without proof a particular case of the SDT given in [2,6,7,1,8], and then we describe a procedure to determine the different components that appear in the spectral decomposition of a Markov operator.

The main assumption in the SDT is the constrictivity of the Markov operator $P^*: \ell^1 \to \ell^1$, which is defined as follows.

Definition 11. We say that $P^*: \ell^1 \to \ell^1$ is *constrictive* if there exists a compact set $F \subset \ell^1$ such that

$$\limsup_{n\to\infty} d(P^{*n}(\mu), F) = 0 \quad \text{for all } \mu \in D,$$

where $d(\nu, F) := \inf\{\|\nu - \rho\|_1 : \rho \in F\}$.

In Theorem 17 we give a simple condition under which a Markov operator is constrictive. Moreover, Theorem 18 characterizes the set $F_0 = \{\nu_1, \ldots, \nu_r\}$ given in the SDT and, finally, Theorem 19 gives a formula to find the number of elements in F_0.

Definition 12. We say that $\mu \in D$ is *P^*-periodic* if there is a positive integer n such that $P^{*n}(\mu) = \mu$.

Theorem 13 (SDT). *Let P^* be a constrictive Markov operator on ℓ^1. Then:*
(a) *There exists*
- *a finite set $F_0 = \{\nu_1, \ldots, \nu_r\}$ of pairwise orthogonal P^*-periodic elements of D,*
- *a set of continuous linear functionals $\lambda_1, \ldots, \lambda_r$ on ℓ^1, and*
- *a permutation σ of the integers $1, \ldots, r$ such that*

(I) $\lim_{n\to\infty} \|P^{*n}(\mu) - \sum_{i=1}^r \lambda_i(\mu)\nu_{\sigma^n(i)}\|_1 = 0$ *for each* $\mu \in \ell^1$.
(II) $P^*(\nu_i) = \nu_{\sigma(i)}$ *for* $i = 1, \ldots, r$.
(b) *The functionals λ_i are positive, that is, $\lambda_i(\mu) \geqslant 0$ if $\mu \geqslant 0$. Moreover,*

$$\sum_{i=1}^r \lambda_i(\nu) = 1 \quad \textit{for } \nu \in D,$$

(III) $|\lambda_i(\mu)| \leqslant \|\mu\|_1$ *for* $\mu \in \ell^1$.
(c) *The sets $\{\nu_1, \ldots, \nu_r\}$ and $\{\lambda_1, \ldots, \lambda_r\}$ satisfying* (I) *and* (II) *are unique.*

Remark 14. If $P = (p_{ij})_{i,j=1}^\infty$ is an infinite Markov matrix, we define $\hat{p}_j := \sup\{p_{ij}: i \in \mathbb{N}\}$ and $\hat{p} := (\hat{p}_j)_{j=1}^\infty$. Further, recall that $P^*: \ell^1 \to \ell^1$ stands for the Markov operator defined by P, i.e., $P^*(\mu) = \mu P$.

In the remainder of this section we suppose the following:

Assumption 15. P is such that $\|\hat{p}\|_1 < \infty$.

Remark 16. If $\hat{p}$ and the rows of P are seen as σ-finite measures on the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$, then Assumption 15 gives that $\hat{p}$ is *tight* (cf. [2]). Moreover, from Theorem 3.2 in [9] it follows that the family of rows of P is also tight.

Theorem 17. *Suppose that Assumption* 15 *is satisfied. Then the matrix P defines a constrictive Markov operator P^* on ℓ^1.*

Proof. By hypothesis, $\hat{p}$ is in ℓ^1. If $\mu = (x_1, x_2, \dots)$ is in D, then $P^*(\mu) \leqslant \hat{p}$ and, therefore,

$$\{P^*(\mu)\colon \mu \in D\} \subset K := \{\nu \in D\colon \nu \leqslant \hat{p}\}.$$

Note that $P^{*n}(\mu)$ is in K for all $n \in \mathbb{N}$. We will next prove that K is compact. In fact, since ℓ^1 is a metric space, it suffices to show that K is *sequentially compact* [10], Theorem 7.4. To prove this, let $(\nu_n)_{n=1}^{\infty}$ be a sequence in K, and write $\nu_n = (y_{n1}, y_{n2}, \dots)$. By definition of K, we have $0 \leqslant y_{nj} \leqslant \hat{p}_j$ for all n. Let us now recursively define the increasing sequences of positive integers $(n_k^1)_{k=1}^{\infty}$, $(n_k^2)_{k=1}^{\infty}, \dots$ as follows: the increasing sequence $(n_k^1)_{k=1}^{\infty}$ is such that $(y_{n_k^1 1})_{k=1}^{\infty}$ converges to a nonnegative number y_1; for $j > 1$, $(n_k^j)_{k=1}^{\infty}$ is a subsequence of $(n_k^{j-1})_{k=1}^{\infty}$ such that $(y_{n_k^j j})_{k=1}^{\infty}$ converges to a nonnegative number y_j. Let $\nu = (y_1, y_2, \dots)$, and observe that $\nu \in K$ and $(y_{n_k^k j})_{k=1}^{\infty}$ converges to y_j for each j. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\sum_{j=N+1}^{\infty} 2\hat{p}_j < \varepsilon$. Now

$$\begin{aligned}
\lim_{k\to\infty} \|\nu_{n_k^k} - \nu\|_1 &= \lim_{k\to\infty} \sum_{j=1}^{\infty} |y_{n_k^k j} - y_j| \\
&= \lim_{k\to\infty} \left(\sum_{j=1}^{N} |y_{n_k^k j} - y_j| + \sum_{j=N+1}^{\infty} |y_{n_k^k j} - y_j| \right) \\
&\leqslant \lim_{k\to\infty} \left(\sum_{j=1}^{N} |y_{n_k^k j} - y_j| + \varepsilon \right) = \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we have $\lim_{k\to\infty} \|\nu_{n_k^k} - \nu\|_1 = 0$; in other words, the sequence $(\nu_n)_{n=1}^{\infty}$ has a subsequence $(\nu_{n_k^k})_{k=1}^{\infty}$ convergent in ℓ^1 to some $\nu \in K$. Therefore, K is sequentially compact, which proves that K is compact in ℓ^1. Hence P^* is a constrictive operator on ℓ^1. □

Theorem 18. *Suppose that Assumption* 15 *holds, and let r be the positive integer given in the SDT. Then there exists an infinite idempotent Markov matrix A such that $(P^{*(r!n)}(\mu))_{n=1}^{\infty}$ converges to $A^*(\mu)$ in ℓ^1 for all $\mu \in \ell^1$. Moreover, the elements $\nu_1, \dots, \nu_r$ given in the SDT are the rows of A which have a positive diagonal entry.*

Proof. By Theorem 17, there exists a finite set $F_0 = \{\nu_1, \dots, \nu_r\}$ of pairwise orthogonal periodic elements of D and a set of continuous linear functionals $\{\lambda_1, \dots, \lambda_r\}$ on ℓ^1 that satisfy the condition of the SDT; in particular,

$$\lim_{n\to\infty} \|P^{*n}(\mu) - \sum_{k=1}^{r} \lambda_k(\mu) \nu_{\sigma^n(k)}\|_1 = 0 \quad \text{for each } \mu \in \ell^1.$$

In addition, since $\sigma^{r!n}(k) = k$, we have

$$\lim_{n\to\infty} \|P^{*(r!n)}(\delta_i.) - \sum_{k=1}^{r} \lambda_k(\delta_i.)v_k\|_1 = 0,$$

where

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let y_{ki} be the ith entry of v_k and let $A = (a_{ij})_{i,j=1}^{\infty}$ be the infinite matrix with rows

$$a_i. = \lim_{n\to\infty} \delta_i.P^{r!n} = \lim_{n\to\infty} P^{*(r!n)}(\delta_i.) = \sum_{k=1}^{r} \lambda_k(\delta_i.)v_k, \tag{11}$$

where, by the SDT, $\sum_{k=1}^{r} \lambda_k(\delta_i.) = 1$ and $0 \leqslant \lambda_k(\delta_i.) \leqslant 1$. Moreover, note that

$$v_k = P^{*(r!n)}(v_k) = \sum_{i=1}^{\infty} y_{ki} \lim_{m\to\infty} P^{*(r!m)}(\delta_i.) = \sum_{i=1}^{\infty} y_{ki} a_i. \;. \tag{12}$$

Observe that A is idempotent and Markov, and $\sum_{k=1}^{\infty} a_{kk} \leqslant \sum_{k=1}^{\infty} \hat{p}_k < \infty$. Therefore, by Theorem 10(b) and (12), the elements $v_1, \ldots, v_r$ are convex combinations of the rows of A which have a positive diagonal entry. Hence, by Eq. (11) and part (c) of the SDT, $v_1, \ldots, v_r$ are the rows of A with positive diagonal entry. □

An argument given in Ref. [7], p. 753, Eq. (1.5) shows that

$$0 < r \leqslant \|\hat{p}\|_1,$$

which gives an *estimate* of the number r of distributions $v_1, \ldots, v_r$ in the SDT. This is important because, even if we do not know r, it allows us to calculate $A^*(\mu)$ as the limit

$$\lim_{n\to\infty} \|P^{*(s!n)}(\mu) - A^*(\mu)\|_1 = 0 \quad \text{for any integer } s \geqslant r,$$

in particular for $s \geqslant \|\hat{p}\|_1$. The following theorem gives the *precise value* of r.

Theorem 19. *Suppose that Assumption 15 holds and let the positive integer r and $A = (a_{ij})_{i,j=1}^{\infty}$ be as in Theorem 18. Then r is given by*

$$r = \sum_{k=1}^{\infty} a_{kk}.$$

Proof. By Theorem 18, $v_1, \ldots, v_r$ are the rows of A that have a positive diagonal entry, and these vectors are pairwise orthogonal, by Theorem 10(a). Let

$$T_n := \{i \in \mathbb{N}: a_{i\cdot} = v_n\} \quad \text{for } 1 \leqslant n \leqslant r.$$

Then, from Theorem 10 and Lemma 5,

$$\sum_{k=1}^{\infty} a_{kk} = \sum_{n=1}^{r} \sum_{\{k \in T_n\}} a_{kk} = \sum_{n=1}^{r} \|v_n\|_1 = \sum_{k=1}^{r} 1 = r. \qquad \square$$

5. P^*-invariant distributions

Let $P^*: \ell^1 \to \ell^1$ be a constrictive Markov operator, and let $D_\infty := \bigcap_{n=1}^{\infty} P^{*n}(D)$ be the set of all the limit points of the sequences $(P^{*n}(\mu))_{n=1}^{\infty}$ with $\mu \in D$. By the SDT, v is in D_∞ if and only if it is a convex combination of the distributions $v_1, \dots, v_r$. That is, D_∞ is the convex hull of $\{v_1, \dots, v_r\}$. We will now identify the set $D_\infty^I \subset D_\infty$ of P^*-*invariant* distributions and the set D_∞^E that consists of all the P^*-*ergodic* distributions (see Definiton 3).

Two integers i and j in $\{1, \dots, r\}$ are said to be *equivalent* (denoted by $i \leftrightarrow j$) if $P^{*k}(v_i) = v_j$ for some positive integer k. Observe that $\leftrightarrow$ is an equivalence relation, and denote by $O_1, O_2, \dots, O_d$ the different equivalence classes of $\{1, \dots, r\}$. Let $\bar{O}_l := \{v_i: i \in O_l\}$. For $j = 1, \dots, d$, let

$$\tau_j := \frac{1}{\#O_j} \sum_{i \in O_j} v_i \tag{13}$$

be the "average" of the elements in $\bar{O}_j$. Observe that $P^*: \bar{O}_j \to \bar{O}_j$ is bijective and that $P^*(v_i) \in \bar{O}_j \iff v_i \in \bar{O}_j$. Therefore,

$$\sum_{i \in O_j} v_i = \sum_{v \in \bar{O}_j} v = \sum_{v \in \bar{O}_j} P^*(v) = \sum_{i \in O_j} P^*(v_i) = P^*\left(\sum_{i \in O_j} v_i\right),$$

which gives that τ_j is a P^*-invariant distribution. Note that $\tau_1, \dots, \tau_d$ are mutually orthogonal. The proof of the following theorem is similar to the proof in Ref. [3], Theorem 10.

Theorem 20 (Ergodic Decomposition Theorem). *Let $P^*: \ell^1 \to \ell^1$ be a constrictive Markov operator defined by an infinite Markov matrix P and let $D_\infty^I \subset D_\infty$ be the set of all the P^*-invariant distributions. Then D_∞^I is a convex set and, in fact, it is the convex hull of $\{\tau_1, \dots, \tau_d\}$ with τ_j as in Eq.* (13), *that is,*

$$D_\infty^I = \{\mu \in \ell^1: \mu = \sum_{j=1}^{d} \alpha_j \tau_j \text{ with } \alpha_j \geqslant 0 \text{ and } \sum_{j=1}^{d} \alpha_j = 1\}. \tag{14}$$

Hence, the collection of all the P^*-ergodic distributions is $D_\infty^E = \{\tau_1, \dots, \tau_d\}$.

Proof. Let C be the convex hull of $\{\tau_1, \ldots, \tau_d\}$. Since τ_j is a P^*-invariant distribution for $j = 1, 2, \ldots, d$, any convex combination

$$\mu = \sum_{j=1}^{d} \alpha_j \tau_j$$

is also a P^*-invariant distribution. Hence $C \subset D^I_\infty$. To prove that $D^I_\infty \subset C$, first note that $D^I_\infty \subset D_\infty$ so that if $\mu \in D$ is a P^*-invariant distribution, then, by the SDT, μ is a convex combination of $v_1, \ldots, v_r$, that is,

$$\mu = \sum_{m=1}^{r} \beta_m v_m. \tag{15}$$

Also note that Eq. (15) is the unique representation of μ as a linear combination of $v_1, v_2, \ldots, v_r$, because these distributions are mutually orthogonal. Now, if i, k are both in O_j, then the coefficients β_i and β_k are equal. Indeed, if $i, k \in O_j$, then there is a nonnegative integer t, such that $v_k = P^{*t}(v_i)$, and so

$$\mu = P^{*t}(\mu) = \sum_{m=1}^{r} \beta_m P^{*t}(v_m) = \sum_{m=1}^{i-1} \beta_m P^{*t}(v_m) + \beta_i v_k + \sum_{m=i+1}^{r} \beta_m P^{*t}(v_m).$$

Hence, as the representation in Eq. (15) is unique, we most have $\beta_i = \beta_k$.

Now, for $j = 1, 2, \ldots, d$, let s_j be an integer in O_j. Then

$$\begin{aligned}\mu &= \sum_{m=1}^{r} \beta_m v_m = \sum_{i \in O_1} \beta_{s_1} v_i + \sum_{i \in O_2} \beta_{s_2} v_i + \cdots + \sum_{i \in O_d} \beta_{s_d} v_i \\ &= \beta_{s_1}(\#O_1)\tau_1 + \beta_{s_2}(\#O_2)\tau_2 + \cdots + \beta_{s_d}(\#O_d)\tau_d.\end{aligned}$$

Finally, if for $j = 1, 2, \ldots, d$ we take $\alpha_j = \beta_{s_j}(\#O_j)$, then we get

$$\mu = \sum_{j=1}^{d} \alpha_j \tau_j,$$

and, moreover, $\alpha_j \geqslant 0$ and $\sum_{j=1}^{d} \alpha_j = \sum_{j=1}^{d} \beta_{s_j}(\#O_j) = \sum_{j=1}^{d} \sum_{i \in O_j} \beta_i = \sum_{m=1}^{r} \beta_m = 1$. Therefore, $D^I_\infty \subset C$, which completes the proof of Eq. (14). This in turn yields the last statement in the theorem, $D^E_\infty = \{\tau_1, \ldots, \tau_d\}$; see for instance, Kifer [5], Theorem 1.1 in Appendix A.1. □

Acknowledgements

The author is grateful to Professor Onésimo Hernández-Lerma for his useful comments and invaluable suggestions concerning this paper.

References

[1] A. Lasota, T.Y. Li, J.A. Yorke, Asymptotic periodicity of the iterates of Markov operators, Trans. Amer. Math. Soc. 286 (2) (1984) 751–764.

[2] J. Komorník, Asymptotic periodicity of the iterates of weakly constrictive Markov operators, Tôhoku Math. J. 38 (1986) 15–27.

[3] C.E. Villarreal, Asymptotic periodicity of Markov matrices, Bol. Soc. Mat. Mex. 4(1)(1998) 147–157.

[4] M.-H. Chi, The long-run behavior of Markov chains, Linear Algebra Appl. 244 (1996) 111–121.

[5] Y. Kifer, Ergodic Theory of Random Transformation, Birkhäuser, Boston, 1986.

[6] J. Komorník, Asymptotic periodicity of Markov and related operators, in: C.K.R.T. Jones et al. (Eds.), Dinamics Reported, vol. 2, Springer, Berlin, 1993, pp. 31–68.

[7] J. Komorník, E.G.F. Thomas, Asymptotic periodicity of Markov operators on signed measures, Math. Bohem. 116 (2) (1991) 174–180.

[8] A. Lasota, M.C. Mackey, Chaos, Fractals, and Noise, 2nd ed., Springer, New York, 1994.

[9] J. González-Hernández, O. Hernández-Lerma, Envelopes of sets of measures, tightness, and Markov control processes, Appl. Math. Optim. (to appear).

[10] J.R. Munkres, Topology, a First Course, Prentice-Hall, Englewood Cliffs, NJ, 1975.