The spectral characterization of $\infty$-graphs

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ABSTRACT

A $\infty$-graph $B(r, s)$ is a graph consisting of two cycles $C_{r+1}$ and $C_{s+1}$ with just a vertex in common. Fan and Luo in Theorem 4.4 [8] alleged that a $\infty$-graph is determined by its adjacency spectrum if it contains no cycle $C_4$. However, according to Theorem 6.3 [16] we find that the result in Theorem 4.4 [8] is not completely correct. In this paper, we prove that $B(r, s)$ ($s \geq r > 7$) is DAS if and only if $s \neq r + 2$, and $B(r, s)$ ($s \geq r > 7$) has a unique cospectral mate if $s = r + 2$.

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1. Introduction

The graphs considered in this paper are simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Let $A(G)$ be the $(0, 1)$-adjacency matrix of $G$, the polynomial $P_G(\lambda) = \det(\lambda I - A(G))$ is the characteristic polynomial of $G$. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers, which will be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and be called as the (adjacency) eigenvalues of $G$. The eigenvalues of $G$ together with their multiplicities is called the adjacency spectrum of $G$. Two graphs $G$ and $H$ are said to be cospectral if they share the same spectrum (i.e., equal characteristic polynomial). A graph $G$ is said to be determined by its adjacency spectrum (DAS for short) if for any graph $H$, $P_G(\lambda) = P_H(\lambda)$ implies that $H$ is isomorphic to $G$. Up to now, numerous examples of cospectral but non-isomorphic graphs are reported. But, only few graphs with very special structures have been proved to be determined by their adjacency spectrum (see [5,7,9,11,13–15,17,18] for references), such as the path $P_n$ and its complement, the complete graph

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$K_n$, the cycle $C_n$, graph $Z_n$, some T-shape trees, lollipop graphs, some dumbbell and $\theta$ graphs etc. van Dam and Haemers proposed the question [5]: which graphs are determined by their spectrum? For a recent survey of the subject, one can consult [6].

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. The $\infty$-graph $B(r, s)$ (see Fig. 1(a), also the equivalent symbol $G_{r+1,s+1}$ in [16]) is a bicyclic graphs obtained by joining two cycles $C_{r+1}$ and $C_{s+1}$ to a common vertex, with out loss of generality, we always assume that $r \leq s$, clearly, $|V(B(r, s))| = r + s + 1 = n$. We denote the graph shown in Fig. 1(b) by $H(a, b, c, d, e)$. Note that in $H(a, b, c, d, e)$ removing the three vertices of degree 3 leaves five disjoint paths $P_a, P_b, P_c, P_d$ and $P_e$, $|V(H(a, b, c, d, e))| = a + b + c + d + e + 3 = n$. We always assume, with out loss of generality, that $a \leq b$ and $c \leq d$. The graph depicted in Fig. 1(c) is denoted by $D(f, g, h, i)$, without loss of generality, we assume $g \leq h$. Let $x_k$ be the number of vertices of degree $k$ of a graph $G$, then we write the degree sequence of $G$ as $\pi(G) = (0^{x_0}, 1^{x_1}, \ldots, k^{x_k}, \ldots, \Delta^{x_{\Delta}})$, clearly, $x_0 + x_1 + \cdots + x_{\Delta} = n$.

Throughout this paper $G - v$ and $G - uv$ denote the graph obtained from $G$ by deleting a vertex $v$ (together with the edges incident to it) and an edge $uv$, respectively. The notation and symbols not defined here are standard, one can also find in [2] for references.

Recently, Cvetković et al. [3,4] intend to build a spectral theory for the signless Laplacian matrix (which is also called Q-theory) of graphs. Recall that the signless Laplacian matrix is defined as $Q(G) = D(G) + A(G)$. Wang et al. [16] studied the signless Laplacian spectral characterization of $\infty$-graphs and obtained the following results:

**Proposition 1.1** (see Theorem 3.1(i) [16]). Graphs $G$ and $H$ are signless Laplacian cospectral graphs iff their subdivision $S(G)$ and $S(H)$ are adjacency cospectral graphs.

**Proposition 1.2** (Theorem 6.3 and Remark 6.4 [16]). All $\infty$-graphs but $G_{r,r+1}(r \geq 3)$ are determined by their signless Laplacian spectra. In addition, the unique signless Laplacian cospectral non-isomorphic graph of $G_{r, r+1}(r \geq 3)$ is $H^{r+1}_5$ (see Fig. 2).

Fan and Luo [8] investigated the adjacency spectral characterization of $\infty$-graphs, and they alleged that a $\infty$-graph without the cycle $C_4$ is determined by its adjacency spectrum (see Theorem 4.4 in their paper). However, from Propositions 1.1 and 1.2 we know that the subdivision graphs of $G_{r, r+1}$ and $H^{r+1}_5 (r \geq 3)$ are adjacency cospectral. This fact indicates that Fan and Luo’s result is not completely right. By the way, it is deserved to point that their another result that the dumbbell graphs without the cycle $C_4$ are determined by the adjacency spectrum (see Theorem 3.4 [8]) is also wrong. See the
papers [17, 18] for the correct result and details. In this paper we will re-study the adjacency spectral characterization of $\infty$-graphs.

The paper is organized as follows. In section 2, some useful lemmas will be summarized. In Section 3, the rough structure of graphs which are cospectral with $\infty$-graphs will be determined. In section 4, the precise structure of a cospectral graph of $\infty$-graph $B(r, s)$ ($s \geq r > 7$) will be obtained. In Section 5, it will be proved that $B(r, s)$ ($s \geq r > 7$) is determined by its adjacency spectrum if and only if $s \neq r + 2$.

2. Preliminaries

First, we give some lemmas that will be used in the next section.

Lemma 2.1 ([2], Interlacing). Suppose that $A$ is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ of a principal submatrix of $A$ of size $m$ satisfy $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$.

Lemma 2.2 [5]. For the adjacency matrix of a graph $G$, the following can be deduced from the spectrum:

(i) The number of vertices.
(ii) The number of edges.
(iii) The number of closed walks of any fixed length.

Let $N_C(H)$ be the number of subgraphs of a graph $G$ which are isomorphic to $H$ and let $N_C(i)$ be the number of closed walks of length $i$ in $G$. Let $N_H(i)$ be the number of closed walks of $H$ of length $i$ which contain all the edges of $H$ and let $S_i(G)$ be the set consisting of all the connected subgraph $H$ of $G$ such that $N_H(i) \neq 0$. It is easy to see that $N_C(i)$ can be expressed by

$$N_C(i) = \sum_{H \in S_i(G)} N_C(H)N_H'(i).$$

Omid [12] uses (1) to obtain some formulae for calculating the number of closed walks of length 2, 3, 4, 5, 7 for any graph. However, we find that an item is missed in the expression of $N_C(7)$. So we correct it and proceed giving formulae to compute the number of closed walks of length 6, 8 for any graph.

Lemma 2.3 [12]. The number of closed walks of length 2, 3, 4, 5 of a graph $G$ are given in the following, where $m$ is number of edges of $G$ and the graphs used are shown in Fig. 3.

(i) $N_C(2) = 2m$, $N_C(3) = 6N_C(K_3)$.
(ii) $N_C(4) = 2m + 4N_C(P_3) + 8N_C(C_4)$.
(iii) $N_C(5) = 30N_C(K_3) + 10N_C(C_5) + 10N_C(G_6)$.

Lemma 2.4. The number of closed walks of length of 6, 7, 8 of a graph $G$ are respectively determined as follows, where $m$ is number of edges of $G$ and the graphs used are shown in Fig. 3.

(i) $N_C(6) = 2m + 12N_C(P_3) + 6N_C(P_4) + 48N_C(C_4) + 12N_C(C_5) + 24N_C(K_3) + 12N_C(K_{1,3}) + 36N_C(G_6) + 12N_C(G_7) + 24N_C(G_8)$.
(ii) $N_C(7) = 126N_C(C_3) + 70N_C(C_5) + 14N_C(C_7) + 84N_C(C_9) + 14N_C(C_6) + 14N_C(G_4) + 14N_C(G_5) + 28N_C(G_6) + 42N_C(G_7) + 28N_C(G_8) + 112N_C(G_9) + 84N_C(H_0)$.
(iii) $N_C(8) = 2m + 28N_C(P_3) + 32N_C(P_4) + 8N_C(P_5) + 16N_C(C_3) + 264N_C(C_4) + 96N_C(C_5) + 16N_C(C_6) + 528N_C(K_4) + 72N_C(K_{1,3}) + 48N_C(K_{1,4}) + 192N_C(K_{2,3}) + 96N_C(K_{2,4}) + 64N_C(C_6) + 64N_C(G_4) + 464N_C(G_5) + 16N_C(G_6) + 112N_C(G_7) + 32N_C(G_8) + 16N_C(G_9) + 16N_C(G_{10}) + 16N_C(G_{11}) + 32N_C(G_{12}) + 96N_C(C_8) + 32N_C(C_9) + 32N_C(G_5) + 32N_C(G_6) + 32N_C(G_7) + 96N_C(H_0)$. 

To verify the above results we give examples in Appendix A which provide the exact datum for complete graph of order 7, 8, 9.

The following corollary immediately follows from Lemma 2.4 and it will be frequently used in the next section.

**Corollary 2.5.** Let $\Gamma_1$ be a connected graph without cycles $C_i$ ($i = 3, 4, 5, 7$). Then

(i) $N_{\Gamma_1}(6) = 2m + 12N_G(P_3) + 6N_G(P_4) + 12N_G(K_{1,3}) + 12N_G(G_6).$

(ii) $N_{\Gamma_1}(8) = 2m + 28N_G(P_3) + 32N_G(P_4) + 72N_G(K_{1,3}) + 8N_G(P_5) + 16N_G(G_1) + 48N_G(K_{1,4}) + 96N_G(G_8) + 16N_G(G_r) + 16N_G(G_8).$

**Lemma 2.6** [2]. Let $uv$ be an edge of a graph $G$, $\varphi(u)$ and $\varphi(uv)$ be the sets of all cycles $Z$ containing $u$ or $uv$, respectively. Then

(i) $P_C(\lambda) = \lambda P_G(\lambda) - \sum_{uv \in E(G)} P_G - u(\lambda) - 2 \sum_{Z \in \varphi(u)} P_G - V(Z)(\lambda).$

(ii) $P_C(\lambda) = P_G - uv(\lambda) - P_G - u(\lambda) - 2 \sum_{Z \in \varphi(uv)} P_G - V(Z)(\lambda).$

For the sake of simplicity, we denote $P_{P_r}(\lambda)$ by $P_r$. By convention, let $P_0 = 1, P_{-1} = 0$ and $P_{-2} = -1$.

**Lemma 2.7** [13]. $P_r = \frac{x^{2r+2} - 1}{x^{2r+2} - x}$ and $P_r(2) = r + 1$, where $x$ satisfies $x^2 - \lambda x + 1 = 0$.

**Lemma 2.8** [13]. Let $\theta(a, b, c)$ be the graph obtained from $P_{a+2}, P_{b+2}$ and $P_{c+2}$ by identifying their initial and terminal vertices, where $a, b, c \geq 0$ and at most one of them is zero. Then

$$P_{\theta(a,b,c)}(\lambda) = \lambda^2 P_a P_b P_c - 2\lambda (P_{a-1} P_b P_c - P_a P_{b-1} P_c + P_a P_b P_{c-1})$$
$$+ 2(P_{a-1} P_b P_{c-1} + P_{a-2} P_b P_c) + P_a P_b P_{c-2} - 2(P_a + P_b + P_c).$$

**Lemma 2.9** [8]. No two non-isomorphic $\infty$-graphs $B(r, s)$ are cospectral.

**Lemma 2.10** [8]. Let $G = B(r, s)$ and $H$ be a graph cospectral with $G$. Then $H$ has no induced subgraph isomorphic to the disjoint union of two cycles.
Let $W_n$ be a graph obtained from the path $P_{n+2}$ (indexed in natural order 1, 2, \ldots, $n + 2$) by adding one pendent edge at vertices 2 and $n + 1$, respectively. An internal path of a graph $G$ is a sequence of vertices $x_1, \ldots, x_k$ such that all $x_i$ are distinct (except possibly $x_1 = x_k$), the vertex degree $d(x_i)$ satisfy $d(x_1) \geq 3$, $d(x_2) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$), $d(x_k) \geq 3$, $x_i$ is adjacent to $x_{i+1}$, $i = 1, \ldots, k - 1$. The following lemma relates the behavior of spectral radius of a graph by subdividing an edge.

**Lemma 2.11** [10], Let $G$ be a connected graph that is not isomorphic to $W_n$ and let $G_{uv}$ be the graph obtained from $G$ by subdividing the edge $uv$ of $G$. If $uv$ lies on an internal path of $G$, then $\lambda_1(G_{uv}) < \lambda_1(G)$.

### 3. Structure of graphs cospectral with $\infty$-graphs

In this section we determine the rough structure of graphs which are cospectral with $\infty$-graphs. Let $H$ be a graph cospectral with the $\infty$-graph $G = B(r, s)$, by Lemma 2.2, $G$ and $H$ have the same number of vertices, edges and closed walks of any given length. Denote by $x_i$ and $y_i$ the number of vertices of degree $i$ in $G$ and $H$, respectively. By counting the number of vertices, edges and closed walks of length 4 in $G$ and $H$, we have the following three equations:

\[
\sum_{i=0}^{4} x_i = n = \sum_{i=0}^{\Delta'} y_i, \tag{3}
\]

\[
\sum_{i=0}^{4} ix_i = 2(n + 1) = \sum_{i=0}^{\Delta'} iy_i, \tag{4}
\]

\[
\sum_{i=0}^{4} ix_i + 4 \sum_{i=0}^{4} \binom{i}{2} x_i + 8n_4 = 6n + 22 + 8n_4 = \sum_{i=0}^{\Delta'} iy_i + 4 \sum_{i=0}^{\Delta'} \binom{i}{2} y_i + 8n_4', \tag{5}
\]

where $n_4 = N_G(C_4)$ and $n_4' = N_H(C_4)$. By adding up these three equations with coefficients $2, -5/2$ and $1/2$, respectively, we have

\[
6 + 4n_4 = \sum_{i=0}^{\Delta'} (i^2 - 3i + 2)y_i + 4n_4'. \tag{6}
\]

Clearly, $n_4 \leq 2$. For $n_4 = 2$ (i.e., $r = s = 3$), $B(3, 3)$ has only seven vertices, it is not difficult to show that $B(3, 3)$ is DAS. For the case of $n_4 = 0$ or $n_4 = 1$, we will determine the degree sequence of $H$ in the following lemma.

**Lemma 3.1.** Let $G = B(r, s)$ be the $\infty$-graph, $H$ be a graph cospectral with $G$. Let $n_4$ and $n_4'$ be the number of $C_4$ in $G$ and $H$ respectively. If $n_4 = 0$ then $n_4' = 0$ and the degree sequence $\pi (H)$ is either $(2^{n-1}, 4^1)$ or $(1^1, 2^{n-4}, 3^3)$; If $n_4 = 1$ then $n_4' = 1$ or $n_4' = 0$, and the degree sequence $\pi (H)$ is either $(2^{n-1}, 4^1)$ or $(1^1, 2^{n-4}, 3^3)$ if $n_4' = 1$; the degree sequence $\pi (H)$ is $(1^2, 2^{n-5}, 3^2, 4^1), (0^1, 2^{n-5}, 3^3)$ or $(1^3, 2^{n-8}, 3^5)$ if $n_4' = 0$.

**Proof.** First suppose that $n_4 = 0$. The Eq. (6) yields $\sum_{i=0}^{\Delta'} (i^2 - 3i + 2)y_i + 4n_4' = 6$, this implies that $2y_0 + 2y_2 + 6y_4 + 4n_4' = 6$ and $y_i = 0$ for $i \geq 5$. We claim that $n_4' = 0$, since otherwise, $n_4' = 1$, thus $y_0 + y_3 + 3y_4 = 1$, and hence $y_4 = 0$. By (3), (4) we have $(y_0, y_1, y_2) = (1 - y_3, 3y_3 - 4, n + 3 - 3y_3)$, which is impossible since all $y_i$ are nonnegative. Now $n_4' = 0$, we obtain $y_0 + y_3 + 3y_4 = 3$. Thus $y_4 = 1$ or 0. If $y_4 = 1$, then $y_0 = y_3 = 0$, and by (3), (4) we get $(y_0, y_1, y_2, y_3, y_4) = (0, 0, -1, 0, 0)$ which gives $\pi (H) = (2^{n-1}, 4^1)$; if $y_4 = 0$, then $y_0 + y_3 = 3$, again by (3) and (4) we obtain $(y_0, y_1, y_2, y_3, y_4) = (3 - y_3, 3y_3 - 8, n + 5 - 3y_3, y_3)$. Easy to find the unique nonnegative solution is $(y_0, y_1, y_2, y_3, y_4) = (0, 1, n - 4, 3, 0)$ which leads to $\pi (H) = (1^1, 2^{n-4}, 3^3)$. 


Next suppose that \( n_4 = 1 \). From equation (6) we have \( \sum_{i=0}^{t-2} (t^2 - 3i + 2)y_i + 4n'_4 = 10 \), this implies that \( 2y_0 + 2y_3 + 6y_4 + 4n'_4 = 10 \) and \( y_i = 0 \) for \( i \geq 5 \). In this case \( n'_4 = 2 \), \( n_4 = 1 \), \( y_0 = 0 \), \( y_0 + y_3 + 3y_4 = 1 \), thus \( y_4 = 0 \) and \( y_0 + y_3 = 1 \). By (3), (4) we have \((y_0, y_1, y_2, y_3, y_4) = (1 - y_3, 3y_3 - 4, n + 3 - 3y_3)\). It is obvious that \( y_1 \) is negative, a contradiction. If \( n'_4 = 1 \), then \( y_0 + y_3 + 3y_4 = 3 \), hence \( y_4 = 1 \) or \( 0 \). Combining (3), (4) we get \((y_0, y_1, y_2, y_3, y_4) = (3 - y_3 - 3y_3, 3y_3 - 3y_4 - 4, n + 5 - 3y_3 - 6y_3, y_4, y_4)\). It is easy to verify that there are two nonnegative solutions: \((0, 0, n - 1, 0, 0)\) and \((0, 1, n - 4, 3, 0)\) which leads to \( \pi(H) = (2^{n - 1}, 4^1) \) and \( \pi(H) = (1, 2^{n - 4}, 3^3) \), respectively. If \( n'_4 = 0 \), then \( y_0 + y_3 + 3y_4 = 5 \), thus \( y_4 = 1 \) or \( 0 \). Again by (3) and (4) we get \((y_0, y_1, y_2, y_3, y_4) = (5 - y_3 - 3y_4, 3y_3 - 8y_4 - 12, n + 7 - 3y_3 - 6y_3, y_4, y_4)\). It is not difficult to obtain all the nonnegative solutions: \((0, 2, n - 5, 2, 1), (1, 0, n - 5, 4, 0)\) and \((0, 3, n - 8, 5, 0)\) which leads to \( \pi(H) = (1^2, 2^{n - 5}, 3^2, 4^1) \), \( \pi(H) = (1^3, 2^{n - 8}, 3^5) \) and \( \pi(H) = (1^1, 2^{n - 4}, 3^3) \), respectively. \( \square \)

**Lemma 3.2.** Let \( G = B(r, s) \) be a \( \infty \)-graph where \( r \) and \( s \) are not equal to three. If a graph \( H \) is cospectral with \( G \), then \( H \in \{B(r, s), H(a, b, c, d, e), D(f, g, h, i)\} \) (see Fig. 1(b) and (c)). Moreover, the graphs in the set contain no cycle \( C_4 \).

**Proof.** Since \( r \) and \( s \) are not equal to three, \( G \) has no cycle \( C_4 \). Thus from Lemma 3.1, we know that \( H \) also has no cycle \( C_4 \) and the degree sequence of \( H \) is \((2^{n - 1}, 4^1) \) or \((1, 2^{n - 4}, 3^3) \). Moreover, Lemma 2.10 implies \( H \) has no induced subgraph containing two disjoint cycles. If \( \pi(H) = (2^{n - 1}, 4^1) \) then \( H \cong B(r', s') \) for some \( n = r' + s' + 1 \). Furthermore, \( H \cong G \) by Lemma 2.9. If \( \pi(H) = (1, 2^{n - 4}, 3^3) \) then \( H \) has no component isomorphic to a tree since \( H \) has a unique vertex of degree one. Thus \( H \) must be connected and so \( H \cong H(a, b, c, d, e) \) or \( D(f, g, h, i) \). \( \square \)

**Remark 1.** Let \( H \) be a graph cospectral with the \( \infty \)-graph \( B(3, s) \) \((s > 3)\). From Lemma 3.1 we see that there are five possible degree sequences for \( H \), and for each possible \( \pi(H) \) we can easily find the corresponding graph. Thus to determine the DAS \( \infty \)-graph \( B(3, s) \) is more difficult than to determine that has no \( C_4 \). In this paper we have no plan to find which \( \infty \)-graph \( B(3, s) \) \((s > 3)\) is DAS. A similar problem that lollipop graphs with \( C_4 \) have been investigated in a very long paper [1]. Furthermore we will count the number of closed walks of length six and eight of the \( \infty \)-graph \( B(r, s) \) in the following passage. In view of Lemma 2.4, we know that if a graph \( G \) has \( C_4 \) and \( C_5 \), then \( N_c(6) \) and \( N_c(8) \) become more complicated. In order to make the paper readable, we restrict ourselves to study which \( \infty \)-graph \( B(r, s) \) with \( s \geq r > 7 \) is DAS in the following text.

Denote by \( c^1 \) the cycle formed by \( v_1, v_3, P_a \) and \( P_b \) of \( H(a, b, c, d, e) \), \( c^2 \) the cycle formed by \( v_1, v_2, v_3, P_a, P_c \) and \( P_d \) of \( H(a, b, c, d, e) \), and \( c^3 \) the cycle formed by \( v_1, v_2, v_3, P_b, P_c \) and \( P_d \) of \( H(a, b, c, d, e) \) (see Fig. 1(b)), respectively. Denote by \( c^4 \) the cycle formed by \( v_1 \) and \( P_f \) of \( D(f, g, h, i) \), \( c^5 \) the cycle formed by \( v_2, v_3, P_g \) and \( P_h \) of \( D(f, g, h, i) \), (see Fig. 1(c)) respectively. For the sake of simplicity, we misuse the notations \( c^i \) \( i = 1, \ldots, 5 \) and \( P_a, P_b, P_c, P_d, P_e, P_f, P_g, P_h, P_i \) to denote the number of vertices in them, respectively. Recall that we always assume \( P_a \leq P_b \) and \( P_c \leq P_d \), thus \( c^2 \leq c^3 \). Denote by \( B(r, s) = \{B(r, s), |r + s| = n, s \leq r > 7\} \), \( \mathbb{B}(a, b, c, d, e) = \{H(a, b, c, d, e) | a + b + c + d + e = 3 = n \) and \( H(a, b, c, d, e) \) has no \( C_3, C_4, C_5, C_7 \} \), \( D(f, g, h, i) = \{D(f, g, h, i) | f + g + h + i = 3 = n \) and \( D(f, g, h, i) \) has no \( C_3, C_4, C_5, C_7 \}, \).

Let \( G \in B(r, s) \) and \( G' \) be cospectral with \( G \). Since \( G \) has no \( C_3, C_4, C_7 \) and its cospectral graph has the same length of the shortest odd cycle, so does \( G' \). In addition, by Lemma 3.2 \( G' \in \mathbb{B}(a, b, c, d, e) \) or \( D(f, g, h, i) \). In order to obtain the precise structure of \( G' \), we need to count the closed walks of length six and eight of graphs in \( B(r, s), \), \( \mathbb{B}(a, b, c, d, e) \), and \( D(f, g, h, i) \), respectively. For any triangle free graph \( \Gamma \), let us define

\[
d(uv) = (d(u) - 1)(d(v) - 1), \quad \text{where } uv \in E(\Gamma).
\]

It is easy to verify \( N_G(P_3) = \sum_{v \in V(\Gamma)} \left( \frac{d(v)}{2} \right), N_G(P_4) = \sum_{uv \in E(\Gamma)} d(uv) \) and \( N_G(K_{1,3}) = \sum_{v \in V(\Gamma)} \left( \frac{d(v)}{3} \right) \).

Set \( \Gamma = G \in \mathbb{B}(r, s) \), we have \( |E(G)| = n + 1, N_G(P_3) = \left( \frac{5}{2} \right) + (r + s) \left( \frac{2}{2} \right) = n + 5, N_G(P_4) = \left( \frac{5}{2} \right) + (r + s) \left( \frac{2}{2} \right) = n + 5 \), and \( N_G(P_4) = \left( \frac{5}{2} \right) + (r + s) \left( \frac{2}{2} \right) = n + 5 \).
Then by Corollary 2.5 (i), we have
\[ N_G(6) = 2|E(G)| + 12N_G(P_3) + 6N_G(P_4) + 12N_G(K_{1,3}) = 20n + 164. \] (7)

Set \( \Gamma = G' e \mathbb{H}(a, b, c, d, e) \) or \( \mathbb{D}(f, g, h, i) \), we obtain \(|E(G')| = n + 1, N_G(P_3) = \left( \frac{5}{3} \right) \times 3 + \left( \frac{2}{3} \right)(n - 4) = n + 5, N_G(K_{1,3}) = \left( \frac{5}{3} \right) \times 3 = 3. \) \( N_G(P_4) \), related to the structure of \( H \), is more complicated.

First suppose that \( G' = H e \mathbb{H}(a, b, c, d, e) \), it is easy to see that for an edge \( uv \in E(H), d(uv) \in \{0, 1, 2, 4\} \). \( uv \) is said to be the edge of \( i \)-type if \( d(uv) = i \), where \( i = 0, 1, 2, 4 \). It is clear that \( uv \) is 0-type if and only if \( uv \) is a pendent edge; \( uv \) is 1-type if and only if \( d(u) = d(v) = 2; uv \) is 2-type if and only if \( d(u), d(v) = \{2, 3\} \); \( uv \) is 4-type if and only if \( d(u) = d(v) = 3 \). Denote by \( m_i(H) \) the number of \( i \)-type edges in \( H \). We have
\[ m_0(H) + m_1(H) + m_2(H) + m_4(H) = |E(H)| = n + 1, \] (8)

and, since \( H \) has only one pendent edge, \( m_0(H) = 1 \). We have
\[ N_H(P_4) = \sum_{uv \in E(\Gamma)} d(uv) = m_1(H) + 2m_2(H) + 4m_4(H) = n + 2m_2(H) + 3m_4(H). \] (9)

Since \( H \) has three vertices of degree three and has no \( C_3, 0 \leq m_4(H) \leq 2 \). Now we have to distinguish the value of \( m_4(H) \) to count \( N_H(P_4) \).

**Case 1.** \( m_4(H) = 0 \);

From the structure of \( H \) in Fig. 1(b), we know that \( m_2(H) = 8 \) if \( P_e = 1 \), and \( m_2(H) = 9 \) if \( P_e > 1 \).

Thus according to (9)
\[ N_H(P_4) = \begin{cases} n + 8 & \text{if } P_e = 1; \\ n + 9 & \text{if } P_e > 1. \end{cases} \]

**Case 2.** \( m_4(H) = 1 \);

From the structure of \( H \) in Fig. 1(b), we know that \( m_2(H) = 6 \) if \( P_e = 1 \), and \( m_2(H) = 7 \) if \( P_e > 1 \).

Thus according to (9)
\[ N_H(P_4) = \begin{cases} n + 9 & \text{if } P_e = 1; \\ n + 10 & \text{if } P_e > 1. \end{cases} \]

**Case 3.** \( m_4(H) = 2 \);

From the structure of \( H \) in Fig. 1(b), we know that \( m_2(H) = 4 \) if \( P_e = 1 \), and \( m_2(H) = 5 \) if \( P_e > 1 \).

Thus according to (9)
\[ N_H(P_4) = \begin{cases} n + 10 & \text{if } P_e = 1; \\ n + 11 & \text{if } P_e > 1. \end{cases} \]

Then by Corollary 2.5 (i), we have
\[
N_H(6) = 2|E(H)| + 12N_H(P_3) + 6N_H(P_4) + 12N_H(K_{1,3}) + 12N_H(C_6)
\]
\[
= 14n + 98 + 6N_H(P_4) + 12N_H(C_6)
\]
\[
= 20n + 146 + 12N_H(C_6) \quad \text{if } N_H(P_4) = n + 8;
\]
\[
= 20n + 152 + 12N_H(C_6) \quad \text{if } N_H(P_4) = n + 9;
\]
\[
= 20n + 158 + 12N_H(C_6) \quad \text{if } N_H(P_4) = n + 10;
\]
\[
= 20n + 164 + 12N_H(C_6) \quad \text{if } N_H(P_4) = n + 11.
\]
Since $0 \leq N_H(C_6) \leq 3$, from the above we know that $N_H(6) = 20n + 164$ if and only if $(N_H(P_4), N_H(C_6)) = (n + 9, 1)$ or $(N_H(P_4), N_H(C_6)) = (n + 11, 0)$. It is easy to see that $(N_H(P_4), N_H(C_6)) = (n + 9, 1)$ corresponds to the following cases (1) or (2), and $(N_H(P_4), N_H(C_6)) = (n + 11, 0)$ corresponds to the following case (3)

1. $c^1 = 6, c^2 > 6, c^3 > 6$ and $N_H(P_4) = n + 9$;
2. $c^1 > 6, c^2 = 6, c^3 > 6$ and $N_H(P_4) = n + 9$;
3. $c^1 > 6, c^2 > 6, c^3 > 6$ and $N_H(P_4) = n + 11$.

Recall that $c^1 = a + b + 2, c^2 = a + c + d + 3, c^3 = b + c + d + 3$ and $c^2 \leq c^3$. We list the above three possible graphs $H$ with $N_H(6) = 20n + 164$ in Table 1, where its $i$-row corresponds the case (i), $i = 1, 2, 3$, respectively.

Next suppose that $G' = D \in \mathcal{D}(f, g, h, i)$, it is also easy to see that $d(uv) \in \{0, 1, 2, 4\}$ for $uv \in E(D)$. Denote by $m_i(D)$ $i = 0, 1, 2, 4$ the number of $i$-type edges in $D$. Similarly as (8) and (9), we have

$$m_0(D) + m_1(D) + m_2(D) + m_4(D) = |E(D)| = n + 1,$$

$$N_D(P_4) = \sum_{uv \in E(D)} d(uv) = m_1(D) + 2m_2(D) + 4m_4(D) = n + m_2(D) + 3m_4(D).$$

Since $w_1w_2 \in E(D)$ and $P_g$ may be zero, it is easy to see that $4 \leq m_2(D) \leq 7, 1 \leq m_4(D) \leq 2$. Now we have to distinguish the value of $m_4(D)$ to count $N_D(P_4)$.

**Case 1.** $m_4(D) = 1$.

From the structure of $D$ in Fig. 1(c), we know that $m_2(D) = 6$ if $P_l = 1$, and $m_2(D) = 7$ if $P_l > 1$. Thus according to (11)

$$N_H(P_4) = \begin{cases} n + 9 & \text{if } P_l = 1; \\
 n + 10 & \text{if } P_l > 1; \end{cases}$$

**Case 2.** $m_4(D) = 2$.

From the structure of $D$ in Fig. 1(c), we know that $m_2(D) = 4$ if $P_l = 1$, and $m_2(D) = 5$ if $P_l > 1$. Thus according to (11)

$$N_H(P_4) = \begin{cases} n + 10 & \text{if } P_l = 1; \\
 n + 11 & \text{if } P_l > 1; \end{cases}$$
Then by Corollary 2.5 (i), we have

\[ N_D(6) = 2|E(D)| + 12N_D(P_3) + 6N_D(P_4) + 12N_D(K_{1,3}) + 12N_D(C_6) \]

\[ = \begin{cases} 
20n + 152 + 12N_D(C_6) & \text{if } N_H(P_4) = n + 9; \\
20n + 158 + 12N_D(C_6) & \text{if } N_H(P_4) = n + 10; \\
20n + 164 + 12N_D(C_6) & \text{if } N_H(P_4) = n + 11; 
\end{cases} \]

since \( 0 \leq N_D(C_6) \leq 2 \), from the above we know that \( N_D(6) = 20n + 164 \) if and only if \( (N_D(P_4), N_D(C_6)) = (n + 9, 1) \) or \( (N_D(P_4), N_D(C_6)) = (n + 11, 0) \). Analogous to \( H \), \( (N_D(P_4), N_D(C_6)) = (n + 9, 1) \) corresponds to the following cases (1) or (2), and \( (N_D(P_4), N_D(C_6)) = (n + 11, 0) \) corresponds to the following case (3)

1. \( c^4 = 6, c^5 > 6 \) and \( N_H(P_4) = n + 9 \);
2. \( c^4 > 6, c^5 = 6 \) and \( N_H(P_4) = n + 9 \);
3. \( c^4 > 6, c^4 > 6 \) and \( N_H(P_4) = n + 11 \).

Recall that \( c^4 = f + 1, c^5 = g + h + 2 \). According the above cases we list the above three possible graphs \( D \) with \( N_D(6) = 20n + 164 \) in Table 2.

From Tables 1 and 2 we know that, given \( G \in \mathbb{B}(r, s) \) where \( s \geq r > 7 \), there are exactly fifteen graphs in \( \mathbb{G}(a, b, c, d, e) \) and \( \mathbb{D}(f, g, h, i) \) such that they have the same number of closed walks of length 4 and 6, respectively. To distinguish whether they are cospectral we have to enumerate the number closed walks of length 8 for these graphs.

For any graph \( \Gamma \) without \( C_3 \) and \( C_4 \), it is easy to verify

\[ N_\Gamma(P_5) = \sum_{v \in V(\Gamma)} \sum_{d(u) = 1} (d(v_i) - 1)(d(v_j) - 1). \]  

By (12) we have \( N_{C_5}(P_5) = 4 \times 3 + 6 \times 1 + r + s + 1 - 5 = n + 13 \). Let \( G \in \mathbb{B}(r, s) \), we have known that \( |E(G)| = n + 1, N_{C_5}(P_3) = n + 5, N_{C_5}(P_4) = n + 9 \), and \( N_{C_5}(K_{1,3}) = 4 \). Moreover, \( N_{C_5}(G_i) = \binom{4}{2} \times 2 = 12 \).

Thus according to Corollary 2.5, we have

\[ N_{C_5}(8) = 2|E(G)| + 28N_{C_5}(P_3) + 32N_{C_5}(P_4) + 72N_{C_5}(K_{1,3}) + 8N_{C_5}(P_5) + 16N_{C_5}(G_i) + 48N_{C_5}(K_{1,4}) \]

\[ = 2(n + 1) + 28(n + 5) + 32(n + 9) + 72 \times 4 + 8(n + 13) + 16 \times 12 + 48 \]

\[ = 70n + 1062. \]

Let \( G_1 = H(a, b, 0, 0, e) \in \mathbb{G}(a, b, c, d, e) \) \( (a > 3, e > 1) \), we have known that \( |E(G_1)| = n + 1, N_{C_5_1}(P_3) = n + 5, N_{C_5_1}(K_{1,3}) = 3, N_{C_5_1}(P_4) = n + 11 \). By (12)

\[ N_{G_1}(P_5) = \begin{cases} 
8 + 5 \times 2 + 2 \times 5 + a - 2 + b - 2 + e - 3 = n + 18 & \text{if } e > 2; \\
8 + 5 \times 2 + 2 \times 4 + a - 2 + b - 2 = n + 17 & \text{if } e = 2. 
\end{cases} \]
Furthermore, $N_{G_1'}(G_i) = 5 + 4 + 4 = 13$, by Corollary 2.5, we have

$$N_{G_1'}(8) = 2|E(G_1')| + 28N_{G_1'}(P_3) + 32N_{G_1'}(P_4) + 72N_{G_1'}(K_{1,3}) + 8N_{G_1'}(P_5)$$
$$+ 16N_{G_1'}(G_i) + 16N_{G_1'}(C_8)$$
$$= 2(n + 1) + 28(n + 5) + 32(n + 11) + 72 \times 3 + 8N_{G_1'}(P_5) + 16N_{G_1'}(G_i)$$
$$+ 16N_{G_1'}(C_8)$$
$$= \begin{cases} 
62n + 710 + 8(n + 18) + 16 \times 13 + 16N_{G_1'}(C_8) \\
62n + 710 + 8(n + 17) + 16 \times 13 + 16N_{G_1'}(C_8) \\
70n + 1062 + 16N_{G_1'}(C_8) & \text{if } e > 2; \\
70n + 1054 + 16N_{G_1'}(C_8) & \text{if } e = 2.
\end{cases}$$

Thus we obtain $N_{G_1'}(8) = 70n + 1062$ if and only if $e > 2$ and $N_{G_1'}(C_8) = 0$.

Let $G_2' = H(1, b, 1, 1, e) \in \mathbb{H}(a, b, c, d, e) (b > 3, e > 1)$, we have known $|E(G_2')| = n + 1$, $N_{G_2'}(P_3) = n + 5$, $N_{G_2'}(K_{1,3}) = 3$, $N_{G_2'}(P_4) = n + 9$,

$$N_{G_2'}(P_5) = \begin{cases} 
3 \times 3 + 4 \times 3 + 2 \times 3 + b - 2 + e - 3 = n + 16 & \text{if } b > 3, e > 2; \\
3 \times 3 + 4 \times 3 + 2 \times 2 + b = n + 15 & \text{if } b > 3, e = 2.
\end{cases}$$

$N_{G_2'}(G_i) = 3 \times 3 = 9$, $N_{G_2'}(C_8) = 1$, $N_{G_2'}(G_r) = 3$, by Corollary 2.5, we have

$$N_{G_2'}(8) = 2|E(G_2')| + 28N_{G_2'}(P_3) + 32N_{G_2'}(P_4) + 72N_{G_2'}(K_{1,3}) + 8N_{G_2'}(P_5)$$
$$+ 96N_{G_2'}(G_i) + 16N_{G_2'}(C_8)$$
$$= 2(n + 1) + 28(n + 5) + 32(n + 9) + 72 \times 3 + 8N_{G_2'}(P_5) + 16N_{G_2'}(G_i)$$
$$+ 96N_{G_2'}(C_8) + 16N_{G_2'}(G_r) + 16N_{G_2'}(C_8)$$
$$= \begin{cases} 
62n + 646 + 8(n + 16) + 16 \times 9 + 96 + 3 \times 16 + 16N_{G_2'}(C_8) \\
62n + 646 + 8(n + 15) + 16 \times 9 + 96 + 3 \times 16 + 16N_{G_2'}(C_8) \\
70n + 1062 + 16N_{G_2'}(C_8) & \text{if } b > 3, e > 2; \\
70n + 1054 + 16N_{G_2'}(C_8) & \text{if } b > 3, e = 2.
\end{cases}$$

So we have $N_{G_2'}(8) = 70n + 1062$ if and only if $b > 3$, $e > 2$ and $N_{G_2'}(C_8) = 0$.

By the same argument one can obtain $N_G(8)$ of the other 13 graphs in Tables 1 and 2. Since it is long and trivial, we just put the details to the Appendix A at the end of this paper, from which we know that six of them have the same number of closed walks of length 8 with that of $G \in \mathbb{B}(r, s)$ and we list them in Table 3.

From Tables 1–3, we have the following lemma.

### Table 3

<table>
<thead>
<tr>
<th>Graph</th>
<th>$H(0, b, 0, d, e)$</th>
<th>$H(a, b, 0, 0, e)$</th>
<th>$H(0, b, 1, 2, 1)$</th>
<th>$H(1, b, 1, 1, e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t.</td>
<td>$b &gt; 6, d &gt; 5$</td>
<td>$a &gt; 5, e &gt; 2$</td>
<td>$b &gt; 6$</td>
<td>$b &gt; 5, e &gt; 2$</td>
</tr>
<tr>
<td>$c_1$, $c_2$, $c_3$</td>
<td>$(&gt; 8, &gt; 8, &gt; 8)$</td>
<td>$(&gt; 8, &gt; 8, &gt; 8)$</td>
<td>$(&gt; 8, &gt; 6, &gt; 8)$</td>
<td>$(&gt; 8, &gt; 8, &gt; 8)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Graph</th>
<th>$H(2, 2, 1, 2, e)$</th>
<th>$D(f, 0, h, i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t.</td>
<td>$e &gt; 2$</td>
<td></td>
</tr>
<tr>
<td>$c_1$, $c_2$, $c_3$, $c_4$, $c_5$</td>
<td>$(6, 8, 8, --, --)$</td>
<td>$(--, --, &gt; 8, &gt; 8)$</td>
</tr>
</tbody>
</table>
Lemma 3.3. Let $G = B(r, s) \in \mathbb{B}(r, s)$ be a $∞$-graph. Let $G'$ be a graph cospectral with $G$, then $G'$ may be one of the graphs shown in Table 3.

4. Spectral characterization of $\mathbb{B}(r, s)$

In this section we focus on the spectral characterization of $∞$-graphs $\mathbb{B}(r, s) = \{B(r, s)|r + s + 1 = n, s \geq r > 7\}$.

It has been shown that the largest signless Laplacian eigenvalue $\kappa(G_{r,s})$ of $∞$-graph $G_{r,s}$ satisfies $\kappa(G_{r,s}) > \frac{16}{3}$ (see Lemma 6.8(i) [16]). According to the relation $\rho(G_{r,s}) = \sqrt{\kappa(G_{r,s})}$ (see Lemma 3.1 [16]) we obtain the following lemma:

Lemma 4.1. For $s \geq r \geq 3$, $\rho(G_{r,s}) > \sqrt{\frac{16}{3}} \approx 2.3094$.

Lemma 4.2. Let $G = B(r, s) \in \mathbb{B}(r, s)$ be a $∞$-graph. Then $G$ is not cospectral with $H(0, b, 1, 2, 1)(b > 10)$.

Proof. It is easy to verify that we can subdivide some edges in the internal path of $H(0, 9, 1, 2, 1)$ to get $H(0, b, 1, 2, 1)$ for $b > 10$. From Lemma 2.11 and Lemma 4.1, $\rho(G) > 2.3094 > 2.3086 \cdots = \rho(H(0, 9, 1, 2, 1)) > \rho(H(0, b, 1, 2, 1))$. Thus $G$ is not cospectral with $H(0, b, 1, 2, 1)(b > 10)$ and the lemma holds. □

Next we use Lemma 2.6 and Lemma 2.7 to calculate the character polynomials of graph $B(r, s)$ and the graphs mentioned in Lemma 3.3.

Lemma 4.3. Let $P_{B(r,s)}(λ)$ be the characteristic polynomial of the $∞$-graph $B(r, s)$, denote by $P_{B}(x) = (x^2 - 1)^2P_{B(r,s)}(λ)$, then $P_{B}(x) = C(n; x) + P_{B}(r, s; x)$ where $n = r + s + 1$, $x$ satisfies $x^2 - \lambda x + 1 = 0$ and

$$C(n; x) = x^{-n} - 3x^{2-n} - 3x^{2+n} + x^{4+n}$$

$$P_{B}(r, s; x) = -2x^{-r} + 2x^{2-r} + x^{3+r} + x^{3+1-r} + x^{3+3-r} + 2x^{r+2} - 2x^{r+4} + x^{r+1-s} + x^{3+3-s} - 2x^{-s} + 2x^{2-s} + 2x^{3+2} - 2x^{3+4}.$$  \(14\)

Proof. Let $B(r, s)$ be shown in Fig. 1(a) and $u$ be the unique vertex of degree four. Using Lemma 2.6, we have

$$P_{B(r,s)}(λ) = λP_{T}P_{s} - 2(P_{T-1}P_{s} + P_{T}P_{s-1} + P_{r} + P_{s}).$$  \(15\)

Then we substitute $P_{r} = \frac{x^{2r+2} - 1}{x^{2r+2} - x}$ to the above equation and by using Maple, we obtain $P_{B}(x) = C(n; x) + P_{B}(r, s; x)$. □

Lemma 4.4. Let $H_{1} = H(1, b, 1, 1, e)$ and $P_{H_{1}}(λ)$ the characteristic polynomial of $H_{1}$, denote by $P_{H_{1}}(x)$, where $x$ satisfies $x^2 - \lambda x + 1 = 0$, then $P_{H_{1}}(x) = C(n; x) + P_{H_{1}}(b, e; x)$ where $n = b + e + 6$, $C(n; x)$ is shown in (13) and

$$P_{H_{1}}(b, e; x) = -2x^{-3-e} + 2x^{1-e} + 2x^{3+e} - 2x^{6} + 2x^{2+b-e} - x^{4+b-e} + x^{6+b-e} + x^{-2-b-e} - x^{-b+e} + 2x^{2-b}.$$  \(16\)

Proof. Let $H = H(a, b, c, d, e)$ be shown in Fig. 1(b) and $v_{2}$ the vertex which joins the path $P_{c}$. Denote by $H^{-} = H - (v_{2} \cup V(P_{c}))$, and $T(a, b, d)$ the T-shape tree induced by vertex set $\{v_{3}\} \cup V(P_{a}) \cup$
Using Lemma 2.6 (i) at \( v_3 \) in \( T(a, b, d) \), we have

\[
P_{T(a, b, d)}(\lambda) = \lambda P_a P_b P_d - P_{a-1} P_b P_d - P_a P_{b-1} P_d - P_a P_b P_{d-1}.
\]

(17)

Again using Lemma 2.6 (i) at \( v_1 \) in \( H^- \), we have

\[
P_{H^-}(\lambda) = \lambda P_c P_{T(a, b, d)}(\lambda) - (P_c P_{T(a-1, b, d)}(\lambda)) + P_c P_{T(a, b-1, d)}(\lambda) + P_c P_{T(a, b, d-1)}(\lambda)) - 2P_c P_d.
\]

(18)

Let \( v_21 \) be the edge in \( H \) which connects \( v_2 \) and \( P_e \). Applying Lemma 2.6 (ii) for \( H \) at \( v_21 \) (see Fig. 1(b)), we have

\[
P_H(\lambda) = P_{\theta(a, b, c, d+1)}(\lambda) P_e - P_{H^-}(\lambda) P_{e-1}.
\]

(19)

By putting (2), (17) and (18) into (19) we obtain the characteristic polynomial of \( H = H(a, b, c, d, e) \) i.e., \( P_H(\lambda) \). If we substitute \( (a, b, c, d, e) = (1, 1, 1, 1, 1) \) and \( P_r = \frac{x^{2r+2}}{x^2 - x^r} \) in \((x^2 - 1)^2 P_H(\lambda), \) then we obtain \( P_{H_1}(x) = C(n; x) + P_{H_1}(b, e; x) \). We have used Maple to perform the above calculations. Since the computation is long, here we omit the process and only give the concrete expression of \( P_{H_1}(x) \). □

If we substitute \( (a, b, c, d, e) = (0, 0, 0, 0, 0), (2, 2, 1, 2, e) \) and \( P_r = \frac{x^{2r+2}}{x^2 - x^r} \) in \((x^2 - 1)^2 P_H(\lambda), \) then, from (19), we get the following two corollaries by using Maple.

**Corollary 4.5.** Let \( H_2 = H(0, 0, 0, d, e) \) and \( P_{H_2}(\lambda) \) the characteristic polynomial of \( H_2 \) and set \( P_{H_2}(x) = (x^2 - 1)^2 P_{H_2}(\lambda) \) where \( x \) satisfies \( x^2 - \lambda x + 1 = 0 \), then \( P_{H_2}(x) = C(n; x) + P_{H_2}(b, e; x) \) where \( n = b + d + e + 3, C(n; x) \) is shown in (13) and

\[
P_{H_2}(b, d, e; x) = -2x^{-e} + 2x^{-e+2} - 2x^{-b-e} + x^{-b-d+e+1} + 2x^{-b+e+2}
\]

\[
+ x^{-b+d+e+3} + 2x^{-e+4} - 2x^{-d-e+1} + 2x^{-d-e+1}
\]

\[
+ x^{-b-d-e+1} + x^{-b+d-e+3} + 2x^{e+2} - 2x^{e+4} + 2x^{d+e+3} - 2x^{d+e+5}.
\]

(20)

**Corollary 4.6.** Let \( H_3 = H(2, 2, 1, 2, e) \) and \( P_{H_3}(\lambda) \) be its characteristic polynomial and set \( P_{H_3}(x) = (x^2 - 1)^2 P_{H_3}(\lambda) \) where \( x \) satisfies \( x^2 - \lambda x + 1 = 0 \), then \( P_{H_3}(x) = C(n; x) + P_{H_3}(e; x) \) where \( n = e + 10, C(n; x) \) is shown in (13) and

\[
P_{H_3}(e; x) = x^{-e} + 2x^{-e+4} - x^{-e+6} - x^{-e+8} + x^{-e+10} + x^{e-6}
\]

\[-x^{-e-4} - x^{-e-2} + 2x^e + x^{e+4}.
\]

(21)

As similar as Lemma 4.4, one can also obtain the following Lemma.

**Lemma 4.7.** Let \( D = D(f, 0, h, i) \) and \( P_D(\lambda) \) be its characteristic polynomial and set \( P_D(x) = (x^2 - 1)^2 P_D(\lambda) \) where \( x \) satisfies \( x^2 - \lambda x + 1 = 0 \), then \( P_D(x) = C(n; x) + P_D(f, h, i; x) \) where \( n = f + h+i+3, C(n; x) \) is shown in (13) and

\[
P_D(f, h, i; x) = 4x^{-i} - 4x^{-i} - 4x^{-i} + 4x^{4+i} + 2x^{3+i+f} - 2x^{f+5+i} + 4x^{4+i+h} - 2x^{6+i+h}
\]

\[-2x^{-i+1+h} + 2x^{-i+3+h} - 2x^{-i-1-f} + 2x^{-i+1-f} + 2x^{i+1-f} - 2x^{3+i-f}
\]

\[-2x^{-i-h} + 4x^{-i-h} - 2x^{i+2-h} - 2x^{-i+h} + x^{-i+h+1-f} + x^{-i-h+1-f}
\]

\[-x^{-i-h+1+f} + x^{-i-h+1-f} + x^{-i+3+h+f} + x^{-i+3-h+f} - x^{-i+h+3-f} + x^{i+h+5-f}.
\]

(22)
Theorem 4.8. Let \( G = B(r, s) \in \mathbb{B}(r, s) \) be a \( \infty \)-graph of order \( n \). Then \( G \) is cospectral with \( H_1 = H(1, b, 1, 1, e) \) if and only if \( r = e + 1, s = e + 3, b = e - 1 \).

Proof. We first show the necessity. Since \( G \) and \( H_1 \) are cospectral, they have the same number of vertices and the same characteristic polynomial, i.e., \( r + s + 1 = b + e + 6 = n \), \( P_b(\lambda) = P_{H_1}(\lambda) \). Thus \( P_b(x) - P_{H_1}(x) = P_b(r, s; x) - P_{H_1}(b, e; x) = 0 \) by Lemma 4.3 and Lemma 4.4. Now we divide two cases to consider.

Suppose that \( s > r \). By Lemma 4.3, we know that the leading term of \( P_b(r, s; x) \) is \(-2x^{s+4}\). By Lemma 4.4, the leading term of \( P_{H_1}(b, e; x) \) may be one of \(-2x^{e+7}, x^{b-e+6}\) (or their summation). Since the difference of the two leading terms must be zero, we have \(-2x^{s+4} = -2x^{e+7}, s+4 = e+7, i.e., s = e + 3\). Furthermore the second leading term of \( P_b(r, s; x) \) may be one among \( \{2x^{s+2}, -2x^{s+4}, 2x^{s+2}\} \) (or their summation), the second leading term of \( P_{H_1}(b, e; x) \) may be one of \( \{2x^{e+3}, x^{b-e+6}\} \) (or their summation). Thus we claim that \( 2x^{s+4} \) must be the second leading term of \( P_{H_1}(b, e; x) \) since the coefficient of the second leading term of \( P_b(r, s; x) \) is two. It implies that \( 2x^{s+2} - 2x^{e+4} = 0 \), and so \( 2x^{s+2} = 2x^{e+3} \), i.e., \( r = e + 1 \). Moreover, since \( r + s + 1 = b + e + 6 = n \) it follows that \( b = e - 1 \).

Next suppose that \( s = r \). In this situation, the leading term of \( P_b(r, s; x) \) is \(-4x^{s+4}\) while that of \( P_{H_1}(b, e; x) \) is some summation of \( \{-2x^{s+2}, x^{b-e+6}\} \). Obviously, their difference can not be zero. So this case disappears.

For the sufficiency, if we substitute \( r = e + 1, s = e + 3 \) and \( b = e - 1 \) in \( P_b(x) \) and \( P_{H_1}(x) \), respectively, then by using maple it is easy to verify that \( P_b(x) - P_{H_1}(x) = 0 \). Hence \( P_{B(r,s)}(\lambda) = P_{H_1}(\lambda) \), i.e., \( B(e + 1, e + 3) \) and \( H(1, e - 1, 1, 1, e) \) are cospectral. □

Remark 2. From Remark 1 we know that in order to avoid some trivial trouble, we restrict ourselves with the condition \( s \geq r \), and this condition is inherited to Theorem 4.8. In fact we can easily verify that for \( 7 \geq r \geq 2, 6 \geq e \geq 1 \) \( B(e + 1, e + 3) \) and \( H(1, e - 1, 1, 1, e) \) are cospectral too.

Lemma 4.9. Let \( G = B(r, s) \in \mathbb{B}(r, s) \) be a \( \infty \)-graph of order \( n \), \( H_2 = H(0, b, 0, d, e) \) \((b > 6, d > 5, e > 2)\) be also of order \( n \). Then \( G \) is not cospectral with \( H_2 \).

Proof. By the way of contradiction, assume \( G \) and \( H_2 \) are cospectral, then they share the same number of vertices and the same characteristic polynomials, i.e.,

\[
n = r + s + 1 = b + d + e + 3,
\]

and \( P_b(\lambda) = P_{H_2}(\lambda) \). Then from Lemma 4.3 and Corollary 4.5 we have \( P_b(r, s; x) = P_{H_2}(b, e; x) \).

Denote by \( P_b'(x) = \frac{d^3P_b(r, s; x)}{dx^3} \) and \( P_{H_2}'(x) = \frac{d^3P_{H_2}(b, e; x)}{dx^3} \), respectively. Then we have \( P_b'(x) = P_{H_2}'(x) \),
in particular, $P''_B(1) = P''_{H_2}(1)$. By simple calculation we have

$$P''_B(1) = r^2 + s^2 - 2rs - 4r - 4s - 5 = b^2 + d^2 + e^2 - 6be - 4b - ed - 12e - 13 = P''_{H_2}(1).$$

(26)

Subtract (26) from (25)'s square we have

$$2rs = bd + 4be + de + eb + d + 6e + 2.$$  

(27)

Next, by carefully checking the terms of $P_{H_2}(b, e; x)$ we find that the first several leading terms must come from the following:

$$-2x^{b+e+4}, -2x^{d+e+5}, 2x^{d+e+3}, -2x^{e+4}, x^{b+d-e+3}, x^{d+e-b+3}.$$  

(28)

From (14) we see that the first several leading terms of $P_B(r, s; x)$ must come from the following:

$$-2x^{s+4}, 2x^{s+2}, -2x^{r+4}, 2x^{r+2}.$$  

(29)

First we claim that $s > r$ (recall that $s \geq r$), since otherwise $s = r$, the leading term of $P_B(r, s; x)$ is $-4x^{s+4}$ and the second leading term is $4x^{s+2}$. Consequently, the leading term of $P_{H_2}(b, e; x)$ must be $-2x^{b+e+4} - 2x^{d+e+5} = -4x^{s+4}$ and the second leading term is some summation of \{2x^{d+e+3}, x^{b+d-e+3}\}. Clearly, the coefficient of the second leading term of $P_{H_2}(b, e; x)$ is not equal to four, so $P_B(r, s; x) \neq P_{H_2}(b, e; x)$, a contradiction. Now we study the first three leading terms of $P_B(r, s; x)$ and $P_{H_2}(b, e; x)$ according to the following three cases.

Suppose that $b + e + 4 < d + e + 5$. Then from (28) we have the leading term of $P_{H_2}(b, d, e; x)$ is $-2x^{d+e+5}$ while that of $P_B(r, s; x)$ is $-2x^{s+4}$. So $s + 4 = d + e + 5$, i.e., $s = d + e + 1$, then from (25) we have $r = b + 1$. Put these two equations into (27) we have $e = \frac{2b+d+4}{d(b+1)}$. Since $b > 6$, $d > 5$ it is easy to see that $1 > \frac{2b+d+4}{d(b+1)} > 0$, contradicting $e > 2$.

Next suppose that $b + e + 4 = d + e + 5$. From (28) we have the leading term of $P_{H_2}(b, d, e; x)$ is $-4x^{d+e+5}$ while that of $P_B(r, s; x)$ is $-2x^{s+4}$, obviously, they are not equal. Thus $P_B(r, s; x) \neq P_{H_2}(b, d, e; x)$, a contradiction.

At last, suppose that $b + e + 4 > d + e + 5$. If $s > r + 2$, then from (29) we have the first and the second leading terms of $P_B(r, s; x)$ are $-2x^{s+4}, 2x^{s+2}$, respectively. From (28) we claim that in $P_{H_2}(b, d, e; x)$, the leading term is $-2x^{b+e+4}$ and the second leading term is some summation of \{-2x^{d+e+5}, x^{b+d-e+3}\}. Clearly, the coefficient of the second leading term of $P_{H_2}(b, e; x)$ may be one of $-2, -1, 1$ and it is not equal to 2. Thus $P_B(r, s; x) \neq P_{H_2}(b, d, e; x)$, a contradiction. If $s = r + 2$, then from (29) we have the first and the second leading terms of $P_B(r, s; x)$ are $-2x^{s+4}, 2x^{s+2} = 2x^{s}$, respectively, and we get a contradiction as the same as the case of $s > r + 2$. If $s = r + 1$, by examining the terms of $P_B(r, s; x)$ we find the first three leading terms are $-2x^{s+4}, -2x^{r+4} = -2x^{s+3}, 2x^{s+2}$, respectively. In this case the first three leading terms of $P_{H_2}(b, d, e; x)$ are $-2x^{b+e+4}, -2x^{d+e+5}, 2x^{d+e+3}$, respectively. Thus $-2x^{s+3} = -2x^{d+e+5}$ and $2x^{s+2} = 2x^{d+e+3}$, which lead to $s = d + e + 2$ and $s = d + e + 1$, a contradiction.  

\begin{lemma}
Let $G = B(r, s) \in B(r, s)$ be an $\infty$-graph of order $n$, $H_3 = H(2, 2, 1, 2, e)$ ($e > 2$) be also of order $n$. Then $G$ is not cospectral with $H_3$.
\end{lemma}

\begin{proof}
Suppose that $G$ and $H_3$ are cospectral, then they share the same number of vertices and the same characteristic polynomials, i.e., $n = r + s + 1 = e + 10 = P_B(\lambda) = P_{H_3}(\lambda)$. Then $P_B(r, s; x) = P_{H_3}(e; x)$. Note that the leading term in $P_B(x)$ is some summation of \{-2x^{s+4}, -2x^{r+4}\}, however, the leading term in $P_{H_3}(e; x)$ is $x^{s+4}$. Clearly, their difference cannot be zero, so $P_B(r, s; x) - P_{H_3}(e; x) \neq 0$, a contradiction.
\end{proof}
Lemma 4.11. Let \( G = B(r, s) \in \mathbb{G}(r, s) \) be an \( \infty \)-graph of order \( n \), \( D = D(f, 0, h, i)(f > 7, h > 6, i > 2) \) be also of order \( n \). Then \( G \) is not cospectral with \( D \).

**Proof.** Suppose that \( G \) and \( D \) are cospectral, then \( P_G(\lambda) = P_D(\lambda) \) and

\[
n = r + s + 1 = f + h + i + 3.
\]  

By Lemma 4.3 and Lemma 4.7 we have \( P_B(r, s; x) = P_D(f, h, i; x) \). Denote by \( P_B''(x) = \frac{d^2P_B(r, s; x)}{dx^2} \) and \( P_D''(x) = \frac{d^2P_D(f, h, i; x)}{dx^2}, \) respectively. Then we have \( P_B''(x) = P_D''(x) \), in particular, \( P_B''(1) = P_D''(1) \). By simple calculation we have

\[
P_B''(1) = r^2 + s^2 - 2rs - 4r - 4s - 5
\]

\[
= f^2 + h^2 + i^2 + 2hi + 2h + 2i - 1 = P_D''(1).
\]  

(31)

Subtract (31) from (30)’s square we have

\[
2rs = fh + fi - h - i - 4.
\]  

(32)

Next by carefully checking the terms of \( P_D(f, h, i; x) \) we find the first several leading terms must come from the following:

\[
-2x^{f+i+5}, -2x^{h+i+6}, 2x^{f+i+3}, 4x^{h+i+4}, x^{f+h+3-i}, x^{f+i+3-h}.
\]  

(33)

First we claim that \( s > r \) (recall that \( s \geq r \)), otherwise suppose that \( s = r \), then by (29), the leading term of \( P_B(r, s; x) \) must be \(-4x^{s+4}\). In this case, the leading term of \( P_D(f, h, i; x) \) must be \(-2x^{f+i+5} - 2x^{h+i+6} = -4x^{s+4}, \) which gives \( s = f + i + 1 = h + i + 2 \), and so \( s = h + 1 \). Moreover from (30) we have \( r + s = 1 + 2 = 2(h + i + 2) = f + h + i + 3 \), this implies \( f = h + i + 2 \) and thus \( s = f, \) this contradicts \( s = f + i + 1 \). Now we study the first three leading terms of \( P_B(r, s; x) \) and \( P_D(f, h, i; x) \) according to the following three cases.

Suppose that \( f + i + 5 < h + i + 6 \). Then from (33) we have the leading term of \( P_B(f, h, i; x) \) is \(-2x^{h+i+6} \) while that of \( P_B(r, s; x) \) is \(-2x^{s+4} \). So \( s + 4 = h + i + 6 \), that is, \( s = h + i + 2 \), then from (30) we have \( r = f \). Put these two equations into (32) we have \( i = -(h + 4), \) a contradiction.

Next suppose that \( f + i + 5 = h + i + 6 \). From (33) we have the leading term of \( P_B(f, h, i; x) \) is \(-4x^{f+i+5} \) while that of \( P_B(r, s; x) \) is \(-2x^{s+4} \), obviously, they are not equal. Thus \( P_B(r, s; x) \neq P_D(f, h, i; x), \) a contradiction.

At last, suppose that \( f + i + 5 > h + i + 6 \). Since \( s > r \), we see that the leading term of \( P_B(r, s; x) \) is \(-2x^{s+4}, \) the leading term of \( P_D(f, h, i; x) \) is \(-2x^{f+i+5} \). Thus we have \( s = f + i + 1 \). Put this equation into (30) we have \( r = h + 1 \). Then by substituting the above two equations into (32) we obtain \( f = \frac{-2hi+3h+3i+6}{i-h-2} \). Since \( f > 0 \) we have \( i > h + 2 \) and thus \( f - h + i + 3 > f + h - i + 3 \). Again by putting the above two equations into (14) we obtain \( P_B(f, h, i; x) \) in the following:

\[
P_B(f, h, i; x) = -2x^{-f-i-1} - 2x^{-1-h} + 2x^{-f-i+1} + x^{-f-i+2-h} + x^{-f-i+3-h} + 2x^{1-h}
\]

\[
+ x^{f+i+h+1} + 2x^{3+h} - 2x^{5+h} + x^{f+i+3+h} + 2x^{f+i+3} - 2x^{f+i+5},
\]

(34)

then from (34) and (22) we obtain the leading term of \( P_B(f, h, i; x) - P_D(f, h, i; x) \) is some summation of \( \{x^{-f-h+i+1}, 2x^{h+i+6}\} \), and it cannot be zero. Thus we have \( P_B(r, s; x) \neq P_D(f, h, i; x), \) a contradiction. \( \square \)

Now, our main result follows from Lemmas 3.3, 4.2, 4.9, 4.10, 4.11 and Theorem 4.8.

**Theorem 4.12.** Let \( G = B(r, s) \in \mathbb{G}(r, s) \) be an \( \infty \)-graph of order \( n \). Then \( G \) is determined by its adjacency spectrum if and only if \( s \neq r + 2 \). If \( s = r + 2 \), then \( G = B(r, r + 2) \) has a unique cospectral mate \( H(1, r - 2, 1, 1, r - 1) \).
Acknowledgments

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A. Appendix

A.1. The number of closed walks of complete graphs

By compiling a program in Matlab, we list the number of some subgraphs (not necessarily induced) of complete graph $K_7$, $K_8$, $K_9$ in Table 4.

Let $A(K_i)$ ($i = 7, 8, 9$) be the adjacency matrices of complete graphs $K_i$ ($i = 7, 8, 9$), respectively. Using Matlab, it is easy to obtain the trace of $A^j(K_i)$ ($i = 7, 8, 9$; $j = 4, \ldots, 8$) in Table 5.

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Table 4
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From Example 1 and Table 5 we see that $N_{K_8}(i)$ is coincident with $tr(A^i(K_8))$ ($i = 4, \ldots, 8$).

**A.2. Counting closed walks of length eight**

Let $G' = H(0, b, 0, d, e) \in \mathbb{H}(a, b, c, d, e)$ ($b > 4, d > 3, e > 1$), we have $|E(G')| = n + 1$, $N_{G'}(P_3) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{2}{3}\right)(n - 4) = n + 5, N_{G'}(K_{1,3}) = \left(\frac{2}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 11$.

$$N_{G'}(P_5) = \begin{cases} \frac{8 + 5 \times 2 + 2 \times 5 + b - 2 + d - 2 + e - 3}{2} = n + 18 & \text{if } e > 2; \\ 8 + 5 \times 2 + 2 \times 4 + b - 2 + d - 2 = n + 17 & \text{if } e = 2. \end{cases}$$

$N_{G'}(G_i) = 5 + 4 + 4 = 13$, by Corollary 2.5, we have

$$N_{G'}(8) = \begin{cases} 70n + 1062 + 16N_{G'}(C_8) & \text{if } e > 2; \\ 70n + 1054 + 16N_{G'}(C_8) & \text{if } e = 2. \end{cases}$$

So we have $N_{G'}(8) = 70n + 1062$ if and only if $b > 4, d > 3, e > 2$ and $N_{G'}(C_8) = 0$.

Let $G' = H(0, 4, c, d, 1) \in \mathbb{H}(a, b, c, d, e)$ ($c > 0, c + d > 3$), we have $|E(G')| = n + 1$, $N_{G'}(P_3) = \left(\frac{2}{3}\right) \times 3 + \left(\frac{2}{3}\right)(n - 4) = n + 5, N_{G'}(K_{1,3}) = \left(\frac{2}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9$.

$$N_{G'}(P_5) = \begin{cases} 1 + 5 \times 2 + 2 \times 6 + c - 2 + d - 2 + 2 = n + 13 & \text{if } c > 1; \\ 1 + 5 \times 2 + 2 \times 4 + 4 + 2 + d - 2 = n + 14 & \text{if } c = 1. \end{cases}$$
\[ N_{G'}(G_i) = 2 + 4 + 4 = 10, \quad N_{G'}(C_6) = 1, \quad N_{G'}(G_r) = 2, \] by Corollary 2.5, we have

\[ N_{G'}(8) = \begin{cases} 
70n + 1038 + 16N_{G'}(C_8) & \text{if } c > 1; \\
70n + 1046 + 16N_{G'}(C_8) & \text{if } c = 1.
\end{cases} \]

So we have \( N_{G'}(8) = 70n + 1062 \) if and only if \( c = 1, d = 4 \) and \( N_{G'}(C_8) = 1 \), but in this case \( |V(G')| = 13 < 17 \leq |V(B(r, s))| \) \((s \geq r > 7)\) and they are not cospectral.

Let \( G' = H(1, 3, c, d, e) \in \mathbb{H}(a, b, c, d, e) \) \((c + d > 2, c > 0, e > 1)\), we have \( |E(G')| = n + 1, N_{G'}(P_2) = \binom{3}{2} \times 3 + \binom{2}{2}(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \binom{3}{3} \times 3 = 3, N_{G'}(P_4) = n + 9, \)

\[ N_{G'}(P_3) = \begin{cases} 
3 \times 3 + 4 + 2 \times 7 + 1 + c - 2 + d - 2 + e - 3 = n + 14 & \text{if } c > 1, e > 2; \\
3 \times 3 + 4 \times 2 + 2 \times 7 + 1 + d - 2 + e - 3 = n + 15 & \text{if } d > c = 1, e > 2; \\
3 \times 3 + 4 + 2 \times 6 + 1 + c - 2 + d - 2 = n + 13 & \text{if } c > 1, e = 2; \\
3 \times 3 + 4 \times 2 + 2 \times 4 + 1 + d - 2 = n + 14 & \text{if } d > c = 1, e = 2.
\end{cases} \]

So we have \( N_{G'}(P_3) \neq 70n + 1062. \)

Let \( G' = H(1, 3, 0, d, 1) \in \mathbb{H}(a, b, c, d, e) \) \((d > 2)\), we have \( |E(G')| = n + 1, N_{G'}(P_3) = \binom{3}{2} \times 3 + \binom{2}{2}(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \binom{3}{3} \times 3 = 3, N_{G'}(P_4) = n + 9, N_{G'}(P_5) = 2 + 5 + 3 + 4 + 2 \times 4 + 1 + d - 2 = n + 13, N_{G'}(G_i) = 3 + 3 + 4 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 2, \) by Corollary 2.5, we have

\[ N_{G'}(8) = 70n + 1038 + 16N_{G'}(C_8). \]

So we have \( N_{G'}(8) \neq 70n + 1062. \)

Let \( G' = H(2, 2, c, d, e) \in \mathbb{H}(a, b, c, d, e) \) \((c > 0, e > 1)\), we have \( |E(G')| = n + 1, N_{G'}(P_3) = \binom{3}{2} \times 3 + \binom{2}{2}(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \binom{3}{3} \times 3 = 3, N_{G'}(P_4) = n + 9, \)

\[ N_{G'}(P_3) = \begin{cases} 
3 \times 3 + 2 \times 9 + c - 2 + d - 2 + e - 3 = n + 13 & \text{if } c > 1, e > 2; \\
3 \times 3 + 2 \times 7 + 4 + d - 2 + e - 3 = n + 14 & \text{if } d > c = 1, e > 2; \\
3 \times 3 + 2 \times 8 + c - 2 + d - 2 = n + 12 & \text{if } c > 1, e = 2; \\
3 \times 3 + 2 \times 6 + 4 + d - 2 = n + 13 & \text{if } d > c = 1, e = 2.
\end{cases} \]

So we have \( N_{G'}(G_i) = 3 \times 3 = 9, N_{G'}(C_6) = 1, N_{G'}(G_r) = 2, \) by Corollary 2.5, we have

\[ N_{G'}(8) = \begin{cases} 
70n + 1022 + 16N_{G'}(C_8) & \text{if } c > 1, e > 2; \\
70n + 1030 + 16N_{G'}(C_8) & \text{if } d > c = 1, e > 2; \\
70n + 1014 + 16N_{G'}(C_8) & \text{if } c > 1, e = 2; \\
70n + 1022 + 16N_{G'}(C_8) & \text{if } d > c = 1, e = 2.
\end{cases} \]
So we have $N_{G'}(8) = 70n + 1062$ if and only if $c = 1, d = 2, e > 2$ and $N_{G'}(C_8) = 2$.

Let $G' = H(2, 2, 0, d, 1) \in \mathbb{H}(a, b, c, d, e) \ (d > 2)$, we have $|E(G')| = n + 1, N_{G'}(P_2) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{3}{2}\right)(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \left(\frac{3}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9, N_{G'}(P_5) = 2 + 3 + 5 + 2 \times 6 + d - 2 = n + 12, N_{G'}(G_i) = 3 + 3 + 4 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 2$, by Corollary 2.5, we have

$$N_{G'}(8) = 70n + 1030 + 16N_{G'}(C_8).$$

So we have $N_{G'}(8) = 70n + 1062$ if and only if $d = 3$ and $N_{G'}(C_8) = 2$, but in this case $|V(G')| = 13 < 17 \leq |V(B_r(s))| (s \geq r > 7)$ and they are not cospectral.

Let $G' = H(0, b, 1, 2, 1) \in \mathbb{H}(a, b, c, d, e) \ (b > 4)$, we have $|E(G')| = n + 1, N_{G'}(P_2) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{3}{2}\right)(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \left(\frac{3}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9, N_{G'}(P_5) = 5 + 5 + 1 + 4 + 2 \times 4 + b - 2 = n + 14, N_{G'}(G_i) = 2 + 4 + 4 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 3$, by Corollary 2.5, we have

$$N_{G'}(8) = 70n + 1062 + 16N_{G'}(C_8).$$

So we have $N_{G'}(8) = 70n + 1062$ if and only if $b > 6$ and $N_{G'}(C_8) = 0$.

Let $G' = H(1, b, 0, 2, 1) \in \mathbb{H}(a, b, c, d, e) \ (b > 3)$, we have $|E(G')| = n + 1, N_{G'}(P_2) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{3}{2}\right)(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \left(\frac{3}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9, N_{G'}(P_5) = 2 + 5 + 3 + 4 + 2 \times 4 + b - 2 = n + 13, N_{G'}(G_i) = 2 + 4 + 4 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 3$, by Corollary 2.5, we have

$$N_{G'}(8) = 70n + 1054 + 16N_{G'}(C_8).$$

So we have $N_{G'}(8) \neq 70n + 1062$.

Let $G' = H(2, b, 0, 0, 1) \in \mathbb{H}(a, b, c, d, e) \ (b > 2)$, we have $|E(G')| = n + 1, N_{G'}(P_2) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{3}{2}\right)(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \left(\frac{3}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9, N_{G'}(P_5) = 2 + 5 + 3 + 4 + 2 \times 4 + b - 2 = n + 13, N_{G'}(G_i) = 2 + 4 + 4 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 3$, by Corollary 2.5, we have

$$N_{G'}(8) = 70n + 1054 + 16N_{G'}(C_8).$$

So we have $N_{G'}(8) \neq 70n + 1062$.

Let $G' = D(5, g, h, 1) \in \mathbb{D}(f, g, h, i) \ (g + h > 4)$, we have $|E(G')| = n + 1, N_{G'}(P_3) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{3}{2}\right)(n - 4) = n + 5, N_{G'}(K_{1, 3}) = \left(\frac{3}{3}\right) \times 3 = 3, N_{G'}(P_4) = n + 9,$

$$N_{G'}(P_5) = \begin{cases} 5 \times 2 + 1 + 2 \times 6 + 3 + g - 2 + h - 2 = n + 13 & \text{if } h \geq g > 1; \\ 5 \times 2 + 1 + 4 + 2 \times 4 + 3 + h - 2 = n + 14 & \text{if } h > 2, g = 1. \end{cases}$$

$N_{G'}(G_i) = 4 \times 2 + 2 = 10, N_{G'}(C_6) = 1, N_{G'}(G_r) = 1$ by Corollary 2.5, we have

$$N_{G'}(8) = \begin{cases} 70n + 1022 + 16N_{G'}(C_8) & \text{if } h \geq g > 1; \\ 70n + 1030 + 16N_{G'}(C_8) & \text{if } h > 2, g = 1. \end{cases}$$

Since $N_{G'}(C_8) \neq 2$, we have $N_{G'}(8) \neq 70n + 1062$. 

Let $G' = D(\ell, 1, 3, 1) \in \mathbb{D}(\ell, g, h, i)$ ($\ell > 5$), we have $|E(G')| = n + 1, N_C'(P_3) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{5}{2}\right)(n - 4) = n + 5, N_C'(K_{1,3}) = \left(\frac{3}{2}\right) \times 3 = 3, N_C'(P_4) = n + 9, N_C'(P_5) = 5 \times 2 + 1 + 4 + 2 \times 4 + 1 + f - 2 = n + 14, N_C'(G_i) = 4 \times 2 + 2 = 10, N_C'(C_6) = 1, N_C'(G_r) = 2$, by Corollary 2.5, we have

$$N_C'(8) = 70n + 1046 + 16N_C'(C_8).$$

So we have $N_C'(8) = 70n + 1062$ if and only if $f = 7$ and $N_C'(C_8) = 1$, but in this case $|V(G')| = 15 < 17 \leq |V(B(r, s))| (s \geq r > 7)$ and they are not cospectral.

Let $G' = D(\ell, 2, 2, 1) \in \mathbb{D}(\ell, g, h, i)$ ($\ell > 5$), we have $|E(G')| = n + 1, N_C'(P_3) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{5}{2}\right)(n - 4) = n + 5, N_C'(K_{1,3}) = \left(\frac{3}{2}\right) \times 3 = 3, N_C'(P_4) = n + 9, N_C'(P_5) = 5 \times 2 + 1 + 2 \times 6 + f - 2 = n + 13, N_C'(G_i) = 4 \times 2 + 2 = 10, N_C'(C_6) = 1, N_C'(G_r) = 2$, by Corollary 2.5, we have

$$N_C'(8) = 70n + 1038 + 16N_C'(C_8).$$

So we have $N_C'(8) \neq 70n + 1062$.

Let $G' = D(\ell, 0, h, i) \in \mathbb{D}(\ell, g, h, i)$ ($\ell > 5, h > 4$), we have $|E(G')| = n + 1, N_C'(P_3) = \left(\frac{3}{2}\right) \times 3 + \left(\frac{1}{2}\right)(n - 4) = n + 5, N_C'(K_{1,3}) = \left(\frac{3}{2}\right) \times 3 = 3, N_C'(P_4) = n + 11,$

$$N_C'(P_5) = \begin{cases} 5 \times 2 + 2 \times 5 + 8 + f - 2 + h - 2 + i - 3 = n + 18 & \text{if } i > 2; \\ 5 \times 2 + 2 \times 4 + 8 + f - 2 + h - 2 = n + 17 & \text{if } i = 2. \end{cases}$$

$$N_C'(G_i) = 4 \times 2 + 5 = 13, N_C'(C_6) = 0, N_C'(G_r) = 0,$$ by Corollary 2.5, we have

$$N_C'(8) = \begin{cases} 70n + 1062 + 16N_C'(C_8) & \text{if } i > 2; \\ 70n + 1054 + 16N_C'(C_8) & \text{if } i = 2. \end{cases}$$

So we have $N_C'(8) = 70n + 1062$ if and only if $f > 7, h > 4, i > 2$ and $N_C'(C_8) = 0$.

References

