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Square roots of semihyponormal operators have scalar extensions

Eungil Ko [a,b,1]

[a] *Department of Mathematics, Ewha Women's University, Seoul 120-750, South Korea*
[b] *Department of Mathematics, Purdue University, W. Lafayette, IN 47907, USA*

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Abstract

In this paper, we study some properties of $(\sqrt{SH})$, i.e., square roots of semihyponormal operators. In particular we show that an operator $T \in (\sqrt{SH})$ has a scalar extension, i.e., is similar to the restriction to an invariant subspace of a (generalized) scalar operator (in the sense of Colojoară–Foiaş). As a corollary, we obtain that an operator $T \in (\sqrt{SH})$ has a nontrivial invariant subspace if its spectrum has interior in the plane.

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1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, complex Hilbert spaces and $\mathcal{L}(\mathcal{H},\mathcal{K})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H}=\mathcal{K}$, we write $\mathcal{L}(\mathcal{H})$ in place of $\mathcal{L}(\mathcal{H},\mathcal{K})$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{ap}(T)$, and $\sigma_e(T)$ for the spectrum, the approximate point spectrum, and the essential spectrum of T, respectively.

An operator T is called p-hyponormal, $0<p\leqslant 1$, if $(T^*T)^p \geqslant (TT^*)^p$ where T^* is the adjoint of T. If $p=1$, T is called hyponormal and if $p=\frac{1}{2}$, T is called semihyponormal. Semihyponormal operators were introduced by Xia (see [14]). There is a vast literature concerning semihyponormal operators. Let (SH) denote the class of semihyponormal

E-mail address: eiko@ewha.ac.kr (E. Ko).
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operators. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *a square root of a semihyponormal operator* if T^2 is semihyponormal. We denote this class by $(\sqrt{SH})$. It is known that if T is hyponormal, then T^2 is semihyponormal (see [1]). Hence every hyponormal operator is contained in $(\sqrt{SH})$. In Example 3.1, we give an example of a square root of a semihyponormal operator which is not semihyponormal. Therefore, this class gives good reasons for the future study.

In this paper, we study some properties of $(\sqrt{SH})$. In particular we show that an operator $T \in (\sqrt{SH})$ has a scalar extension, i.e., is similar to the restriction to an invariant subspace of a (generalized) scalar operator (in the sense of Colojoară–Foiaş). As a corollary, we obtain that an operator $T \in (\sqrt{SH})$ has a nontrivial invariant subspace if its spectrum has interior in the plane.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to satisfy the single valued extension property if for any open set U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U, \mathcal{H}) \to \mathcal{O}(U, \mathcal{H})$$

defined by the obvious pointwise multiplication is one-to-one where $\mathcal{O}(U, \mathcal{H})$ denote the Fréchet space of $\mathcal{H}$-valued analytic functions on U with respect to uniform topology. If T has the single valued extension property, then for any $x \in \mathcal{H}$ there exists a unique maximal open set $\rho_T(x)(\supset \rho(T)$, the resolvent set) and a unique $\mathcal{H}$-valued analytic function f defined in $\rho_T(x)$ such that

$$(z - T)f(z) = x, \quad z \in \rho_T(x).$$

Moreover, if $F \subset \mathbf{C}$ is a closed set and $\sigma_T(x) = \mathbf{C} \backslash \rho_T(x)$, then $H_T(F) = \{x \in \mathcal{H}\colon \sigma_T(x) \subset F\}$ is a linear subspace (not necessarily closed) of $\mathcal{H}$ and obviously $H_T(F) = H_T(F \cap \sigma(T))$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to satisfy the property (β) if for every open subset G of $\mathbf{C}$ and every sequence $f_n : G \to \mathcal{H}$ on $\mathcal{H}$-valued analytic function such that $(z - T)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G.

A bounded linear operator S on $\mathcal{H}$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism,

$$\Phi : C_0^m(\mathbf{C}) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leqslant m \leqslant \infty$. An operator is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

Let z be the coordinate in $\mathbf{C}$ and let $\mathrm{d}\mu(z)$ denote the planar Lebesgue measure. Fix a separable, complex Hilbert space $\mathcal{H}$ and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(U, \mathcal{H})$ the Hilbert space of measurable functions $f : U \to \mathcal{H}$, such that

$$\|f\|_{2,U} = \left\{ \int_U \|f(z)\|^2 \,\mathrm{d}\mu(z) \right\}^{1/2} < \infty.$$

The space of functions $f \in L^2(U, \mathcal{H})$ which are analytic functions in U (i.e., $\bar{\partial} f = 0$) is denoted by

$$A^2(U, \mathcal{H}) = L^2(U, \mathcal{H}) \cap \mathcal{O}(U, \mathcal{H}).$$

$A^2(U, \mathcal{H})$ is called the Bergman space for U.

We will use the following version of Green's formula for the plane, also known as the Cauchy–Pompeiu formula. Define $C^p(\overline{U}, \mathcal{H})$ in the exactly the same way as $C^p(\overline{U})$ except that the functions in the space are now $\mathcal{H}$-valued.

Cauchy–Pompeiu formula 2.1. Let D be an open disc in the plane, let $z \in D$ and $f \in C^2(\overline{D}, \mathcal{H})$. Then

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta + \bar{\partial} f * \left(-\frac{1}{\pi z}\right)$$

where $*$ denotes the convolution product.

Remark 2.2. The function

$$g(z) = \int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

appearing in Cauchy–Pompeiu formula is analytic in D and extends continuously to $\overline{D}$ as can be seen by examining the $\int_D$ term. So, $g \in A^2(D, \mathcal{H})$ for $f \in C^2(\overline{D}, \mathcal{H})$.

Let us define now a special Sobolev type space. Let U again a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The vector valued Sobolev space $W^m(U, \mathcal{H})$ with respect to $\bar{\partial}$ and order m will be the space of those functions $f \in L^2(U, \mathcal{H})$ whose derivatives $\bar{\partial} f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathcal{H})$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \left\|\bar{\partial}^m f\right\|_{2,U}^2$$

$W^m(U, \mathcal{H})$ become a Hilbert space contained continuously in $L^2(U, \mathcal{H})$.

We next discuss the fact concerning the multiplication operator by z on $W^m(U, \mathcal{H})$. The linear operator M of multiplication by z on $W^m(U, \mathcal{H})$ is continuous and it has a spectral distribution of order m, defined by the relation

$$\Phi_M : C_0^m(\mathbf{C}) \to \mathcal{L}\big(W^m(U, \mathcal{H})\big), \quad \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m.

Let $V : W^m(U, \mathcal{H}) \to \bigoplus_0^m L^2(U, \mathcal{H})$ be the operator defined by

$$V(f) = \big(f, \bar{\partial} f, \ldots, \bar{\partial}^m f\big).$$

Since

$$\|Vf\|^2 = \|f\|_{W^m}^2 = \sum_{i=0}^{m} \left\|\bar{\partial}^m f\right\|_{2,U}^2,$$

an operator V is an isometry such that $VM = (\bigoplus_0^m N_z)V$, where N_z is the multiplication operator on $L^2(U,\mathcal{H})$. Since $\bigoplus_0^m N_z$ is normal, M is a subnormal operator.

3. Subscalarity

In this section we show that every square root of a semihyponormal operator has a scalar extension. For this we start with an example of a square root of a semihyponormal operator which is not semihyponormal.

Example 3.1. If T is any semihyponormal operator in $\mathcal{L}(\mathcal{H})$, consider the following operator matrix

$$A = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

Then $A \in (\sqrt{SH})$, but it is easy to show that A is not semihyponormal.

The following lemma is elementary.

Lemma 3.2 [9, Lemma 4.3]. *Let T be in $(\sqrt{SH})$. If $\{f_n\}$ is a sequence in $L^2(D,\mathcal{H})$ such that* $\lim_{n\to\infty} \|(z^2 - T^2) f_n\|_{2,D} = 0$ *for all $z \in D$, then* $\lim_{n\to\infty} \|(z^2 - T^2)^* f_n\|_{2,D} = 0$.

The following proposition is an essential step to prove our main theorem.

Proposition 3.3. *For every bounded disk D in $\mathbf{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$ and $f \in W^4(D,\mathcal{H})$ we have*

$$\big\|(I-P)f\big\|_{2,D} \leqslant C_D \sum_{i=2}^{4} \big\|\big(z^2 - T^2\big)^* \bar\partial^i f\big\|_{2,D},$$

where P denotes the orthogonal projection of $L^2(D,\mathcal{H})$ onto the Bergman space $A^2(D,\mathcal{H})$.

Proof. Let s_1 and s_2 be in $C^\infty(\overline{D},\mathcal{H})$ such that $s_i \equiv 1$ on $D - D$ for $i = 1,2$. Let $f_n \in C^\infty(\overline{D},\mathcal{H})$ be a sequence which approximates f in the norm W^4. Then for a fixed n we have

$$\bar\partial^2\left[f_n - \frac{1}{2}\big(z^2 - T^2\big)^* \bar\partial^2 f_n\right] = \bar\partial^2 f_n - \frac{1}{2}\sum_{k=0}^{2}\binom{2}{k}\bar\partial^k\big(z^2 - T^2\big)^*\bar\partial^{4-k} f_n$$
$$= -\frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\bar\partial^k\big(z^2 - T^2\big)^*\bar\partial^{4-k} f_n. \qquad (1)$$

By Cauchy–Pompeiu formula and Eq. (1), we get

$$\bar{\partial}\left[f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n\right] = \frac{1}{2\pi i} \int_{\partial D} \frac{\bar{\partial}[f_n(\zeta) - \frac{1}{2}(\zeta^2 - T^2)^* \bar{\partial}^2 f_n(\zeta)]}{\zeta - z}\, \mathrm{d}\zeta + \left[-\frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\bar{\partial}^k\left(z^2 - T^2\right)^*\bar{\partial}^{4-k} f_n\right] * \left(-\frac{s_1}{\pi z}\right). \tag{2}$$

Set

$$g_{1,n}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\bar{\partial}[f_n(\zeta) - \frac{1}{2}(\zeta^2 - T^2)^* \bar{\partial}^2 f_n(\zeta)]}{\zeta - z}\, \mathrm{d}\zeta.$$

Then $g_{1,n} \in A^2(D, \mathcal{H})$ by Remark 2.2. Thus

$$\bar{\partial}\left[f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n\right] = g_{1,n} - \left[-\frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\bar{\partial}^k\left(z^2 - T^2\right)^*\bar{\partial}^{4-k} f_n\right] * \left(-\frac{s_1}{\pi z}\right). \tag{3}$$

Again we apply the Cauchy–Pompeiu formula. Then from Eq. (3), we obtain

$$f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta) - \frac{1}{2}(\zeta^2 - T^2)^* \bar{\partial}^2 f_n(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \bar{\partial}\left[f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n\right] * \left(-\frac{s_2}{\pi z}\right). \tag{4}$$

Set

$$g_{2,n}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta) - \frac{1}{2}(\zeta^2 - T^2)^* \bar{\partial}^2 f_n(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

Again, $g_{2,n} \in A^2(D, \mathcal{H})$ by Remark 2.2. Thus from Eq. (4) we get

$$f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n = g_{2,n} + \bar{\partial}\left[f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n\right] * \left(-\frac{s_2}{\pi z}\right) \tag{5}$$

Hence from Eqs. (3) and (5) we obtain

$$f_n - \frac{1}{2}\left(z^2 - T^2\right)^* \bar{\partial}^2 f_n = g_{2,n} + \left[g_{1,n} + \left\{-\frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\bar{\partial}^k\left(z^2 - T^2\right)^*\bar{\partial}^{4-k} f_n\right\} * \left(-\frac{s_1}{\pi z}\right)\right] * \left(-\frac{s_2}{\pi z}\right)$$

$$= g_{2,n} + g_{1,n} * \left(-\frac{s_2}{\pi z}\right) - \left[\frac{1}{2}\sum_{k=0}^{1}\binom{2}{k}\bar{\partial}^k\big(z^2 - T^2\big)^*\bar{\partial}^{4-k} f_n\right] * \left(-\frac{s_1}{\pi z}\right) * \left(-\frac{s_2}{\pi z}\right). \tag{6}$$

Since

$$\bar{\partial}^k\big(z^2 - T^2\big)^*\bar{\partial}^{4-k} f_n * \left(-\frac{s_1}{\pi z}\right) = \bar{\partial}^{k-1}\big(z^2 - T^2\big)^*\bar{\partial}^{4-k} f_n - \bar{\partial}^{k-1}\big(z^2 - T^2\big)^*\bar{\partial}^{4-k+1} f_n * \left(-\frac{s_1}{\pi z}\right),$$

from Eq. (6) we have

$$f_n - \frac{1}{2}\big(z^2 - T^2\big)^*\bar{\partial}^2 f_n = g_{2,n} + g_{1,n} * \left(-\frac{s_2}{\pi z}\right) + \frac{1}{2}\big(z^2 - T^2\big)^*\bar{\partial}^4 f_n * \left(-\frac{s_1}{\pi z}\right) * \left(-\frac{s_2}{\pi z}\right) - \big(z^2 - T^2\big)^*\bar{\partial}^3 f_n * \left(-\frac{s_2}{\pi z}\right).$$

Set $g_n = g_{2,n} + g_{1,n} * (-s_2/\pi z)$. Since $g_{1,n} * (-s_2/\pi z)$ is analytic, $g_n \in A^2(D, \mathcal{H})$. Hence

$$f_n - g_n = \frac{1}{2}\big(z^2 - T^2\big)^*\bar{\partial}^2 f_n - \big(z^2 - T^2\big)^*\bar{\partial}^3 f_n * \left(-\frac{s_2}{\pi z}\right) + \frac{1}{2}\big(z^2 - T^2\big)^*\bar{\partial}^4 f_n * \left(-\frac{s_1}{\pi z}\right) * \left(-\frac{s_2}{\pi z}\right).$$

Taking the norm, we get

$$\|f_n - g_n\|_{2,D} \leqslant \frac{1}{2}\big\|\big(z^2 - T^2\big)^*\bar{\partial}^2 f_n\big\|_{2,D} + \big\|\big(z^2 - T^2\big)^*\bar{\partial}^3 f_n\big\|_{2,D}\left\|-\frac{s_2}{\pi z}\right\|_{2,D} + \frac{1}{2}\big\|\big(z^2 - T^2\big)^*\bar{\partial}^4 f_n\big\|_{2,D}\left\|\left(-\frac{s_1}{\pi z}\right) * \left(-\frac{s_2}{\pi z}\right)\right\|_{2,D}.$$

Set $C_D = \max\{1/2,\ \|-s_2/\pi z\|_{2,D},\ 1/2\|(-s_1/\pi z) * (-s_2/\pi z)\|_{2,D}\}$. Then

$$\|f - g\|_{2,D} \leqslant \|f - f_n\|_{2,D} + \|f_n - g_n\|_{2,D} \leqslant \|f - f_n\|_{2,D} + C_D\sum_{i=2}^{4}\big\|\big(z^2 - T^2\big)^*\bar{\partial}^i f\big\|_{2,D}.$$

By passing to the limit we conclude

$$\|f - Pf\|_{2,D} \leqslant C_D\sum_{i=2}^{4}\big\|\big(z^2 - T^2\big)^*\bar{\partial}^i f\big\|_{2,D}.$$

So we complete our proof. □

Lemma 3.4. *Let T be in $(\sqrt{SH})$. Then for a bounded disk D which contains $\sigma(T)$, the operator $V:\mathcal{H}\to H(D)$ defined by*

$$Vh=\widetilde{1\otimes h}\quad\bigl(=1\otimes h+\overline{(z-T)W^4(D,\mathcal{H})}\bigr)$$

is one-to-one and has closed range, where $H(D)=W^4(D,\mathcal{H})/\overline{(z-T)W^4(D,\mathcal{H})}$ and $1\otimes h$ denotes the constant function sending any $z\in D$ to h.

Proof. Let $h_i\in\mathcal{H}$ and $f_i\in W^4(D,\mathcal{H})$ be sequences such that

$$\lim_{i\to\infty}\bigl\|(z-T)f_i+1\otimes h_i\bigr\|_{W^4}=0. \tag{7}$$

Then by the definition of the norm of Sobolev space, Eq. (7) implies

$$\lim_{i\to\infty}\bigl\|(z-T)\bar\partial^j f_i\bigr\|_{2,D}=0 \tag{8}$$

for $j=1,2,3,4$. From Eq. (8), we get

$$\lim_{i\to\infty}\bigl\|\bigl(z^2-T^2\bigr)\bar\partial^j f_i\bigr\|_{2,D}=0$$

for $j=1,2,3,4$. Since T^2 is semihyponormal, by Lemma 3.2

$$\lim_{i\to\infty}\bigl\|\bigl(z^2-T^2\bigr)^*\bar\partial^j f_i\bigr\|_{2,D}=0 \tag{9}$$

for $j=1,2,3,4$. Then by Proposition 3.3, we have

$$\lim_{i\to\infty}\bigl\|(I-P)f_i\bigr\|_{2,D}=0 \tag{10}$$

where P denotes the orthogonal projection of $L^2(D,\mathcal{H})$ onto $A^2(D,\mathcal{H})$. By (7) and (10), we have

$$\lim_{i\to\infty}\bigl\|(z-T)Pf_i+1\otimes h_i\bigr\|_{2,D}=0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z\in\Gamma$

$$\lim_{i\to\infty}\bigl\|Pf_i(z)+(z-T)^{-1}(1\otimes h_i)\bigr\|=0$$

uniformly. Hence, by Riesz–Dunford functional calculus,

$$\lim_{i\to\infty}\left\|\frac{1}{2\pi i}\int_\Gamma Pf_i(z)\,\mathrm{d}z+h_i\right\|=0.$$

But since $\int_\Gamma Pf_i(z)\,\mathrm{d}z=0$ by Cauchy's theorem, $\lim_{i\to\infty}h_i=0$. □

Now we are ready to prove our main theorem.

Theorem 3.5. *An operator $T\in(\sqrt{SH})$ is subscalar of order* 4.

Proof. Consider an arbitrary bounded open disk D in $\mathbf{C}$ which contains $\sigma(T)$ and the quotient space

$$H(D)=W^4(D,\mathcal{H})/\overline{(z-T)W^4(D,\mathcal{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Let $M(=M_z)$ be the multiplication operator by z on $W^4(D,\mathcal{H})$. Then M is a scalar operator of order 4 and its spectral distribution is

$$\Phi : C_0^4(\mathbf{C}) \to \mathcal{L}\big(W^4(D,\mathcal{H})\big), \quad \Phi(f) = M_f,$$

where M_f is the multiplication operator with f. Since M commutes with $z-T$, $\widetilde{M}$ on $H(D)$ is still a scalar operator of order 4, with $\widetilde{\Phi}$ as a spectral distribution.

Let V be the operator

$$Vh = \widetilde{1 \otimes h} \quad \big(= 1 \otimes h + \overline{(z-T)W^4(D,\mathcal{H})}\big),$$

from $\mathcal{H}$ into $H(D)$, denoting by $1 \otimes h$ the constant function h. Then $VT = \widetilde{M}V$. Since V is one-to-one and has closed range by Lemma 3.4, T is subscalar of order 4. □

Recall that if U is a non-empty open set in $\mathbf{C}$ and if $\Omega \subset U$ has the property that

$$\sup_{\lambda\in\Omega} |f(\lambda)| = \sup_{\beta\in U} |f(\beta)|$$

for every function f in $H^\infty(U)$ (i.e. for all f bounded and analytic on U), then Ω is said to be dominating for U.

Corollary 3.6. *Let T be in $(\sqrt{SH})$. If $\sigma(T)$ has the property that there exists some non-empty open set U such that $\sigma(T)\cap U$ is dominating for U, then T has a nontrivial invariant subspace.*

Proof. This follows from Theorem 3.5 and [5]. □

The following corollary shows that, exactly as for subnormal operators, the spectrum $\sigma(T)$ is obtained from $\sigma(\widetilde{M})$ by filling some bounded connected components of $\mathbf{C}\backslash\sigma(\widetilde{M})$.

Corollary 3.7. *Let T be in $(\sqrt{SH})$. With the same notation of the proof of Theorem* 3.5*,*

$$\partial\sigma(T) \subset \sigma(\widetilde{M}) \subset \sigma(T).$$

Proof. Since

$$\sigma(\widetilde{M}) \subset \sigma(M|_{W^4(D,\mathcal{H})}) \subset \overline{D},$$

we conclude $\sigma(\widetilde{M}) \subset \sigma(T)$. Since $\partial\sigma(T) \subset \sigma_{ap}(T)$ and $\sigma_{ap}(T) \subset \sigma_{ap}(\widetilde{M})$, we complete the proof. □

Corollary 3.8. *Let T be in $(\sqrt{SH})$ and let f be a function analytic in a neighborhood of $\sigma(T)$. Then $f(T)$ is subscalar.*

Proof. With the same notation of the proof of Theorem 3.5, $Vf(T) = f(\widetilde{M})V$, where $f \to f(T)$ is the functional calculus morphism. The result follows from the fact that $f(\widetilde{M})$ is scalar. □

Recall that an $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be a quasiaffine transform of an operator $T \in \mathcal{L}(\mathcal{K})$ there exists a quasiaffinity $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $XA = TX$. Furthermore, operators A and T are said to be quasisimilar if there are quasiaffinities X and Y such that $XA = TX$ and $AY = YT$.

Lemma 3.9. *If $T \in (\sqrt{SH})$ is quasinilpotent, then it is nilpotent.*

Proof. Since $\sigma(T) = \{0\}$, from Corollary 3.7 $\widetilde{M}$ is quasinilpotent. Then by [3], $\widetilde{M}$ is nilpotent. Since V is one-to-one and $VT = \widetilde{M}V$, T is nilpotent. □

Theorem 3.10. *Let T be in $(\sqrt{SH})$. If A is a quasiaffine transform of T and $\sigma(A) = \{0\}$, then A is subscalar.*

Proof. Assume there exists a one-to-one X with dense range such that $XA = TX$. Then since T is subscalar by Theorem 3.5 it follows from [8] that $\sigma(T) \subset \sigma(A)$. Hence T is quasinilpotent. By Lemma 3.9, T is nilpotent, say $T^n = 0$. Then $XA^n = 0$. Since X is one-to-one, A is a nilpotent operator of order n. Therefore, A is a subscalar operator by [7]. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *power regular* if $\lim_{n\to\infty} \|T^n x\|^{1/n}$ exists for every $x \in \mathcal{H}$ (see [2]).

Theorem 3.11. *If T is in $(\sqrt{SH})$, it is power regular.*

Proof. It is known from Theorem 3.5 that every square root of a semihyponormal operator is the restriction of a scalar operator to one of its invariant subspace. Since a scalar operator is power regular and the restriction of power regular operators to their invariant subspaces clearly remains power regular, every square root of a semihyponormal operator is power regular. □

Theorem 3.12. *If T is in $(\sqrt{SH})$, it satisfies the property (β). Hence it satisfies the single valued extension property.*

Proof. Since every scalar operator satisfies the property (β) and the property (β) is transmitted from an operator to its restriction to closed invariant subspaces, it follows from Theorem 3.5 that every square root of a semihyponormal operator satisfies the property (β). Hence it satisfies the single valued extension property. □

Corollary 3.13. *Let A and T be in $(\sqrt{SH})$. If they are quasisimilar, then $\sigma(A) = \sigma(T)$ and $\sigma_e(A) = \sigma_e(T)$.*

Proof. Since A and T satisfy the property by Theorem 3.12, the proof follows from [12]. □

Corollary 3.14. *If T be in $(\sqrt{SH})$, then $H_T(F) = \{x \in \mathcal{H}\colon\ \sigma_T(x) \subset F\}$ is a closed subspace for every closed set F in* **C** *and*

$$\sigma(T|_{H_T(F)}) \subset \sigma(T) \cap F.$$

Proof. The first statement follows from Theorem 3.12 and [11, Lemma 5.2]. The second statement follows from Theorem 3.12 and [3, Proposition 3.8]. □

Corollary 3.15. *If T_1 and T_2 are in $(\sqrt{SH})$ and $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ satisfies $AT_1 = T_2A$, then $AH_{T_1}(F) \subset H_{T_2}(F)$ for every closed set $F \subset$* **C**.

Proof. It is known that T_1 and T_2 satisfy the single valued extension property from Theorem 3.12. If $x \in H_{T_1}(F)$, then $\sigma_{T_1}(x) \subset F$. Thus $F^c \subset \rho_{T_1}(x)$. Hence there exists an analytic $\mathcal{H}$-valued function f defined on F^c such that

$$(z - T_1)f(z) \equiv x, \quad z \in F^c.$$

Therefore,

$$(z - T_2)Af(z) = A(z - T_1)f(z) \equiv Ax, \quad z \in F^c.$$

Since $Af : F^c \to \mathcal{K}$ is analytic, $F^c \subset \rho_{T_2}(Ax)$. Thus $Ax \in H_{T_2}(F)$. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* if T can be written as a sum $T = T_0 + K$ where T_0 is a triangular operator and K is a compact operator in $\mathcal{L}(\mathcal{H})$. Moreover, T is called *biquasitriangular* if both T and T^* are quasitriangular.

Corollary 3.16. *Let T be in $(\sqrt{SH})$. If T has no nontrivial invariant subspace, then T is biquasitriangular.*

Proof. If T has no nontrivial invariant subspace, then $\sigma_p(T^*) = \phi$. Hence T^* satisfies the single valued extension property. Since T also satisfies the single valued extension property by Theorem 3.12, T is biquasitriangular from [10, Theorem 2.3.21]. □

Theorem 3.17. *Let $T \in (\sqrt{SH})$. Then there exists a positive integer I such that for all positive integers $i \geqslant I$, T^{2i} has a nontrivial invariant subspace.*

Proof. Since T^2 is semihyponormal, the result follows from [6]. □

The following theorem explains the structure of some square roots of semihyponormal operators.

Theorem 3.18. *Let T be in $(\sqrt{SH})$. If T is compact or $m(\sigma(T)) = 0$ where m is the planar Lebesgue measure, then*

$$T = A \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix},$$

where A and B are normal and C is a positive one-to-one operator commuting with B.

Proof. If T is compact, then T^2 is compact and semihyponormal. By [4], T^2 is normal. If $m(\sigma(T)) = 0$ where m is the planar Lebesgue measure, then T^2 is normal by [4]. Since T^2 is normal in any cases, by [13, Theorem 1]

$$T = A \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix},$$

where A and B are normal and C is a positive one-to-one operator commuting with B. □

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