MATHEMATICS

CLOSED SET COUNTABILITY AXIOMS [1)]

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Introduction

In this paper countable base axioms for open sets containing closed sets are introduced as follows.

Definition 1. *A topological space* $(X, \mathscr{T})$ *satisfies* D_1 *if every closed set has a countable base for the open sets containing it. A set* M *has a countable base for the open sets containing it, if there exists a family of open sets* $\{G_n\}$, $M \subset G_n$ *for each* n *such that if* V *is an open set,* $M \subset V$, *then there exists* n *such that* $G_n \subset V$.

Definition 2. *A topological space* $(X, \mathscr{T})$ *satisfies* D_2 *if it has a countable base* $\{U_n\}$ *such that each closed set has a countable base for the open sets containing it which is a subfamily of* $\{U_n\}$.

It will be proved that regular D_1 spaces are perfectly normal and collectionwise normal, perfectly normal countably compact spaces satisfy D_1; T_3, D_1 spaces with a finite number of isolated points are countably compact and T_3, D_1 spaces are metrizable iff the topology has a ploint countable base. Metrizable spaces that are the union of a compact set and isolated points are characterized by the D_1 property, and a T_3 space is compact and metrizable iff it satisfies D_2. Unless otherwise noted, the definitions of KELLEY [9] will be used. When "a" is used as a subscript, it is understood that "a" is a member of an index set A. M^* will mean the complement of M. $\bar{M}^*$ means the complement of M closure whereas $\overline{M^*}$ means the closure of the complement of M.

Clearly D_1 and D_2 are closely related to the first and second countability axioms, C_1 and C_2, respectively; the D_1 axiom is obtained by replacing points by closed sets in the statement of C_1. The relations are expressed in the following theorem.

Theorem 1. *Every* D_1, T_1, *space satisfies* C_1. *Every* D_2 *space satisfies* C_2 *and* D_1.

Relation to perfectly normal spaces

By a sequence of theorems, we will show that regular D_1 spaces are perfectly normal and countably compact perfectly normal spaces satisfy D_1.

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1) Part of research done at Kent State University.

Theorem 2. *In a D_1, T_1, space, every closed set is a G_δ. In a regular D_1 space, every closed set is the intersection of a countable number of closed neighborhoods.*

Proof: Since in a T_1 space, every set is the intersection of open sets, the first statement is immediate. Let $\{G_n\}$ be the countable base for the open sets containing a closed set F in a regular space. For each $x \notin F$, there is an open set $G(x)$ such that $x \notin \overline{G(x)}$ and $F \subset G(x)$. $F = \cap \overline{G(x)} = \cap \overline{G_n}$.

ISHIKAWA [8] has proved that a topological space is countably paracompact (countably metacompact) iff for every decreasing sequence $\{F_n\}$ of closed sets with vacuous intersection, there exists a decreasing sequence of open sets $\{G_n\}$, $F_n \subset G_n$ such that $\cap \overline{G}_n = \phi (\cap G_n = \phi)$. For alternative proofs see HAYASHI, [6] and [7]. As a consequence of these results and Theorem 2, we have,

Theorem 3. *Every regular space such that the closed sets are the intersection of a countable number of closed neighborhoods is countably paracompact. Every D_1 regular (T_1) space is countably paracompact (countably metacompact).*

Proof: Let $\{F_n\}$ be a family of closed sets such that $\cap F_n = \phi$ and let $(X, \mathscr{T})$ be D_1 and T_1. Let $\{G_{mn}\}$ be a G_δ for F_n. Set $G_n = \bigcap_{m=1}^{n} \bigcap_{k=1}^{n} G_{mk}$. $F_n \cap G_n$ and $\cap G_n = \phi$; so $(X, \mathscr{T})$ is countably metacompact. A similar proof shows that a regular D_1 space is countably paracompact.

Theorem 4. *Every regular D_1 space is normal. Every regular space such that the closed sets are the intersection of a countable number of closed neighborhoods is normal.*

Proof: Let F and B be disjoint closed sets of $(X, \mathscr{T})$. Let $\{C_n\}$ be a countable family of closed neighborhoods such that $F = \cap C_n$. $\{C_n{}^*\}$ covers B. $\{C_n{}^*\}$ and B^* cover X and have an open locally finite refinement by Theorem 3. Let $\{V_a\}$ be the subfamily of this refinement intersecting B. Since each V_a is contained in some $C_n{}^*$, $\overline{V}_a \cap F = \phi$ for each a. The disjoint open sets containing F and B are $(\cup \overline{V}_a)^* = (\overline{\cup V_a})^*$ and $\cup V_a$ respectively. The second statement is proved and the first statement follows from Theorem 2.

Corollary 4. *Topological spaces satisfying either of the conditions of Theorem 4 are perfectly normal.*

It might be noted that a perfectly normal space may be characterized as a regular space such that closed sets are the intersection of a countable number of closed neighborhoods. All perfectly normal spaces, however are not D_1. For instance the real line with the usual topology is not D_1, since the set of integers do not have a countable base for the open sets containing them. Here it is proved that a perfectly normal countably compact space is D_1. This is analogous to a theorem of ALEXANDROFF [1] that a regular countably compact space such that every closed set is a G_δ satisfies C_1.

Theorem 5. *Every perfectly normal, countably compact space is* D_1.

Proof: From the original G_δ, $\{U_n\}$, containing a closed set F construct a G_δ, $\{V_n\}$ such that $\overline{V}_n \subset U_n$. Then construct a nested G_δ, $\{W_n\}$ such that $W_n = \bigcap_{k=1}^{n} V_k$. It can be shown that $\{W_n\}$ is the desired base for the open sets containing F. Let T be an open set containing F. $\{\overline{W}_n{}^*\}$ is an open cover of T^* and may be replaced by a finite subcover. Let k be the largest subscript of this subcover. Then $T^* \subset \overline{W}_k{}^*$ so that $W_k \subset T$. Later we will prove a partial converse to this theorem.

Collectionwise normality of D_1 *spaces*

Collectionwise normality was introduced by BING [4].

Definition 3. *A topological space is collectionwise normal if for every locally−discrete family of sets* $\{N_a\}$, *there is a family of pairwise disjoint open sets* $\{V_a\}$ *such that* $M_a \subset V_a$.

We will need two preliminary theorems, before we show the collectionwise normality of D_1 spaces.

Theorem 6. *Let* $\{M_n\}$ *be a countable locally−discrete family of sets in a normal space. Then, there is a family of pairwise disjoint open sets* $\{W_n\}$ *such that* $M_n \subset W_n$. [1])

Proof: Let $P_n = \cup \{M_k : k \neq n\}$. There exists U_n and V_n such that $\overline{M_n} \subset U_n$ and $\overline{P_n} \subset V_n$. Let $W_1 = U_1$; otherwise let $W_n = U_n \cap \bigcap_{k=1}^{n-1} V_k$. $\{W_n\}$ is the desired family of pairwise disjoint open sets.

Theorem 7. *Let* $\{F_a\}$ *be a closed locally discrete family of sets in a regular* D_1 *space,* $(X, \mathcal{T})$. *Then all but a finite number of* $\{F_a\}$ *are open.*

Proof: Assume there is a denumerably infinite family $\{F_n\}$ of closed locally discrete sets in X such that no F_n is open. By the normality of X there is a denumerable family of pairwise open sets $\{G_n\}$ such that $F_n \subset G_n$. Let $F = \cup F_n$. F is closed and we will show that the assumption that there is a countable base for the open sets containing F leads to a contradiction. There is no restriction in assuming that the base is nested. Let $\{U_k\}$ be a countable nested base for the open sets containing F. Set $V_{nk} = G_n \cap U_k$. Let $\{W_{nk}\}$ consist of the distinct V_{nk} for each n. $\{W_{nk}\}$ is infinite since F_n is not open. Set $W_k = \cup W_{nk}$. If $\{U_k\}$ is a countable base for the open sets containing F, then $\{W_k\}$ is also countable base. The set $T = \bigcup_{i=1}^{\infty} W_{ii}$ does not contain any W_k. Hence all but a finite number of F_n are open and the theorem is proved.

[1]) A slight modification of a Theorem of K. ISEKI.

Theorem 8. *Every regular D_1 space is collectionwise normal.*

Proof: Theorems 6 and 7.

ARHANGELSKII [2] has shown that every, perfectly normal T_1, collectionwise normal space with a σ-point finite base is metrizable.

Corollary 8. *A D_1, T_3 space with a σ-point finite base is metrizable.*

Proof: Theorems 4 and 8.

Later we will show that the last condition may be replaced by a point-countable base.

From theorems 5 and 8 it follows that every countably compact perfectly normal space is collectionwise normal. However one may obtain a better result from the following characterization.

Theorem 9. *A T_4 space is countably compact iff it has no infinite locally discrete families.*

Corollary 9. *Every T_4 countably compact space is collectionwise normal.*

Relation of D_1 to countably compactness

We now consider a partial converse to Theorem 5. The author is indebted to P. Doyle and D. Fisk for the theorem that a D_1 connected metric space is compact. Theorem 7 and the next theorem depend heavily on their methods.

Theorem 10. *A T_3, D_1 topological space with at most a finite number of isolated points is countably compact. Every T_3, D_1 space is the union of a countably compact set and isolated points.*

Proof: In a T_1 space that is not countably compact, there is a denumerably infinite locally discrete set of points. Theorem 7 shows that this space has an infinite number of isolated points.

Corollary 10. *A perfectly normal T_1 space with a finite number of isolated points is countably compact iff it is D_1.*

Spaces with a point-countable base

Recently there has been a renewed interest in topological spaces with a point-countable base. MISCENKO [10] showed that a T_2 compact space with a point-countable base is metrizable, the point-countable base being countable. CORSON and MICHAEL [5] showed that the compact condition may be replaced by countably compact.

Definition 4. *A topological space has a point-countable base if there is a base for the topology such that every point is in a countable number of members of the base.*

Before proving a metrization theorem for D_1 spaces, we will prove a theorem about compact sets of a space with a point-countable base.

Theorem 11. *In a T_2 space $(X, \mathscr{T})$ with a point-countable base each compact set has a countable base for the open sets containing it.*

Proof: Let M be a compact set; M is also closed. The subspace M will also have a point-countable base and hence will have a countable base. In the topology for X, the members of the point countable base containing points of M will also be countable. By the compactness of M, any open set containing M will contain finite unions of members of the base which will in turn contain M. These finite unions form a countable base for the open sets containing M.

Theorem 12. *Let a point-countable T_3 space $(X, \mathscr{T})$ be the union of a compact set and isolated points. Then $(X, \mathscr{T})$ satisfies D_1 and is metrizable.*

Proof: Let M be the compact set, and F a closed set. Set

$$F = (F \cap M) \cup (F \cap M^*).$$

$F \cap M$ has a countable base $\{U_n\}$ by Theorem 11, for the open set containing $F \cap M$. $\{(F \cap M^*) \cup U_n\}$ is then a countable base for the open sets containing F. Since F is an arbitrary closed set $(x, \mathscr{T})$ is D_1. A base for the topology for X consists of the members of the point-countable base intersecting M and the isolated points. The base for the topology consists of a countable family and of isolated points. Clearly this is a σ-point finite family and is hence metrizable, by Corollary 8.

Corollary 12. *A D_1, T_3 space is metrizable iff it has a point-countable base. Furthermore the D_1 property characterizes the metrizable spaces that are the union of a compact set and isolated points.*

Metrizable spaces that are D_1 may not even be locally compact. Example. *Let X be the real line. Let all points except "0" be isolated. Let a base for the topology consist of the isolated points and open intervals with rational end points containing "0".*

This space is the union of a compact set, [0], and isolated points, but no neighborhood of [0] is compact.

As there are compact perfectly normal T_1 spaces that are not metrizable, not every T_3, D_1 space is metrizable.

D_2 spaces

Unlike D_1 spaces, it will be shown that every D_2, T_3 space is not only metrizable, but also compact.

Theorem 13. *A T_3 topological space $(X, \mathscr{T})$ is metrizable and compact iff it satisfies D_2.*

Proof: Let $\{U_n\}$ be a countable base for $(X, \mathscr{T})$. Construct a new countable base $\{V_k\}$ consisting of all finite unions of members of $\{U_n\}$. Let F be a closed set and let G be an open set containing F. For each

$x \in F$, there exists U_n such that $x \in U_n \subset G$. Since F is compact a finite number of these U_n cover F and hence $\{V_k\}$ will form a base for the open sets containing F. Thus a metrizable compact space satisfies D_2.

A D_2, T_3 space is clearly metrizable by Theorem 1. Since it is also D_1, it is the union of a compact set and isolated points. Let M be the compact set. If there is an infinite closed set consisting of the union of isolated points not contained in M, a non-denumerable family of open sets will be needed to insure that every closed set has a countable base for the open sets containing it. Thus there is no infinite closed set consisting of the union of isolated points not contained in M. Then any cover of M covers all but a finite number of points of X and hence X is compact.

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