ELSEVIER International Journal of Approximate Reasoning 26 (2001) 157–159
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
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Discussion

Comments to "Paradoxes of fuzzy logic, revisited"

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Received 1 November 1999; received in revised form 1 November 2000; accepted 1 November 2000

What follows are concrete comments to [3] concerning the main goal of [2], the study in Fuzzy Set Theory of the law:

$$\neg(P_1 \wedge \neg P_2) \equiv P_2 \vee (\neg P_1 \wedge \neg P_2). \tag{$*$}$$

(i) Provided the system in [1] contains P_0 such that $t(P_0) = 0$ (if not, what?), (*) yields $\mathrm{Max}(t(P),\ 1 - t(P)) = 1$ for any assertion P. Hence, from the very beginning the system can only contain assertions whose truth values are in $\{0, 1\}$ and obviously, either $t(P_1) = t(P_2)$ or $t(P_1) = 1 - t(P_2)$ for any P_1, P_2. With "numerical" truth values Elkan's theoretical argument is a triviality that says nothing on Fuzzy Logic. But posed with fuzzy sets, as it was done in Ref. [11] in [2], the question is not so trivial and deserves to be reconsidered.

The system in [1], although called there "formal system", is one whose "form" (structure, initial laws) is actually hidden. Hence, it is unknown how to make inferences within it, and easy to impose much simpler laws than (*) to obtain only the numerical truth values 0 and 1.

(ii) In [3], Prof. Elkan asserts that his theorem is not about fuzzy sets but on something simpler and of great importance: fuzzy truth values attached to individual logical assertions. In [1,3] he also asserts that his theorem applies to fuzzy set theory, and that provided $A(x), B(x), \ldots$ refer to truth values of

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0888-613X/01/$ - see front matter
PII: S0888-613X(00)00066-9

individual assertions, our Theorem 1 is correct. Since "fuzzy truth values" means "numerical truth values" and "individual" before "logical assertions" means nothing, what is the *meaning* of Elkan's claims?

There is a difference between the interpretation of $t(A)$ and $A(x)$ that Elkan did not point out. The system in [1] just supposes that "A logically equivalent to B" *implies* $t(A) = t(B)$, but in any theory of fuzzy sets what is supposed is "$A = B$" *if and only if* $A(x) = B(x)$ for all x. Consequently, provided P_1, P_2 are interpreted as fuzzy sets in $([0,1]^X$, Min, Max, $1 - \mathrm{i}d)$, (*) is *equivalent* to our inequality. Only the hypothesis that (*) does hold simultaneously for any pair (P_1, P_2) in $\{(A,B),(A',B),(A,B'),(A',B'),(B,A),(B',A),(B,A'),(B',A')\}$ transforms our inequalities for the symmetrical case $B \cap B' \leqslant A$ and $A \cap A' \leqslant B$ into the equality $B \cap B' = A$.

(iii) Elkan is right when in [3] he claims that the assertions in [2], "But in Elkan context, both statements "$A \to B$ is the same as $B \to A$", etc", are not in [1]. Nevertheless, it is rather obvious that these quotation marks do not refer to Elkan's but to *our* words as given in paragraph 2.2 in [2] where, when quoting Elkan, we not only use quotation marks but also the expression "to the letter".

An implicative interpretation of (*) is not reached for all the implications used in Fuzzy Logic. For example, with Mandani's or Larsen's implications $P_1 \to P_2$ is neither equivalent to $P_2 \vee (\neg P_1 \wedge \neg P_2)$ nor to $\neg(P_1 \vee \neg P_2)$ and many applications of Fuzzy Logic are made with these implications.

(iv) Section 3 in [2] does *not* concern generalized versions of fuzzy set theory having "nothing to do with" (see [3] paper [1]). On the contrary, this section studies (*) in lattices with an involution and, in particular, Boole and DeMorgan Algebras. And the theory of fuzzy sets obtained with Min, Max and $1 - \mathrm{i}d$ is a DeMorgan Algebra.

In a DeMorgan Algebra (*) is equivalent to $P_2 \wedge (\neg P_2) \leqslant P_1$. Hence, in Boolean Algebras (*) is a law since $P_2 \wedge (\neg P_2) = 0$. It seems that in most interesting lattices with involution the validity of (*) as a law, depends on the Excluded-Middle Principle.

(v) Section 4 in [2] does *not* concern generalized versions of fuzzy set theory, at least as they are understood since 1980. It concerns standard theories (only one is a lattice) and it is shown that there the validity of (*) as a law is more closely related to the duality principles than to the Excluded-Middle Principle.

Actually, generalized theories are those introduced in [4]. They have only one self-contradiction, verify the Excluded-Middle Principle, and it is not difficult to see in which of them (*) is a law.

(vi) Certainly, Section 5 in [2] deals with some generalized fuzzy set theories (obtained by mixing standard ones) that, in our view, are of interest when dealing with predicates on several universes of discourse whose "logics" are not coincidental, as often happens in applications. Theorem 3 in this section is completely proven in Ref. [13] given in [2], currently under publication in [5]. Nevertheless, in Ref. [7] in [2], a good part of that proof is published and it is enough to see the clear objective of Section 5: there are uncountable many fuzzy set theories where (*) is a law.

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