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Leray–Schauder results for inward acyclic and approximable maps defined on Fréchet spaces

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Abstract

New Leray–Schauder type results are presented for inward type mappings defined on a Fréchet space E. The proofs rely on Leray–Schauder results in Banach spaces and viewing E as the projective limit of a sequence of Banach spaces.

Keywords: Inward; Acyclic; Approximable; Fixed point theory

1. Introduction

This work presents a continuation theorem for inward type mappings defined between Fréchet spaces. The theory is based on the notion of an essential map and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n \in N}$ (here $N = \{1, 2, \ldots\}$). The usual continuation theory in the non-normable situation is rarely of interest from an application viewpoint (this point seems to be overlooked by many authors) since the set constructed is usually open and bounded and so has empty interior. We present results for inward acyclic and inward approximable maps. Also we present results for inward Kakutani–Mönch type maps.

For the remainder of this section we present some definitions and some known facts. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values.

Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V)-approximate continuous selection [1] of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

We say $F : X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U, V)-approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 respectively.

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Let Q be a subset of a Hausdorff topological space X and $x \in X$. The *inward set* $I_Q(x)$ is defined by

$$I_Q(x) = \{x + r(y - x) : y \in Q, r \geq 0\}.$$

If Q is convex and $x \in Q$ then

$$I_Q(x) = x + \{r(y - x) : y \in Q, r \geq 1\}.$$

Let (X, d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by (here $A \in \Omega_X$)

$$\alpha(A) = \inf\{r > 0 : A \subseteq \cup_{i=1}^{n} A_i \text{ and } diam(A_i) \leq r\}.$$

Let S be a nonempty subset of X. For each $x \in X$, define $d(x, S) = \inf_{y \in S} d(x, y)$. We say a set is countably bounded if it is countable and bounded. Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i) countably k-set contractive (here $k \geq 0$) if $G(S)$ is bounded and $\alpha(G(W)) \leq k\alpha(W)$ for all countably bounded sets W of S,
(ii) countably condensing if $G(S)$ is bounded, G is countably 1-set contractive and $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \neq 0$,
(iii) hemicompact if each sequence $\{x_n\}_{n \in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [2].

Theorem 1.1. *Let (Y, d) be a metric space, D a nonempty, complete subset of Y, and $G : D \to 2^Y$ a countably condensing map. Then G is hemicompact.*

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall \alpha, \beta \in I, \alpha \leq \beta\right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [3, p. 439] $\cap_{\alpha \in I} E_\alpha$).

Existence in Section 2 is based on the following continuation theory for $AcAp$ maps. A map is said to be $AcAp$ if it is either acyclic or approximable. In our next definitions E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$ and $0 \in U$. In our definitions $\overline{U}$ and ∂U denote the closure and the boundary of U in C respectively.

Definition 1.1. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is a closed $AcAp$ countably condensing map with $F(x) \subseteq I_C(x)$ for $x \in \overline{U}$.

Definition 1.2. A map $F \in A_{\partial U}(\overline{U}, E)$if $F \in A(\overline{U}, E)$ with $x \notin Fx$ for $x \in \partial U$.

Definition 1.3. A map $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ if for every $G \in A_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in Gx$.

The following result was established in [4].

Theorem 1.2. *Let E, C, U_0, U be as above, $0 \in U$ and $F \in A(\overline{U}, E)$ with*

$$x \notin \lambda Fx \quad \text{for } x \in \partial U \quad \text{and} \quad \lambda \in (0, 1]. \tag{1.1}$$

Then F is essential in $A_{\partial U}(\overline{U}, E)$.

Remark 1.1. The proof of Theorem 1.2 is based on the fact that the zero map is essential in $A_{\partial U}(\overline{U}, E)$ and $F \cong 0$ in $A_{\partial U}(\overline{U}, E)$.

Remark 1.2. If the map F in Theorem 1.2 was Kakutani then in fact $F(x) \subseteq I_C(x)$ for $x \in \overline{U}$ could be replaced by $F(x) \cap I_C(x) \neq \emptyset$ for $x \in \overline{U}$; see [6] for details.

We will also consider maps which are more general than countably condensing maps, i.e. we will discuss Kakutani–Mönch maps. Let E, C, U_0 and U be as in Definition 1.1 and let $0 \in U$.

Definition 1.4. We say $F \in K(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is upper semicontinuous, $F(\overline{U})$ is bounded, $F(x) \subseteq I_C(x)$ for $x \in \overline{U}$, and if $D \subseteq E$ with $D \subseteq co(\{0\} \cup F(D \cap U))$ and $\overline{D} = \overline{B}$ with $B \subseteq D$ countable then $\overline{D \cap U}$ is compact; here $CK(E)$ denotes the family of nonempty convex compact subsets of E.

Definition 1.5. A map $F \in K_{\partial U}(\overline{U}, E)$ if $F \in K(\overline{U}, E)$ with $x \notin Fx$ for $x \in \partial U$.

Definition 1.6. A map $F \in K_{\partial U}(\overline{U}, E)$ is essential in $K_{\partial U}(\overline{U}, E)$ if for every $G \in K_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in Gx$.

The following result was established in [5].

Theorem 1.3. *Let E, C, U_0, U be as before in Definition 1.1, $0 \in U$ and $F \in K(\overline{U}, E)$ with (1.1) holding. Then F is essential in $K_{\partial U}(\overline{U}, E)$. In fact there exists a compact set Σ of $\overline{U}$ and a $x \in \Sigma$ with $x \in Fx$.*

Remark 1.3. In [5] we showed there exists $x \in U$ with $x \in Fx$. Of course

$$x \in \Sigma = \{y \in \overline{U} : y \in Fy\}.$$

We now show that Σ is compact. First notice Σ is closed since F is upper semicontinuous. Now let $\{y_n\}_1^\infty$ be a sequence in Σ, and let $C = \{y_n\}_1^\infty$. Notice C is countable and $C \subseteq co(\{0\} \cup F(C \cap U))$ since $y_n \in Fy_n \subseteq F(C \cap U)$; note $y_n \in U$ from (1.1). Now since $F \in K(\overline{U}, E)$ we have (take $D = C$) that $\overline{C \cap U}$ is compact (so sequentially compact). Thus there exists a subsequence N_1 of N and a $y \in \overline{C \cap U}$ with $y_n \to y$ as $n \to \infty$ in N_1. This together with $y_n \in Fy_n$ and the upper semicontinuity of F guarantees that $y \in Fy$, so $y \in \overline{\Sigma} = \Sigma$. Consequently Σ is sequentially compact, so compact.

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots \quad \text{for every } x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. With E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied, the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \quad \text{there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii) Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq \cdots \quad \text{and for each } n \in N, |x|_n \le |x|_{n+1} \ \forall x \in E_{n+1}. \tag{2.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [3]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\mu_{n,m} j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [6]

$$\textit{pseudo-int}(X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = \textit{pseudo-int}(X)$.

We begin with a result for Volterra type operators.

Theorem 2.1. *Let E and E_n be as described above, C a closed convex subset of E and V a bounded pseudo-open subset of E. Let $U = V \cap C$ with $0 \in U$, and $F : \overline{U} \to 2^E$ (here 2^E denotes the family of nonempty subsets of E). Suppose the following conditions are satisfied:*

$$\begin{cases} \text{for each } n \in N,\ F : \overline{U_n} \to K(E_n) \text{ is a} \\ \text{closed AcAp countably condensing map; here} \\ U_n = V_n \cap \overline{C_n} \text{ and } \overline{U_n} \text{ denotes the} \\ \text{closure of } U_n \text{ in } \overline{C_n} \end{cases} \tag{2.5}$$

$$\text{for each } n \in N,\ F(x) \subseteq I_{\overline{C_n}}(x) \text{ for each } x \in \overline{U_n} \tag{2.6}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin \lambda F y \text{ in } E_n \text{ for all} \\ \lambda \in (0,1) \text{ and } y \in \partial U_n; \text{ here } \partial U_n \\ \text{denotes the boundary of } U_n \text{ in } \overline{C_n} \end{cases} \tag{2.7}$$

and

$$\begin{cases} \text{for each } n \in \{2,3,\ldots\} \text{ if } y \in \overline{U_n} \text{ solves } y \in Fy \text{ in } E_n \\ \text{then } y \in \overline{U_k} \text{ for } k \in \{1,\ldots,n-1\}. \end{cases} \tag{2.8}$$

Then F has a fixed point in E.

Remark 2.2. If the map F in (2.5) was Kakutani for each $n \in N$, then for each $n \in N$, $F(x) \subseteq I_{\overline{C_n}}(x)$ for $x \in \overline{U_n}$ can be replaced by $F(x) \cap I_{\overline{C_n}}(x) \ne \emptyset$ for $x \in \overline{U_n}$ (see Remark 1.2).

Proof. Fix $n \in N$. We would like to apply Theorem 1.2. To do so we need to show

$$\overline{C_n} \ \text{ is convex} \tag{2.9}$$

and

$$V_n \ \text{ is a bounded open subset of } \ E_n \ \text{ and } \ 0 \in U_n. \tag{2.10}$$

First we check (2.9). To see this let $\hat{x}, \hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x} + (1-\lambda)\hat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$ so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex. Thus (2.9) holds.

Now since V is pseudo-open and $0 \in V$ then $0 \in$ *pseudo-int* V so $0 = j_n\mu_n(0) \in \overline{V_n} \setminus \partial V_n$ (here $\overline{V_n}$ and ∂V_n denote the closure and boundary of V_n in E_n respectively). Of course

$$\overline{V_n} \setminus \partial V_n = (V_n \cup \partial V_n) \setminus \partial V_n = V_n \setminus \partial V_n$$

so $0 \in V_n \setminus \partial V_n$, and in particular $0 \in V_n$. Thus $0 \in V_n \cap \overline{C_n} = U_n$. Next notice V_n is bounded since V is bounded (note if $y \in V_n$ then there exists $x \in V$ with $y = j_n\mu_n(x)$). It remains to show V_n is open. First notice $V_n \subseteq \overline{V_n} \setminus \partial V_n$ since if $y \in V_n$ then there exists $x \in V$ with $y = j_n\mu_n(x)$ and this together with $V =$ *pseudo-int* V yields $j_n\mu_n(x) \in \overline{V_n} \setminus \partial V_n$, i.e. $y \in \overline{V_n} \setminus \partial V_n$. In addition notice

$$\overline{V_n} \setminus \partial V_n = (int\, V_n \cup \partial V_n) \setminus \partial V_n = int\, V_n \setminus \partial V_n = int\, V_n$$

since $int V_n \cap \partial V_n = \emptyset$. Consequently

$$V_n \subseteq \overline{V_n} \setminus \partial V_n = int V_n, \text{ so } V_n = int V_n.$$

As a result V_n is open in E_n. Thus (2.10) holds.

Theorem 1.2 guarantees that there exists $y_n \in \overline{U_n}$ with $y_n \in F y_n$. Let us look at $\{y_n\}_{n \in N}$. Notice $y_1 \in \overline{U_1}$ and $y_k \in \overline{U_1}$ for $k \in N \setminus \{1\}$ from (2.8). As a result $y_n \in \overline{U_1}$ for $n \in N$, $y_n \in F y_n$ in E_n together with (2.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{U_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \overline{U_2}$ for $n \in N_1$ together with (2.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{U_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (2.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \ldots, \qquad N_k^\star \subseteq \{k, k+1, \ldots\}$$

and $z_k \in \overline{U_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \ldots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : \overline{U_k} \to K(E_k)$ is closed (note $y_n \in \overline{U_k}$ for $n \in N_k$) implies $y \in F y$ in E_k. We can do this for each $k \in N$ so as a result we have $y \in F y$ in E. □

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (2.16).

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Let $U = V \cap C$ with $0 \in U$, and $F : \overline{U} \to 2^E$. Suppose the following conditions are satisfied:*

$$\overline{U_1} \supseteq \overline{U_2} \supseteq \cdots \tag{2.11}$$

$$\begin{cases} \text{for each } n \in N,\ F_n : \overline{U_n} \to K(E_n) \text{ is a} \\ \text{closed AcAp countably condensing map} \end{cases} \tag{2.12}$$

$$\text{for each } n \in N,\ F_n(x) \subseteq I_{\overline{C_n}}(x) \text{ for each } x \in \overline{U_n} \tag{2.13}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin \lambda F_n y \text{ in } E_n \text{ for all} \\ \lambda \in (0,1) \text{ and } y \in \partial U_n \end{cases} \tag{2.14}$$

$$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{U_n} \to 2^{E_n}, \text{ given by} \\ q\mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \text{ (see Remark 2.3), is} \\ \text{countablycondensing} \end{cases} \tag{2.15}$$

and

$$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in \overline{U_n} \text{ and } y_n \in F_n y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \ldots.\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in Fw \text{ in } E. \end{cases} \tag{2.16}$$

Then F has a fixed point in E.

Remark 2.3. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{U_n}$ and $y \notin \overline{U_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{U_{n+1}}$ and $y \notin \overline{U_{n+2}}$ then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Theorem 1.2 guarantees that there exists $y_n \in \overline{U_n}$ with $y_n \in F_n y_n$ in E_n. Let us look at $\{y_n\}_{n\in N}$. Now Theorem 1.1 (with $Y = E_1$, $G = \mathcal{K}_1$, $D = \overline{U_1}$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n \in F_n y_n$ in E_n; here $d_1(x, Z) = \inf_{y\in Z} |x - y|_1$ for $Z \subseteq Y$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Now Theorem 1.1 (with $Y = E_2$, $G = \mathcal{K}_2$ and $D = \overline{U_2}$) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (2.16) implies $y \in Fy$ in E. □

Now we consider Mönch maps.

Theorem 2.3. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Let $U = V \cap C$ with $0 \in U$, and $F : \overline{U} \to 2^E$. Suppose the following conditions are satisfied:*

$$\begin{cases} \text{for each } n \in N,\ F : \overline{U_n} \to CK(E_n) \text{ is} \\ \text{upper semicontinuous and } F(\overline{U_n}) \text{ is bounded} \end{cases} \tag{2.17}$$

$$\begin{cases} \text{for each } n \in N,\ D \subseteq E_n \text{ with } D \subseteq co(\{0\} \cup F(D \cap U_n)) \\ \text{and } \overline{D} = \overline{B} \text{ with } B \subseteq D \text{ countable implies} \\ \overline{D \cap U_n} \text{ is compact} \end{cases} \tag{2.18}$$

$$\text{for each } n \in N,\ F(x) \subseteq I_{\overline{C_n}}(x) \text{ for each } x \in \overline{U_n} \tag{2.19}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin \lambda F y \text{ in } E_n \text{ for all} \\ \lambda \in (0, 1] \text{ and } y \in \partial U_n \end{cases} \tag{2.20}$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \overline{U_n} \text{ solves } y \in Fy \text{ in } E_n \\ \text{then } y \in \overline{U_k} \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \tag{2.21}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Let $\Sigma_n = \{x \in \overline{U_n} : x \in Fx \text{ in } E_n\}$. Now Theorem 1.3 (see Remark 1.3) guarantees there exists $y_n \in \Sigma_n$ with $y_n \in F y_n$. Let us look at $\{y_n\}_{n\in N}$. Now $y_1 \in \Sigma_1$. Also $y_k \in \Sigma_1$ for $k \in N \setminus \{1\}$ since $y_k \in \overline{U_1}$ from (2.21) (see also (2.4)). As a result $y_n \in \Sigma_1$ for $n \in N$ and since Σ_1 is compact (see Remark 1.3) there exists a subsequence $N_1^\star$ of N and a $z_1 \in \Sigma_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \Sigma_2$ for $n \in N_1$ so there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \Sigma_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (2.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in \Sigma_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Essentially the same reasoning as in Theorem 2.1 guarantees that $y \in Fy$ in E. □

Theorem 2.4. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E and V a bounded pseudo-open subset of E. Let $U = V \cap C$ with $0 \in U$, and $F : \overline{U} \to 2^E$. Suppose the following conditions are satisfied:*

$$\overline{U_1} \supseteq \overline{U_2} \supseteq \cdots \tag{2.22}$$

$$\begin{cases} \textit{for each } n \in N,\ F_n : \overline{U_n} \to CK(E_n) \textit{ is} \\ \textit{upper semicontinuous and } F_n(\overline{U_n}) \textit{ is bounded} \end{cases} \tag{2.23}$$

$$\begin{cases} \textit{for each } n \in N,\ D \subseteq E_n \textit{ with } D \subseteq co(\{0\} \cup F_n(D \cap U_n)) \\ \textit{and } \overline{D} = \overline{B} \textit{ with } B \subseteq D \textit{ countable implies} \\ \overline{D \cap U_n} \textit{ is compact} \end{cases} \tag{2.24}$$

$$\textit{for each } n \in N,\ F_n(x) \subseteq I_{\overline{C_n}}(x) \textit{ for each } x \in \overline{U_n} \tag{2.25}$$

$$\begin{cases} \textit{for each } n \in N,\ y \notin \lambda F_n y \textit{ in } E_n \textit{ for all} \\ \lambda \in (0,1] \textit{ and } y \in \partial U_n \end{cases} \tag{2.26}$$

$$\begin{cases} \textit{for each } n \in N,\ \textit{the map } \mathcal{K}_n : \overline{U_n} \to 2^{E_n},\ \textit{given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \textit{ satisfies if } C \subseteq \overline{U_n} \textit{ is} \\ \textit{countable with } C \subseteq \mathcal{K}_n(C) \textit{ then } \overline{C} \textit{ is compact} \end{cases} \tag{2.27}$$

and

$$\begin{cases} \textit{if there exists a } w \in E \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in \overline{U_n} \textit{ and } y_n \in F_n y_n \text{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \ldots\} \textit{ of } N \textit{ with } y_n \to w \textit{ in } E_k \\ \textit{as } n \to \infty \textit{ in } S,\ \textit{then } w \in Fw \textit{ in } E. \end{cases} \tag{2.28}$$

Then F has a fixed point in X.

Proof. Fix $n \in N$. Let $\Sigma_n = \{x \in \overline{U_n} : x \in F_n x \text{ in } E_n\}$. Now Theorem 1.3 guarantees that there exists $y_n \in \Sigma_n$ with $y_n \in F_n y_n$ in E_n. Let us look at $\{y_n\}_{n \in N}$. Note $y_n \in \overline{U_1}$ for $n \in N$ from (2.22). Now with $C = \{y_n\}_1^{\infty}$ we have from assumption (2.27) that $\overline{C}(\subseteq E_1)$ is compact; note $y_n \in \mathcal{K}_1(y_n)$ in E_1 for each $n \in N$. Thus there exists a subsequence $N_1^{\star}$ of N and a $z_1 \in \overline{U_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^{\star}$. Let $N_1 = N_1^{\star} \setminus \{1\}$. Proceed inductively to obtain subsequences of integers $N_1^{\star} \supseteq N_2^{\star} \supseteq \ldots,\ N_k^{\star} \subseteq \{k, k+1, \ldots\}$ and $z_k \in \overline{U_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^{\star}$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^{\star} \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Essentially the same reasoning as in Theorem 2.2 guarantees that $y \in Fy$ in E. □

References

[1] H. Ben-El-Mechaiekh, P. Deguire, Approachability and fixed points for nonconvex set valued maps, J. Math. Anal. Appl. 170 (1992) 477–500.
[2] R.P. Agarwal, M. Frigon, D. O'Regan, A survey of recent fixed point theory in Fréchet spaces, in: Nonlinear Analysis and Applications: To V. Lakshmikantham on His 80th Birthday, vol. 1, Kluwer Acad. Publ., Dordrecht, 2003, pp. 75–88.
[3] L.V. Kantorovich, G.P. Akilov, Functional Analysis in Normed Spaces, Pergamon Press, Oxford, 1964.
[4] D. O'Regan, Homotopy and Leray–Schauder type results for admissible inward multimaps, J. Concrete Appl. Math. 2 (2004) 67–76.
[5] R.P. Agarwal, J.H. Dshalalow, D. O'Regan, Leray–Schauder principles for inward Kakutani Mönch type maps, Nonlinear Funct. Anal. Appl. 10 (2005) 325–330.
[6] M. Frigon, Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations, Bull. Soc. Math. Belgique 9 (2002) 23–37.