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# Tournaments and colouring

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A *tournament* is a complete graph with its edges directed, and *colouring* a tournament means partitioning its vertex set into transitive subtournaments. For some tournaments $H$ there exists $c$ such that every tournament not containing $H$ as a subtournament has chromatic number at most $c$ (we call such a tournament $H$ a *hero*); for instance, all tournaments with at most four vertices are heroes. In this paper we explicitly describe all heroes.



## 1. Introduction

A *tournament* is a digraph such that for every two distinct vertices $u, v$ there is exactly one edge with ends $\{u, v\}$ (so, either the edge $uv$ or $vu$ but not both), and in this paper, all tournaments are finite. If $G$ is a tournament, we say $X \subseteq V(G)$ is *transitive* if the subtournament $G|X$ induced on $X$ has no directed cycle. If $k \geqslant 0$, a *k-colouring* of a tournament $G$ means a map $\phi : V(G) \to \{1, \dots, k\}$, such that for $1 \leqslant i \leqslant k$, the subset $\{v \in V(G)\colon\ \phi(v) = i\}$ is transitive. The *chromatic number* $\chi(G)$ of a tournament $G$ is the minimum $k$ such that $G$ admits a $k$-colouring.

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If $G, H$ are tournaments, we say $G$ *contains* $H$ if $H$ is isomorphic to a subtournament of $G$, and otherwise $G$ is $H$*-free*. Let us say a tournament $H$ is a *hero* if there exists $c$ (depending on $H$) such that every $H$-free tournament has chromatic number at most $c$. Thus for instance, the cyclic triangle is a hero; every tournament not containing it is 1-colourable.

Incidentally, one could ask the same question for graphs; for which graphs $H$ is it true that all graphs not containing $H$ as an induced subgraph have bounded chromatic number? But it is easy to see that the only such graphs are the cliques with at most two vertices, so this question is not interesting. (That is, not interesting if we only exclude one graph; but excluding a *set* of graphs is a different matter. See [3].) For tournaments, on the other hand, the question is interesting, as we shall see. Evidently we have

**1.1.** *Every subtournament of a hero is a hero.*

Our objective is to find all heroes explicitly, but to state our main result we need some more definitions. We denote by $T_k$ the transitive tournament with $k$ vertices. If $G$ is a tournament and $X, Y$ are disjoint subsets of $V(G)$, and every vertex in $X$ is adjacent to every vertex in $Y$, we write $X \Rightarrow Y$. We write $v \Rightarrow Y$ for $\{v\} \Rightarrow Y$, and $X \Rightarrow v$ for $X \Rightarrow \{v\}$. If $G$ is a tournament and $(X, Y, Z)$ is a partition of $V(G)$ into nonempty sets satisfying $X \Rightarrow Y$, $Y \Rightarrow Z$, and $Z \Rightarrow X$, we call $(X, Y, Z)$ a *trisection* of $G$. If $A, B, C, G$ are tournaments, and there is a trisection $(X, Y, Z)$ of $G$ such that $G|X, G|Y, G|Z$ are isomorphic to $A, B, C$ respectively, we write $G = \Delta(A, B, C)$. It is convenient to write $k$ for $T_k$ here, so for instance $\Delta(1, 1, 1)$ means $\Delta(T_1, T_1, T_1)$, and $\Delta(H, 1, k)$ means $\Delta(H, T_1, T_k)$. A tournament is *strong* if it is strongly connected. Now we can state our main result, the following.

**1.2.** *A tournament is a hero if and only if all its strong components are heroes. A strong tournament with more than one vertex is a hero if and only if it equals $\Delta(H, k, 1)$ or $\Delta(H, 1, k)$ for some hero $H$ and some integer $k \geqslant 1$.*

One could also ask for a weaker property; let us say a tournament $H$ is a *celebrity* if there exists $c > 0$ such that every $H$-free tournament $G$ has a transitive subset of cardinality at least $c|V(G)|$. Evidently every hero is a celebrity; but we shall prove the converse as well. Thus we have:

**1.3.** *A tournament is a celebrity if and only if it is a hero.*

This suggests a connection with the Erdős–Hajnal conjecture [5]. For $0 \leqslant \epsilon \leqslant 1$, let us say a tournament $H$ is $\epsilon$*-timid* if there exists $c$ such that $\chi(G) \leqslant c|V(G)|^{\epsilon}$ for every $H$-free tournament $G$. Thus the 0-timid tournaments are the heroes. The Erdős–Hajnal conjecture is equivalent [1] to the following.

**1.4 Conjecture.** *For every tournament $H$, there exists $\epsilon < 1$ such that $H$ is $\epsilon$-timid.*

This remains open.

## 2. Tournaments with large chromatic number

We begin with two constructions of tournaments with large chromatic number. Every hero has to be a subtournament of both of them, and this criterion severely restricts the possibilities for heroes (indeed, we shall see that every tournament that meets this criterion is indeed a hero). The two constructions are contained in the proofs of 2.1 and 2.3.

**2.1.** *If $H$ is a strong hero with at least two vertices then $H = \Delta(P, Q, 1)$ for some choice of non-null heroes $P, Q$.*

**Proof.** Define a sequence $S_i$ $(i \geqslant 1)$ of tournaments as follows. $S_1$ is the one-vertex tournament. Inductively, for $i \geqslant 2$, let $S_i = \Delta(S_{i-1}, S_{i-1}, 1)$.

(1) *For* $i \geqslant 1$, $\chi(S_i) \geqslant i$.

We prove this by induction on $i$, and may assume that $i > 1$. Let $T = S_i$, and let $(X, Y, Z)$ be a trisection of $T$ such that $T|X$ and $T|Y$ are both isomorphic to $S_{i-1}$ and $|Z| = 1$. Let $Z = \{z\}$. Suppose that there is an $(i-1)$-colouring $\phi$ of $T$; and let $\phi(z) = i-1$ say. Since $S_{i-1}$ does not admit an $(i-2)$-colouring, from the inductive hypothesis, there exists $x \in X$ with $\phi(x) = i-1$, and similarly there exists $y \in Y$ with $\phi(y) = i-1$. But $T|\{x, y, z\}$ is a cyclic triangle, contradicting that $\{v \in V(T)\colon \phi(v) = i-1\}$ is transitive. This proves (1).

From (1) and since $H$ is a hero, there exists $i \geqslant 1$, minimum such that $S_i$ contains $H$. Since $H$ has at least two vertices, it follows that $i > 1$. Let $T = S_i$, and let $(X, Y, Z)$ be a trisection of $S_i$ such that $T|X, T|Y$ are both isomorphic to $S_{i-1}$, and $|Z| = 1$. Choose $W \subseteq V(T)$ such that $T|W$ is isomorphic to $H$. Since $S_{i-1}$ does not contain $H$ it follows that $W \nsubseteq X$ and $W \nsubseteq Y$; and since $H$ is strong, it follows that $W$ has nonempty intersection with each of $X, Y, Z$. Since $|Z| = 1$ it follows that $|W \cap Z| = 1$. But then $(W \cap X, W \cap Y, W \cap Z)$ is a trisection of $T|W$, and since $T|W$ is isomorphic to $H$, it follows that $H = \Delta(P, Q, 1)$ where $P = T|(W \cap X)$ and $Q = T|(W \cap Y)$. Both $P, Q$ are heroes by 1.1. This proves 2.1. $\square$

Let $(v_1, \dots, v_n)$ be an enumeration of $V(G)$, for a tournament $G$. If $v_i v_j$ is an edge of $G$ and $j < i$ we call $v_i v_j$ a *backedge* (under the given enumeration). Let $B$ be the graph with vertex set $V(G)$ in which for $1 \leqslant i < j \leqslant n$, $v_i$ and $v_j$ are adjacent in $B$ if and only if $v_j v_i$ is an edge of $G$. We call $B$ the *backedge graph*. We need the following lemma. (The *girth* of a graph is the length of its shortest cycle, or infinity for a forest.)

**2.2.** *Let $G$ be a tournament and let $(v_1, \dots, v_n)$ be an enumeration of $V(G)$, with backedge graph $B$. If $B$ has girth at least four, and $W \subseteq V(G)$ is transitive in $G$ then $W$ is the union of two stable sets of $B$.*

**Proof.** We may assume that $W = V(G)$. Let $X$ be the set of vertices $v \in W$ that are not the head of any backedge, and let $Y$ be the set that are not the tail of any backedge. Thus $X, Y$ are both stable sets of the backedge graph. Suppose that there exists $v \in W \setminus (X \cup Y)$; and so there exist $u, w \in W$ such that $uv, vw$ are both backedges. Since $W$ is transitive it follows that $uw$ is an edge of $G$, and hence a backedge; but then the backedge graph has a cycle of length three, a contradiction. This proves that $X \cup Y = W$, and therefore proves 2.2. $\square$

If $G$ is a tournament, we denote by $\alpha(G)$ the cardinality of the largest transitive subset of $V(G)$.

**2.3.** *If $H$ is a celebrity, then its vertex set can be numbered $\{v_1, \dots, v_n\}$ in such a way that the backedge graph is a forest.*

**Proof.** By a theorem of Erdős [4], for every $k \geqslant 0$ there is a graph $G_k$, such that every stable set $A$ of $G_k$ satisfies $|A| < |V(G)|/(2k)$, and in which every cycle has more than $\max(3, |V(H)|)$ vertices. (In this paper, all graphs are finite and simple.) Number the vertices of $G_k$ in some arbitrary order, say $\{v_1, \dots, v_n\}$. Let $R_k$ be the tournament with vertex set $V(G_k)$, in which for $1 \leqslant i < j \leqslant n$, $v_j v_i$ is an edge of $R_k$ if $v_i, v_j$ are adjacent in $G_k$, and otherwise $v_i v_j$ is an edge of $R_k$. Thus $G_k$ is the backedge graph of $R_k$ under the enumeration $(v_1, \dots, v_n)$.

(1) $\alpha(R_k) < |V(R_k)|/k$.

Every set transitive in $R_k$ is the union of two stable sets of $G_k$, by 2.2, since $G_k$ has girth at least four; and so $G_k$ has a stable set $A$ of cardinality at least $\alpha(R_k)/2$. Since $|A| < |V(G)|/(2k)$ from the choice of $G_k$, this proves (1).

Since $H$ is a celebrity, there exists $k$ such that $R_k$ contains $H$; let $R_k|X$ be isomorphic to $H$. Let $(v_1, \dots, v_n)$ be the enumeration of $V(G_k)$ used to construct $R_k$. Now $G_k|X$ is a forest, since $|X| = |V(H)|$, and every cycle of $G_k$ has more than $|V(H)|$ vertices. But $G_k|X$ is the backedge graph of $R_k|X$ under the enumeration of its vertex set induced by $(v_1, \dots, v_n)$; and so there is an enumeration of the vertex set of $H$ such that its backedge graph is a forest. This proves 2.3. $\square$

We only need 2.3 for one application, the following. Let $C_3$ denote the tournament $\Delta(1, 1, 1)$.

**2.4.** *Every celebrity is two-colourable, and hence $\Delta(C_3, C_3, 1)$ is not a celebrity.*

**Proof.** Let $H$ be a celebrity. By 2.3 we can enumerate its vertex set $(v_1, \dots, v_n)$ so that the backedge graph $B$ is a forest and hence $V(H)$ is the union of two stable sets of $B$. But every stable set of $B$ is transitive in $H$, and so $H$ is two-colourable. Since $\Delta(C_3, C_3, 1)$ is not two-colourable, this proves 2.4. $\square$

This allows us to strengthen 2.1 as follows.

**2.5.** *If $H$ is a strong hero with at least two vertices then $H = \Delta(J, k, 1)$ or $H = \Delta(J, 1, k)$ for some non-null hero $J$ and for some $k \geqslant 1$.*

**Proof.** By 2.1, there are non-null heroes $P, Q$ such that $H = \Delta(P, Q, 1)$. But $H$ does not contain $\Delta(C_3, C_3, 1)$, since $\Delta(C_3, C_3, 1)$ is not a celebrity (by 2.4) and therefore not a hero; and so one of $P, Q$ is transitive. This proves 2.5. $\square$

Incidentally, is the following true?

**2.6 Conjecture.** *For all $k \geqslant 0$ there exists $c$ such that, if $G$ is a tournament in which the set of out-neighbours of each vertex has chromatic number at most $k$, then $\chi(G) \leqslant c$.*

We were unable to decide this even for $k = 2$.

## 3. Strong components of heroes

In this section we prove the first assertion of 1.2, the following:

**3.1.** *A tournament is a hero if and only if all its strong components are heroes.*

The "only if" assertion is clear by 1.1. To prove the "if" assertion, it is enough to prove that if $H_1, H_2$ are heroes then $H_1 \Rightarrow H_2$ is a hero. (If $H_1, H_2$ are tournaments, $H_1 \Rightarrow H_2$ denotes a tournament $G$ such that $X \Rightarrow Y$ and $G|X$, $G|Y$ are isomorphic to $H_1, H_2$ respectively, for some partition $(X, Y)$ of $V(G)$.) For an application later in the paper, it is helpful to prove a more general result. If $\mathcal{H}$ is a set of tournaments, we say a tournament $G$ is *$\mathcal{H}$-free* if no subtournament of $G$ is isomorphic to a member of $\mathcal{H}$. If $\mathcal{H}_1$, $\mathcal{H}_2$ are two sets of tournaments, the set

$$\{H_1 \Rightarrow H_2\colon\ H_1 \in \mathcal{H}_1,\ H_2 \in \mathcal{H}_2\}$$

is denoted by $\mathcal{H}_1 \Rightarrow \mathcal{H}_2$. We shall prove the following, which immediately implies 3.1.

**3.2.** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be sets of tournaments, such that every member of $\mathcal{H}_1 \cup \mathcal{H}_2$ has at most $c$ vertices, where $c \geqslant 3$. Let $G$ be an $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free tournament, such that for $i = 1, 2$, every $\mathcal{H}_i$-free subtournament of $G$ has chromatic number at most $c$. Then*

$$\chi(G) \leqslant (2c)^{4c^2}.$$

The proof of 3.2 is by means of a double induction on the values of $r, s$ such that $G$ contains an "$(r, s)$-clique", so next we define this. If $e = uv$ is an edge of a tournament, $C(e)$ denotes the set of all vertices $w \neq u, v$ such that $w$ is adjacent to $u$ and adjacent from $v$. For $r \geqslant 1$, we define an *$r$-mountain* in a tournament $G$, and an *$r$-heavy edge*, and an *$(r, s)$-clique*, inductively on $r$ as follows. A 1-mountain is a one-vertex tournament. For $r \geqslant 1$,

- an edge $e$ is *$r$-heavy* if $G|C(e)$ contains an $r$-mountain;
- an *$(r, s)$-clique* of $G$ is a subset $X \subseteq V(G)$ such that $|X| = s$, and every edge of $G|X$ is $r$-heavy in $G$;
- an *$(r + 1)$-mountain* in $G$ is a minimal subset $M \subseteq V(G)$ such that the tournament $S = G|M$ contains an $(r, r + 1)$-clique (of $S$).

(Note that in the third bullet we are not just requiring that $M$ include an $(r, r + 1)$-clique of $G$; the edges of the clique must be $r$-heavy in $S$, not just in $G$.) Thus a 2-mountain is a copy of $\Delta(1, 1, 1)$. We observe:

**3.3.** *Every $r$-mountain has chromatic number at least $r$, and has at most $(r!)^2$ vertices.*

The proof is easy by induction on $r$, and we leave it to the reader.

If $G$ is a tournament and $X \subseteq V(G)$, let $A(X), B(X)$ be respectively the sets of vertices $u \in V(G) \setminus X$ such that $X \Rightarrow u$, and $u \Rightarrow X$. If $v \in V(G)$, we write $A(v)$ for $A(\{v\})$, and $B(v)$ for $B(\{v\})$. If $X \subseteq V(G)$, we write $\chi(X)$ for $\chi(G|X)$. The inductive steps in the proof of 3.2 are contained in the following lemma.

**3.4.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be sets of tournaments, such that every member of $\mathcal{H}_1 \cup \mathcal{H}_2$ has at most $h$ vertices. Let $G$ be an $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free tournament, such that for $i = 1, 2$, every $\mathcal{H}_i$-free subtournament of $G$ has chromatic number at most $c$. Let $r \geqslant 1$ and $s \geqslant 2$, and suppose that*

- *$G$ contains no $(r, s)$-clique,*
- *every subtournament of $G$ containing no $r$-mountain has chromatic number at most $p$,*
- *every subset $X$ of $V(G)$ including no $(r, s - 1)$-clique of $G$ has $\chi(X) \leqslant q$.*

*Then*

$$\chi(G) \leqslant \max\bigl(2q + 2c, ph^2 + c(h + 1)\bigr).$$

**Proof.** For a vertex $v$, let $N(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are adjacent to or from $v$ by an $r$-heavy edge. We deduce:

(1) *For $v \in V(G)$, $\chi(N(v)) \leqslant q$.*

Since $G$ contains no $(r, s)$-clique, it follows that $N(v)$ contains no $(r, s - 1)$-clique of $G$, so the subtournament induced on this set has chromatic number at most $q$.

(2) *For every vertex $v$, either $\chi(A(v)) \leqslant c + ph$ or $\chi(B(v)) \leqslant c + q$; and either $\chi(A(v)) \leqslant c + q$ or $\chi(B(v)) \leqslant c + ph$.*

To prove the first claim, we may assume that $\chi(B(v) \setminus N(v)) > c$, for otherwise $\chi(B(v)) \leqslant c + q$ by (1) and the claim holds. Choose $X \subseteq B(v) \setminus N(v)$ such that $G|X$ is isomorphic to some member of $\mathcal{H}_1$. Now let $W$ be the set of all vertices in $A(v)$ that belong to $C(e)$ for some edge $e$ with tail in $X$ and head $v$. Since from the choice of $X$, each such edge $e$ is not $r$-heavy, it follows that $G|C(e)$ has no $r$-mountain, and so $\chi(C(e)) \leqslant p$; and since there are at most $h$ edges from $X$ to $v$, we deduce that $\chi(W) \leqslant ph$. Now $\chi(A(X)) \leqslant c$ since $G|A(X)$ is $\mathcal{H}_2$-free (because $G$ is $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free);

but $A(v) \setminus W \subseteq A(X)$, and so $\chi(A(v) \setminus W) \leqslant c$. Consequently $\chi(A(v)) \leqslant c + ph$. This proves the first claim of (2), and the second follows by symmetry. This proves (2).

Let $P$ be the set of all vertices $v$ with $\chi(A(v)) \leqslant c + ph$, and $Q$ the set with $\chi(B(v)) \leqslant c + ph$. If $P \cup Q \neq V(G)$ then by (2) there is a vertex $v$ with $\chi(A(v)) \leqslant c + q$ and $\chi(B(v)) \leqslant c + q$, and hence $\chi(G) \leqslant 2c + 2q$ as required. Thus we may assume that $P \cup Q = V(G)$. Suppose that $G|P$ contains a member of $\mathcal{H}_2$, and choose $X \subseteq P$ such that $G|X$ is isomorphic to a member of $\mathcal{H}_2$. Every vertex of $V(G) \setminus X$ either belongs to $A(v)$ for some $v \in X$, or to $B(X)$. Each set $A(v) \cup \{v\} (v \in X)$ has chromatic number at most $c + ph$, and $\chi(B(X)) \leqslant c$ since $G$ is $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free. Thus

$$\chi(G) \leqslant |X|(c + ph) + c \leqslant ph^2 + c(h + 1)$$

as required. So we may assume that $G|P$ is $\mathcal{H}_2$-free, and so $\chi(P) \leqslant c$, and similarly $\chi(Q) \leqslant c$; but then $\chi(G) \leqslant 2c$ and again the theorem holds. This proves 3.4. $\square$

We deduce the following, by induction on $s$, using 3.4.

**3.5.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be sets of tournaments, such that every member of $\mathcal{H}_1 \cup \mathcal{H}_2$ has at most $h \geqslant 1$ vertices. Let $G$ be an $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free tournament, such that for $i = 1, 2$, every $\mathcal{H}_i$-free subtournament of $G$ has chromatic number at most $c$, where $c \geqslant 1$. Let $r \geqslant 1$, and suppose that*

- *$G$ contains no $(r + 1)$-mountain, and*
- *every subtournament of $G$ containing no $r$-mountain has chromatic number at most $p$.*

*Then $\chi(G) \leqslant 2^{r-1}(ph^2 + c(h + 3))$.*

**Proof.** Let $f(1) = 0$, and for $s \geqslant 2$ let $f(s) = 2^{s-2}(ph^2 + c(h + 3)) - 2c$. We prove by induction on $s$ that

(1) *For $1 \leqslant s \leqslant r + 1$, if $X \subseteq V(G)$ contains no $(r, s)$-clique then $\chi(X) \leqslant f(s)$.*

For $s = 1$ this is trivial, since a tournament containing no $(r, 1)$-clique has no vertices. If $s \geqslant 2$, then by 3.4, it suffices to check that

$$f(s) \geqslant \max\bigl(2f(s - 1) + 2c, ph^2 + c(h + 1)\bigr),$$

which is easily seen (in fact equality holds). This proves (1).

Since $G$ has no $(r + 1)$-mountain and hence no $(r, r + 1)$-clique, we may set $s = r + 1$ in (1) to deduce the theorem. This proves 3.5. $\square$

Now by induction on $r$, we obtain the following.

**3.6.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be sets of tournaments, such that every member of $\mathcal{H}_1 \cup \mathcal{H}_2$ has at most $h \geqslant 3$ vertices. Let $G$ be an $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free tournament, such that for $i = 1, 2$, every $\mathcal{H}_i$-free subtournament of $G$ has chromatic number at most $c$, where $c \geqslant 1$. For $r \geqslant 1$, if $G$ contains no $(r + 1)$-mountain then*

$$\chi(G) \leqslant 2^{\frac{1}{2}r(r-1)+1}h^{2r-2}c.$$

**Proof.** Let $f(1) = 1$ and for $r \geqslant 2$ let

$$f(r) = 2^{\frac{1}{2}r(r-1)+1}h^{2r-2}c - c.$$

We prove by induction on $r$ that if $G$ has no $(r+1)$-mountain then $\chi(G) \leqslant f(r)$ (the extra term $-c$ is included to make the induction work). For $r = 1$, every tournament with no 2-mountain is transitive, and so the result holds. We assume that $r > 1$ and the result holds for $r - 1$. By 3.5, it suffices to check that $f(r) \geqslant 2^{r-1}(f(r - 1)h^2 + c(h + 3))$, and this is easily seen (using that $h \geqslant 3$). This proves 3.6. $\square$

Now we can prove 3.2, which we restate.

**3.7.** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be sets of tournaments, such that every member of $\mathcal{H}_1 \cup \mathcal{H}_2$ has at most $c$ vertices, where $c \geqslant 3$. Let $G$ be an $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free tournament, such that for $i = 1, 2$, every $\mathcal{H}_i$-free subtournament of $G$ has chromatic number at most $c$. Then*

$$\chi(G) \leqslant (2c)^{4c^2}.$$

**Proof.** Suppose first that $G$ does not contain a $(2c+1)$-mountain. By 3.6, taking $r = 2c$ and $h = c$, it follows that

$$\chi(G) \leqslant 2^{c(2c-1)+1} c^{4c-1} \leqslant (2c)^{4c^2}$$

as required. Thus we may assume that $G$ contains a $(2c+1)$-mountain. Hence by 3.3 there exists $M \subseteq V(G)$ with $|M| \leqslant (2c+1)!^2$ and with $\chi(M) \geqslant 2c+1$. Let $P$ be the set of vertices $v \in V(G) \setminus M$ such that $G|(A(v) \cap M)$ contains a member of $\mathcal{H}_2$, and let $Q$ be the set of all $v \in V(G) \setminus M$ such that $G|(B(v) \cap M)$ contains a member of $\mathcal{H}_1$. Every vertex $v \in V(G) \setminus M$ belongs to one of $P \cup Q$; for if $v \notin P$ then $\chi(A(v) \cap M) \leqslant c$, and if $v \notin Q$ then $\chi(B(v) \cap M) \leqslant c$, and not both these hold since $\chi(M) \geqslant 2c+1$.

For each $Y \subseteq M$ with $|Y| = c$, if $G|Y$ contains a member of $\mathcal{H}_2$, let $P(Y) = P \cap B(Y)$, and otherwise let $P(Y) = \emptyset$. We claim that $\chi(P(Y)) \leqslant c$ for each choice of $Y$. For if $G|Y$ contains a member of $\mathcal{H}_2$, then $G|P(Y)$ is $\mathcal{H}_1$-free (since $G$ is $(\mathcal{H}_1 \Rightarrow \mathcal{H}_2)$-free), and so $\chi(P(Y)) \leqslant c$; while if $G|Y$ is $\mathcal{H}_2$-free then $P(Y) = \emptyset$ and the claim is trivial. It follows that $\chi(Y \cup P(Y)) \leqslant c$ for each $Y \subseteq M$ with $|Y| = c$. Now every vertex of $P \cup M$ belongs to $Y \cup P(Y)$ for some choice of $Y$; and since there are at most $|M|^c$ choices of $Y$, and $|M| \leqslant (2c+1)!^2 \leqslant (2c)^{4c-1}$, it follows that $\chi(M \cup P) \leqslant c(2c)^{c(4c-1)}$, and similarly $\chi(M \cup Q) \leqslant c(2c)^{c(4c-1)}$.

Hence $\chi(G) \leqslant (2c)^{c(4c-1)+1} \leqslant (2c)^{4c^2}$. This proves 3.7 and hence 3.2, and so completes the proof of 3.1. □

## 4. Growing a hero

In this section we complete the proof of 1.2. In view of 3.1 and 2.5 (and symmetry under reversing all edges) it suffices to prove the following.

**4.1.** *If $H$ is a hero and $k \geqslant 1$ is an integer, then $\Delta(H, 1, k)$ is a hero.*

Indeed, by 1.1 it suffices to prove this for all $k \geqslant 3$, which is slightly more convenient. We begin with some lemmas.

**4.2.** *Let $G$ be a tournament, and let $(X_1, X_2, \dots, X_n)$ be a partition of $V(G)$. Suppose that*

- *$\chi(X_i) \leqslant d$ for $1 \leqslant i \leqslant n$, and*
- *for $1 \leqslant i < j \leqslant n$, if there is an edge $uv$ with $u \in X_j$ and $v \in X_i$, then*

$$\chi(X_{i+1} \cup X_{i+2} \cup \dots \cup X_j) \leqslant d.$$

*Then $\chi(G) \leqslant 2d$.*

**Proof.** We may assume that $n \geqslant 1$. We define $t \geqslant 1$ and $k_1, \dots, k_t$ with $1 = k_1 < k_2 < \dots < k_t \leqslant n$ as follows. Let $k_1 = 1$. Inductively, having defined $k_s$, if there exists $j$ with $k_s < j \leqslant n$ and

$$\chi\left(\bigcup (X_i \colon k_s \leqslant i \leqslant j)\right) > d,$$

let $k_{s+1}$ be the least such $j$; and otherwise let $t = s$ and the definition is complete. For $1 \leqslant s < t$, let $Y_s = \bigcup(X_i \colon k_s \leqslant i < k_{s+1})$, and $Y_t = \bigcup(X_i \colon k_t \leqslant i \leqslant n)$. Thus $Y_1, \dots, Y_t$ are pairwise disjoint and have union $V(G)$.

(1) *For $1 \leqslant s \leqslant t$, $\chi(Y_s) \leqslant d$; and for $2 \leqslant s \leqslant t-1$, there is no edge from $Y_{s+1} \cup Y_{s+2} \cup \cdots \cup Y_t$ to $Y_1 \cup \cdots \cup Y_{s-1}$.*

By hypothesis, $\chi(X_{k_s}) \leqslant d$, and so $\chi(Y_s) \leqslant d$ from the definition of $k_{s+1}$. This proves the first claim. For the second, suppose that $2 \leqslant s \leqslant t-1$ and there is an edge $uv$ with $u \in X_j$ for some $j \geqslant k_{s+1}$, and $v \in X_h$ for some $h < k_s$. Then $\chi(\bigcup(X_i\colon\ h < i \leqslant j)) \leqslant d$ by hypothesis; but $\chi(\bigcup(X_i\colon\ k_s \leqslant i \leqslant k_{s+1})) > d$ from the choice of $k_{s+1}$, a contradiction. This proves (1).

From (1) it follows that the sets $\bigcup(Y_i\colon\ 1 \leqslant i \leqslant t, i \text{ odd})$ and $\bigcup(Y_i\colon\ 1 \leqslant i \leqslant t, i \text{ even})$ both have chromatic number at most $d$, and so $\chi(G) \leqslant 2d$. This proves 4.2. □

We need the following result of Stearns [7] (it is easily proved by induction on $k$).

**4.3.** *For each integer $k \geqslant 1$, every tournament with at least $2^{k-1}$ vertices contains $T_k$.*

Let $X_1, \ldots, X_n$ be a sequence of subsets of $V(G)$, pairwise disjoint. We say an edge $uv$ of $G$ is a *backedge* (with respect to this sequence) if $u \in X_j$ and $v \in X_i$ for some $i, j$ with $1 \leqslant i < j \leqslant n$. The *backedge graph* is the graph with vertex set $X_1 \cup \cdots \cup X_n$ and edges all pairs $\{u, v\}$ of distinct vertices such that one of $uv, vu$ is a backedge.

**4.4.** *Let $k \geqslant 1$, let $G$ be a $\Delta(H, 1, k)$-free tournament, and let every $H$-free subtournament of $G$ have chromatic number at most $c$. Let $(X_1, \ldots, X_n)$ be a partition of $V(G)$, such that for $1 \leqslant i \leqslant n$, $\chi(X_i) \leqslant c$ and for each $v \in X_i$,*

$$\chi\big(A(v) \cap (X_1 \cup \cdots \cup X_{i-1})\big) \leqslant c,$$

*and*

$$\chi\big(B(v) \cap (X_{i+1} \cup \cdots \cup X_n)\big) \leqslant c.$$

*Then $\chi(G) \leqslant c(k+3)2^k$.*

**Proof.** For each backedge $uv$, we define its *span* to be $j - i$, where $u \in X_j$ and $v \in X_i$. For each vertex $u$, if there are at most $2^{k-1} - 2$ backedges with tail $u$, let $F_u$ be the set of all backedges with tail $u$; and if there are at least $2^{k-1} - 1$ backedges with tail $u$, let $F_u$ be a set of $2^{k-1} - 1$ such backedges with spans as large as possible. Let $F = \bigcup(F_u : u \in V(G))$.

(1) *For every backedge $uv \notin F$, if $u \in X_j$ and $v \in X_h$ then $\chi(\bigcup(X_i\colon\ h < i \leqslant j)) \leqslant c(k+3)$.*

Let $W = \bigcup(X_i\colon\ h < i \leqslant j)$. From the definition of $F_u$, since there is a backedge with tail $u$ not in $F_u$, it follows that $|F_u| = 2^{k-1} - 1$. Thus the set of heads of edges in $F_u \cup \{uv\}$ has cardinality $2^{k-1}$, and therefore includes a copy of $T_k$ by 4.3, say with vertex set $Y$. From the definition of $F_u$, it follows that $Y \subseteq X_1 \cup \cdots \cup X_h$. Let $P$ be the set of vertices in $W \setminus X_j$ that are adjacent to a member of $Y$ or adjacent from $u$, and let $Q = W \setminus (P \cup X_j)$. Now if $p \in P$ and $p$ is adjacent to some $y \in Y$, then the edge $py$ is a backedge (because $Y \subseteq X_1 \cup \cdots \cup X_h$ and $p \in X_i$ where $h < i < j$); and so for each $y$, the set of all $p \in P$ adjacent to $y$ has chromatic number at most $c$, by hypothesis. Similarly, if $p \in P$ and $p$ is adjacent from $u$ then the edge $up$ is a backedge; and so the set of all such $p$ again has chromatic number at most $c$. Consequently $\chi(P) \leqslant c(|Y| + 1) = c(k+1)$. On the other hand, $G|Q$ does not contain $H$, since otherwise this copy of $H$ together with $Y \cup \{u\}$ would form $\Delta(H, 1, k)$; and so $\chi(Q) \leqslant c$. Since $\chi(X_j) \leqslant c$ by hypothesis, we deduce that

$$\chi\Big(\bigcup(X_i\colon\ h < i \leqslant j)\Big) \leqslant c(k+3).$$

This proves (1).

Now let $B$ be the graph with vertex set $V(G)$ in which $u, v$ are adjacent if one of $uv, vu \in F$. Every nonempty subgraph of $B$ has a vertex of degree at most $2^{k-1} - 1$ (the vertex in $X_i$ with $i$ maximum), and so $B$ is $2^{k-1}$-graph-colourable. Take a partition $(Z_1, \ldots, Z_{2^{k-1}})$ of $V(G)$ into $2^{k-1}$ sets each stable in $B$. For each $Z_i$, (1) and 4.2 applied to the sequence

$$X_1 \cup Z_i, X_2 \cap Z_i, \ldots, X_n \cap Z_i$$

imply that $\chi(Z_i) \leqslant 2c(k+3)$. It follows that $\chi(G) \leqslant c(k+3)2^k$. This proves 4.4. □

Let $H, K$ be tournaments, and let $a \geqslant 1$ be an integer. An $(a, H, K)$-*jewel* in a tournament $G$ is a subset $X \subseteq V(G)$ such that $|X| = a$, and for every partition $(A, B)$ of $X$, either $G|A$ contains $H$ or $G|B$ contains $K$. An $(a, H, K)$-*jewel-chain* of length $t$ is a sequence $Y_1, \ldots, Y_t$ of $(a, H, K)$-jewels, pairwise disjoint, such that $Y_i \Rightarrow Y_{i+1}$ for $1 \leqslant i < t$. We need the following lemma:

**4.5.** *Let $H, K$ be tournaments, and let $a \geqslant 1$ be an integer. Then there are integers $\lambda_1, \lambda_2 \geqslant 0$ with the following property. For every $\Delta(H, 1, K)$-free tournament $G$, if*

- *$c_1$ is such that every $H$-free subtournament of $G$ has chromatic number at most $c_1$, and every $K$-free subtournament of $G$ has chromatic number at most $c_1$, and*
- *$c_2$ is such that every subtournament of $G$ containing no $(a, H, K)$-jewel-chain of length four has chromatic number at most $c_2$,*

*then $G$ has chromatic number at most $\lambda_1 c_1 + \lambda_2 c_2$.*

**Proof.** Let $k = \max(|V(H)|, |V(K)|)$, let $b = 2a^{k+1}$, and let $\lambda_1 = (3bk + 2b + 1)(k+3)2^k$, and $\lambda_2 = (k+1)(k+3)2^k$. We claim that they satisfy the theorem.

We may assume that $G$ contains an $(a, H, K)$-jewel-chain of length four, since otherwise $\chi(G) \leqslant c_2$ and we are done. Choose an $(a, H, K)$-jewel-chain $X_1, \ldots, X_n$ with $n \geqslant 1$ maximum. (Thus $n \geqslant 4$.) Let $X = X_1 \cup \cdots \cup X_n$ and $W = V(G) \setminus X$. We recall that $A(v)$ denotes the set of out-neighbours of a vertex $v$, and $B(v)$ its set of in-neighbours.

(1) *If $v \in X_i$, then for $h < i$, $A(v) \cap X_h$ is $K$-free, and so $G|(B(v) \cap X_h)$ contains $H$. Also, for $j > i$, $B(v) \cap X_j$ is $H$-free, and so $G|(A(v) \cap X_j)$ contains $K$.*

Suppose that there exists $h < i$ such that $G|(A(v) \cap X_h)$ contains $K$, and choose $h$ maximum. Then $h \leqslant i - 2$, since $X_{i-1} \Rightarrow X_i$. Hence $h + 1 < i$, and by the choice of $h$ this implies that $G|(A(v) \cap X_{h+1})$ does not contain $K$. Consequently $G|(B(v) \cap X_{h+1})$ contains $H$, since $X_{h+1}$ is an $(a, H, K)$-jewel; but then the copy of $K$ in $X_h$, the copy of $H$ in $X_{h+1}$, and $v$, induce a copy of $\Delta(H, 1, K)$, a contradiction. This proves the first statement of (3) and the second follows from symmetry. This proves (1).

(2) *For each $v \in W$, there exists $i$ with $1 \leqslant i \leqslant n$, such that*
- *for $1 \leqslant h < i$, $A(v) \cap X_h$ is $K$-free, and so $G|(B(v) \cap X_h)$ contains $H$,*
- *for $i < j \leqslant n$, $B(v) \cap X_j$ is $H$-free, and so $G|(A(v) \cap X_j)$ contains $K$.*

Let $P, Q$ be respectively the sets of $i \in \{1, \ldots, n\}$ such that $G|(B(v) \cap X_i)$ contains $H$, and $G|(A(v) \cap X_i)$ contains $K$. Since each $X_i$ is an $(a, H, K)$-jewel, it follows that $P \cup Q = \{1, \ldots, n\}$. Suppose that there exist $h, j$ with $1 \leqslant h < j \leqslant n$ and $h \in Q$ and $j \in P$, and choose $h, j$ with $j - h$ minimum. If $j > h + 1$, then $h + 1 \notin Q$ (since otherwise $h + 1, j$ is a better pair) and $h + 1 \notin P$ (since otherwise $h, h + 1$ is a better pair), a contradiction. Thus $j = h + 1$; but since $X_h \Rightarrow X_{h+1}$, the copy of $K$ in $G|(A(v) \cap X_h)$, the copy of $H$ in $G|(B(v) \cap X_{h+1})$, and $v$, form a copy of $\Delta(H, 1, K)$, a contradiction. This proves that there do not exist $h, j$ with $1 \leqslant h < j \leqslant n$ and $h \in Q$ and $j \in P$. We deduce that for some $i \in \{1, \ldots, n\}$, every $h < i$ belongs to $P \setminus Q$ and every $j > i$ belongs to $Q \setminus P$. This proves (2).

For each $v \in W$, choose a value of $i$ as in (2), say $c(v)$; if there is more than one choice for $c(v)$, choose $c(v)$ in addition such that $v$ has both an out-neighbour in $X_{c(v)}$ and an in-neighbour in $X_{c(v)}$, if possible. Let $W_i$ be the set of all $v \in W$ with $c(v) = i$. For $1 \leqslant i \leqslant n$, let $Z_i = X_i \cup W_i$; then $Z_1, \ldots, Z_n$ are disjoint, and have union $V(G)$.

(3) *If $i > 1$ and $v \in W_i$ and $v \Rightarrow X_i$ then $X_{i-1} \Rightarrow v$; and if $i < n$ and $v \in W_i$ and $X_i \Rightarrow v$ then $v \Rightarrow X_{i+1}$.*

For then $B(v) \cap H_i$ is $H$-free, and so $i-1$ is an alternate choice of $c(v)$. Since $v$ has no in-neighbour in $X_i$, our choice of $c(v) = i$ implies that $v$ does not have both an in-neighbour and an out-neighbour in $X_{i-1}$. But $G|(B(v) \cap X_{i-1})$ contains $H$, and so $v$ has an in-neighbour in $X_{i-1}$; and consequently $v$ has no out-neighbour in $X_{i-1}$. This proves (3).

(4) *For $1 \leqslant i \leqslant n$, $\chi(Z_i) \leqslant 2bc_1 + c_2$.*

Fix $i$ with $1 \leqslant i \leqslant n$. Let $P$ be the set of all $v \in Z_i$ with an out-neighbour in $X_{i-2}$, if $i \geqslant 3$, and let $P = \emptyset$ if $i \leqslant 2$. Let $P_1$ be the set of $v \in P$ such that $G|(B(v) \cap X_{i-1})$ contains $K$, and $P_2 = P \setminus P_1$.

If $v \in P_1$, then $v$ has an out-neighbour $x \in X_{i-2}$ and there exists $Y \subseteq X_{i-1}$ with $Y \Rightarrow v$ such that $G|Y$ is isomorphic to $K$. Now for each $Y \subseteq X_{i-1}$ such that $G|Y$ is isomorphic to $K$, the set of all $v \in P_1$ with $Y \Rightarrow v \Rightarrow x$ is $H$-free (since $G$ is $\Delta(H, 1, K)$-free); and consequently the set of all $v \in P_1$ with $Y \Rightarrow v \Rightarrow x$ has chromatic number at most $c_1$. Since there are at most $a^{k+1}$ choices for the pair $(x, Y)$ (because there are at most $a$ choices of $x \in X_{i-2}$, and at most $a^k$ choices of $Y \subseteq X_{i-1}$), it follows that $\chi(P_1) \leqslant a^{k+1}c_1$.

If $v \in P_2$, then $G|(A(v) \cap X_{i-1})$ contains $H$, and so there exists $Y \subseteq X_{i-1}$ such that $G|Y$ is isomorphic to $H$ and $v$ is adjacent to every vertex in $Y$. In particular $v \in W_i$. Since $v \notin P_1$ and therefore $X_{i-1} \not\Rightarrow v$, (2) implies that there exists $x \in X_i$ adjacent to $v$. For each $Y \subseteq X_{i-1}$ such that $G|Y$ is isomorphic to $H$, and each $x \in X_i$, the set of all $v \in P_2$ with $x \Rightarrow v \Rightarrow Y$ is $K$-free (because $G$ is $\Delta(H, 1, K)$-free), and so has chromatic number at most $c_1$; and since (as before) there are at most $a^{k+1}$ choices for the pair $(x, Y)$, we deduce that $\chi(P_2) \leqslant a^{k+1}c_1$. Adding, we deduce that $\chi(P) \leqslant 2a^{k+1}c_1 = bc_1$.

Let $Q$ be the set of all $v \in Z_i$ with an in-neighbour in $X_{i+2}$, if $i \leqslant n-2$, and let $Q = \emptyset$ if $i \geqslant n-1$. Then similarly, $\chi(Q) \leqslant bc_1$. Finally, let $R$ be the set of all $v \in Z_i$ such that either $i \leqslant 1$ or $X_{i-2} \Rightarrow v$, and either $i \geqslant n-1$ or $v \Rightarrow X_{i+2}$. Suppose that $G|R$ contains an $(a, H, K)$-jewel-chain of length four, say $Y_1, Y_2, Y_3, Y_4$. Since $n \geqslant 4$, it follows that either $i \geqslant 3$ or $i \leqslant n-2$, and from the symmetry we may assume that $i \leqslant n-2$. If also $i \geqslant 3$ then the sequence

$$X_1, \ldots, X_{i-2}, Y_1, Y_2, Y_3, Y_4, X_{i+2}, \ldots, X_n$$

contradicts the maximality of $n$; and if $i \leqslant 2$ then the sequence

$$Y_1, Y_2, Y_3, Y_4, X_{i+2}, \ldots, X_n$$

contradicts the maximality of $n$. Thus $R$ does not contain an $(a, H, K)$-jewel-chain of length four, and so $\chi(R) \leqslant c_2$. Since $Z_i = P \cup Q \cup R$, this proves (4).

(5) *For each $v \in X_i$,*

$$\chi\big(A(v) \cap (Z_1 \cup Z_2 \cdots \cup Z_{i-1})\big) \leqslant 3bc_1 + c_2,$$

*and*

$$\chi\big(B(v) \cap (Z_{i+1} \cup Z_{i+2} \cdots \cup Z_n)\big) \leqslant 3bc_1 + c_2.$$

From the symmetry, it is sufficient to prove the first statement. This is trivial if $i = 1$, so we assume that $i \geqslant 2$. Let $P = A(v) \cap (Z_1 \cup Z_2 \cdots \cup Z_{i-2})$. Let $P_1$ be the set of all $w \in P$ such that $G|(A(w) \cap X_{i-1})$ contains $H$, and $P_2 = P \setminus P_1$. Now for each $w \in P_1$, there exists $Y \subseteq X_{i-1}$ such that $w \Rightarrow Y$ and $G|Y$ is isomorphic to $H$. For each such choice of $Y$, the set of $w \in P_1$ such that $w \Rightarrow Y$ is $K$-free (since $G$ is $\Delta(H, 1, K)$-free), and so has chromatic number at most $c_1$; and since there are at most $a^k$ choices for $Y$, it follows that $\chi(P_1) \leqslant a^k c_1$.

For each $w \in P_2$, there exists $Y \subseteq X_{i-1}$ such that $Y \Rightarrow w$ and $G|Y$ is isomorphic to $K$. Also, $i \geqslant 3$ (since $w \in P$); and $X_{i-2} \not\Rightarrow w$ (this is clear if $w \in Z_h$ for some $h < i-2$, while if $w \in Z_{i-2}$ then it follows from (3) since $w \not\Rightarrow X_{i-1}$). Thus there exists $x \in X_{i-2}$ such that $w$ is adjacent to $x$. For each choice of $Y$ and $x$, the set of all $w \in P_2$ such that $Y \Rightarrow w \Rightarrow x$ is $H$-free (since $G$ is $\Delta(H,1,K)$-free) and so has chromatic number at most $c_1$. Since there are at most $a^{k+1}$ choices for the pair $(x, Y)$, it follows that $\chi(P_2) \leqslant a^{k+1}c_1$.

Hence $\chi(P) \leqslant 2a^{k+1}c_1 = bc_1$, and since $\chi(A(v) \cap Z_{i-1}) \leqslant 2bc_1 + c_2$ by (4), this proves the first claim of (5), and the second follows from the symmetry.

(6) *For each $v \in W_i$,*

$$\chi\bigl(A(v) \cap (Z_1 \cup Z_2 \cdots \cup Z_{i-1})\bigr) \leqslant (3bk + 2b + 1)c_1 + (k+1)c_2,$$

*and*

$$\chi\bigl(B(v) \cap (Z_{i+1} \cup Z_{i+2} \cdots \cup Z_n)\bigr) \leqslant (3bk + 2b + 1)c_1 + (k+1)c_2.$$

By the symmetry it suffices to prove the first claim. Thus we may assume that $i \geqslant 2$. Choose $Y \subseteq X_{i-1}$ such that $G|Y$ is isomorphic to $H$ and $Y \Rightarrow v$. Let $P = A(v) \cap (Z_1 \cup Z_2 \cdots \cup Z_{i-2})$. The set of all $w \in P$ such that $w \Rightarrow Y$ is $K$-free (since $G$ is $\Delta(H,1,K)$-free), and so has chromatic number at most $c_1$; while for each $y \in Y$, the set of $w \in P$ that are adjacent from $y$ has chromatic number at most $3bc_1 + c_2$ by (5). Hence $\chi(P) \leqslant c_1 + k(3bc_1 + c_2)$. From (4), we deduce that

$$\chi\bigl(A(v) \cap (Z_1 \cup Z_2 \cdots \cup Z_{i-1})\bigr) \leqslant c_1 + k(3bc_1 + c_2) + 2bc_1 + c_2.$$

This proves (6).

From (4), (5), (6), and 4.4, applied to the sequence $Z_1, \dots, Z_n$, it follows that

$$\chi(G) \leqslant (k+3)2^k\bigl((3bk + 2b + 1)c_1 + (k+1)c_2\bigr).$$

This proves 4.5. □

Now we deduce the main theorem of this section, the following.

**4.6.** *If $H$ is a hero and $K$ is a transitive tournament then $\Delta(H,1,K)$ and $\Delta(K,1,H)$ are heroes.*

**Proof.** Let $k = |V(K)|$, and let $c$ be such that every $H$-free tournament has chromatic number at most $c_1$; and we may assume that $c_1 \geqslant 2^k$. Every $K$-free tournament has at most $2^k$ vertices, by 4.3, and so has chromatic number at most $c_1$. Let $a = |V(H)|2^k$.

(1) *Every tournament not containing an $(a,H,K)$-jewel has chromatic number at most $2^k|V(H)| + c_1$.*

Let $G$ be a tournament not containing an $(a,H,K)$-jewel. Choose pairwise vertex-disjoint subtournaments $H_1, \dots, H_n$ of $G$, each isomorphic to $H$, with $n$ maximum, and let the union of their vertex sets be $W$. If $n \geqslant 2^k$, then $V(H_1) \cup \cdots \cup V(H_{2^k})$ is an $(a,H,K)$-jewel by 4.3, a contradiction, and so $n < 2^k$. Then

$$\chi(W) \leqslant |W| \leqslant 2^k\bigl|V(H)\bigr|,$$

and $\chi(G \setminus W) \leqslant c_1$ since $G \setminus W$ is $H$-free. But then $\chi(G) \leqslant 2^k|V(H)| + c_1$. This proves (1).

By (1) and 3.2 (taking $\mathcal{F}_1 = \mathcal{F}_2$ to be the set of all $(a,H,K)$-jewels) there exists $c_0$ such that every tournament not containing an $(a,H,K)$-jewel-chain of length two has chromatic number at most $c_0$; and then by 3.2 again (now taking $\mathcal{F}_1 = \mathcal{F}_2$ to be the set of all tournaments in the class just produced), there exists $c_2$ such that every tournament not containing an $(a,H,K)$-jewel-chain of length four has chromatic number at most $c_2$. By 4.5, there exists $c_3$ such that every $\Delta(H,1,K)$-free tournament has chromatic number at most $c_3$, and so $\Delta(H,1,K)$ is a hero; and similarly so is $\Delta(K,1,H)$. This proves 4.6, and hence proves 4.1. □

## 5. Minimal non-heroes

Since every subtournament of a hero is a hero by 1.1, one might ask for the list of minimal tournaments that are not heroes. It turns out that there are only five of them:

- Let $H_1$ be the tournament with five vertices $v_1, \dots, v_5$, in which $v_i$ is adjacent to $v_{i+1}$ and $v_{i+2}$ for $1 \leqslant i \leqslant 5$ (reading subscripts modulo 5).
- Let $H_2$ be the tournament obtained from $H_1$ by replacing the edge $v_5v_1$ by an edge $v_1v_5$.
- Let $H_3$ be the tournament with five vertices $v_1, \dots, v_5$ in which $v_i$ is adjacent to $v_j$ for all $i, j$ with $1 \leqslant i < j \leqslant 4$, and $v_5$ is adjacent to $v_1, v_3$ and adjacent from $v_2, v_4$.
- Let $H_4$ be the tournament $\Delta(2, 2, 2)$.
- Let $H_5$ be the tournament $\Delta(C_3, C_3, 1)$, where $C_3$ denotes the tournament $\Delta(1, 1, 1)$.

To prove this, we use an unpublished result of Gaku Liu [6], the following (we give a proof for the reader's convenience).

**5.1.** *Every strong tournament with more than one vertex containing none of $H_1, H_2, H_3$ admits a trisection.*

**Proof.** Let $G$ be a strong tournament containing none of $H_1, H_2, H_3$, with $n > 1$ vertices. We prove by induction on $n$ that $G$ admits a trisection. Suppose it does not, for a contradiction. Thus $n \geqslant 4$, since tournaments with at most three vertices satisfy the theorem. We say $X \subseteq V(G)$ is a *homogeneous set* if $1 < |X| < |V(G)|$ and for every vertex $v \in V(G) \setminus X$, either $v \Rightarrow X$ or $X \Rightarrow v$.

(1) *We may assume that there is no homogeneous set.*

Suppose that $X$ is a homogeneous set, and choose $x \in X$. Let $G' = G|((V(G) \setminus X) \cup \{x\})$. Then $G'$ is strong, with more than one vertex, and so by induction admits a trisection $(A', B', C')$ say, where $x \in C'$; but then $(A', B', C' \cup X)$ is a trisection of $G$ as required. This proves (1).

Since every tournament with four vertices has a homogeneous set, it follows from (1) that $n \geqslant 5$. Since $G$ has a cyclic triangle, we may choose a strong subtournament $H$ of $G$ with $V(H) \neq V(G)$, maximal with these properties. Thus $|V(H)| \geqslant 2$, and since $V(H)$ is not a homogeneous set, there is a vertex $v \notin V(H)$ with an out-neighbour and an in-neighbour in $V(H)$. From the maximality of $H$ it follows that $V(H) \cup \{v\} = V(G)$, and so $G \setminus v$ is strong.

From the inductive hypothesis, $G \setminus v$ admits a trisection $(P, Q, R)$ say. Let $A, B$ be respectively the sets of out-neighbours and in-neighbours of $v$. Since $A, B \neq \emptyset$, we may assume (by cyclically permuting $P, Q, R$ if necessary) that $B \cap P, A \cap Q \neq \emptyset$. Choose $p \in B \cap P$ and $q \in A \cap Q$.

If there exist $r \in A \cap R$ and $r' \in B \cap R$ then the subtournament induced on $\{v, p, q, r, r'\}$ is one of $H_1, H_2$ (depending on the direction of the edge between $r, r'$), a contradiction. So one of $A \cap R, B \cap R$ is empty; and from the symmetry under reversing edges we may assume that $A \cap R = \emptyset$. Choose $r \in B \cap R$. Since $P \cup \{v\}$ is not a homogeneous set, $B \cap Q \neq \emptyset$; choose $q' \in B \cap Q$. Then the subtournament induced on $\{v, p, r, q, q'\}$ is one of $H_2, H_3$ (depending on the direction of the edge between $q, q'$), a contradiction. This proves 5.1. $\square$

We deduce:

**5.2.** *A tournament is a hero if and only if it contains none of $H_1, \dots, H_5$ as a subtournament.*

**Proof.** Since $H_1, \dots, H_4$ are strongly connected and do not admit a trisection as in 2.5, it follows $H_1, \dots, H_4$ are not heroes, and by 2.4, $H_5$ is not a hero. By 1.1, this proves the "only if" part of the theorem.

For the "if" part, we need to show that every tournament $H$ containing none of $H_1, \dots, H_5$ is a hero, and we prove this by induction on $|V(H)|$. We may assume that $|V(H)| > 3$. If $H$ is not strong then its strong components are heroes by the inductive hypothesis, and hence so is $H$ by 3.1. Thus we may assume that $H$ is strong.

From 5.1, it follows that $H$ admits a trisection $(A, B, C)$. If $|A|, |B|, |C| > 1$, then $G$ contains $H_4$, a contradiction, so we may assume that $|C| = 1$. If neither of $A, B$ is transitive, then $G$ contains $H_5$, a contradiction, so from the symmetry we may assume that $B$ is transitive, and therefore $H = \Delta(H|A, |B|, 1)$. But $H|A$ is a hero from the inductive hypothesis, and hence so is $H$ by 4.1, as required. This proves 5.2. $\square$

## 6. Transitive subtournaments of linear size

We recall that a tournament $H$ is a *celebrity* if there exists $c > 0$ such that $\alpha(G) \geqslant c|V(G)|$ for every $H$-free tournament $G$. In this section we prove 1.3, which we restate.

**6.1.** *A tournament is a celebrity if and only if it is a hero.*

For the moment, let us assume the following lemma.

**6.2.** *The tournament* $\Delta(2, 2, 2)$ *is not a celebrity.*

**Proof of 6.1, assuming 6.2.** Certainly every hero is a celebrity; we prove that every celebrity $H$ is a hero, by induction on $|V(H)|$. We may assume that $|V(H)| \geqslant 2$. Suppose that $H$ is not strong. Each strong component $J$ of $H$ is a celebrity, since every subtournament of a celebrity is a celebrity; and so each such $J$ is a hero, from the inductive hypothesis; and hence so is $H$, from 3.1. Thus we may assume that $H$ is strong.

Next, we need a modification of the argument of 2.1. Define a sequence $D_i$ $(i \geqslant 0)$ of tournaments as follows. $D_0$ is the one-vertex tournament. Inductively, for $i \geqslant 1$, let $D_i = \Delta(D_{i-1}, D_{i-1}, D_{i-1})$.

(1) *For* $i \geqslant 1$, $\alpha(D_i) \leqslant 2^i$.

We prove this by induction on $i$. Let $T = D_i$, and let $(X, Y, Z)$ be a trisection of $T$ such that $T|X$, $T|Y$, $T|Z$ are each isomorphic to $D_{i-1}$. If $W \subseteq V(T)$ is transitive, then not all of $W \cap X$, $W \cap Y$, $W \cap Z$ are nonempty, and so we may assume that $W \subseteq X \cup Y$. But from the inductive hypothesis, $|W \cap X|$, $|W \cap Y| \leqslant 2^{i-1}$, and so $|W| \leqslant 2^i$. This proves (1).

Since $|V(D_i)| = 3^i$ for each $i$, and $H$ is a celebrity, (1) implies that there exists $i \geqslant 0$, minimum such that $D_i$ contains $H$. Since $H$ has at least two vertices, it follows that $i > 0$. Let $T = D_i$, and let $(X, Y, Z)$ be a trisection of $D_i$ such that $T|X$, $T|Y$, $T|Z$ are each isomorphic to $D_{i-1}$. Choose $W \subseteq V(T)$ such that $T|W$ is isomorphic to $H$. Since $D_{i-1}$ does not contain $H$ it follows that $W$ is not a subset of any of $X$, $Y$, $Z$; and since $H$ is strong, it follows that $W$ has nonempty intersection with each of $X$, $Y$, $Z$. By 6.2, not all of $W \cap X$, $W \cap Y$, $W \cap Z$ have at least two elements, and so we may assume that $|W \cap Z| = 1$. Moreover, at least one of $X$, $Y$ is transitive, since $\Delta(C_3, C_3, 1)$ is not a celebrity, by 2.4. It follows that $H = \Delta(J, k, 1)$ or $\Delta(J, 1, k)$ for some tournament $J$ and some integer $k \geqslant 1$. Since $J$ is a celebrity, the inductive hypothesis implies that $J$ is a hero, and hence so is $H$, from 4.1. This proves 6.1. $\square$

Thus it remains to prove 6.2. We shall in fact prove a stronger statement, the following.

**6.3.** *For every real* $\epsilon > 0$, *and all sufficiently large integers* $n$ (*depending on* $\epsilon$) *there is a tournament* $T$ *with* $n$ *vertices, not containing* $\Delta(2, 2, 2)$, *such that*

$$\alpha(T) \leqslant \frac{n}{(\ln(n))^{-\epsilon} + 1/3}.$$

(We use ln for the natural logarithm.) We need several lemmas, and begin with the following.

**6.4.** *Let $a_1, \ldots, a_k$ be real numbers with $0 \leqslant a_1 < a_2 < \cdots < a_k \leqslant 1$. Then*

$$\sum_{1\leqslant i<j\leqslant k} (a_j - a_i)^{-1} \geqslant k^2 \ln(k/3).$$

**Proof.** We may assume that $k \geqslant 4$ (for otherwise $\log(k/3) \leqslant 0$ and the result is trivially true). Let $1 \leqslant h \leqslant k-1$. Then there are $k-h$ pairs $(i, j)$ with $1 \leqslant i < j \leqslant k$ and $j - i = h$. Let $P$ be the set of all such pairs. For each $x$ with $0 \leqslant x \leqslant 1$, there are at most $h$ pairs $(i, j) \in P$ with $a_i \leqslant x \leqslant a_j$, and so

$$\sum_{(i,j)\in P} (a_j - a_i) \leqslant h.$$

By the Cauchy–Schwarz inequality,

$$\sum_{(i,j)\in P} (a_j - a_i) \sum_{(i,j)\in P} (a_j - a_i)^{-1} \geqslant |P|^2,$$

and so

$$\sum_{(i,j)\in P} (a_j - a_i)^{-1} \geqslant \frac{(k-h)^2}{h}.$$

Summing for $h = 1, \ldots, k-1$, we deduce that

$$\sum_{1\leqslant i<j\leqslant k} (a_j - a_i)^{-1} \geqslant \sum_{1\leqslant h\leqslant k-1} \frac{(k-h)^2}{h}.$$

The right side of this inequality equals

$$\left(\sum_{1\leqslant h\leqslant k-1} \frac{k^2}{h}\right) - \frac{3}{2}k(k-1) \geqslant \sum_{3\leqslant h\leqslant k-1} \frac{k^2}{h} \geqslant k^2 \ln(k/3).$$

This proves 6.4. □

For each integer $k \geqslant 1$, let $S(k)$ be the set of all permutations of $\{1, \ldots, k\}$. For $\sigma \in S(k)$, and for $1 \leqslant i < j \leqslant k$, we say $(i, j)$ is an *inversion* of $\sigma$ if $\sigma(i) > \sigma(j)$. Let $I(\sigma)$ be the set of inversions of $\sigma$. We need the following lemma.

**6.5.** *Let $0 \leqslant c < 1$, and for $k \geqslant 1$ let $W_k(c) = \sum_{\sigma\in S(k)} c^{|I(\sigma)|}$. Then $W_k(c) \leqslant (\frac{1}{1-c})^k$.*

**Proof.** It is easy to see that

$$W_k(c) = W_{k-1}(c)\big(1 + c + c^2 + \cdots + c^{k-1}\big) = W_{k-1}(c)\frac{1-c^k}{1-c}$$

for all $k \geqslant 2$. Consequently $W_k(c) \leqslant W_{k-1}(c)/(1-c)$ (since $0 \leqslant c < 1$) for $k \geqslant 2$, and since $W_1(c) = 1$, it follows that $W_k(c) \leqslant (\frac{1}{1-c})^k$ for all $k \geqslant 1$. This proves 6.5. □

Let $\mathbb{Z}$ denote the set of integers. Let $G$ be a tournament, and let $\phi : V(G) \to \mathbb{Z}$ be an injective map. Let $B$ be the graph with vertex set $V(G)$ and edge set all pairs $\{u, v\}$ such that $uv$ is an edge of $G$ and $\phi(u) > \phi(v)$; as before, we call $B$ the *backedge graph*. We speak of the edges of $B$ as *backedges*. (Earlier we spoke of some of the edges of $G$ as backedges, but the latter are ordered pairs, while the edges of $B$ are undirected, so this should cause no confusion.) If $e = \{u, v\}$ is an edge of $B$, we write $\phi(e) = |\phi(u) - \phi(v)|$. Let $r, s \geqslant 1$ be integers. Two distinct edges $e, f$ of $B$ are said to be *$(r, s)$-comparable* (under $\phi$) if

- there is a path $P$ of $B$ with at most $s$ edges, with $e, f \in E(P)$, and
- $\phi(e) \leqslant r\phi(f)$ and $\phi(f) \leqslant r\phi(e)$.

We use 6.4 and 6.5 to prove the following.

**6.6.** *For all integers $r, s \geqslant 1$, and every real number $\epsilon > 0$, and all sufficiently large $n$ (depending on $r, s, \epsilon$), there is a tournament $T$ with $n$ vertices and the following properties:*

- *there is an injective map from $V(T)$ into $\mathbb{Z}$ such that no two edges of the backedge graph are $(r, s)$-comparable, and*
- *$\alpha(T) \leqslant n(\ln(n))^{-1/s+\epsilon}$.*

**Proof.** Let $n \geqslant (re)^2$ be an integer, and let $\delta \geqslant 0$ be a real number, with $\delta \leqslant 1/4$. (We shall specify $\delta$ later.) Construct a tournament $G$ with vertex set $\{1, \dots, 2n\}$ as follows. Independently for each pair $(i, j)$ of vertices of $G$ with $i < j$, let $ji$ be an edge with probability $\delta/(j-i)$, and otherwise let $ij$ be an edge. For $X \subseteq \{1, \dots, 2n\}$, let $p(X)$ denote the probability that $uv$ is an edge of $G$ for all $u, v \in X$ with $u < v$.

(1) *For $X \subseteq V(G)$, $p(X) \leqslant e^{-\delta|X|^2 \ln(|X|/3)/(2n)}$.*

Let $X = \{x(1), \dots, x(k)\}$ say, where $x(1) < x(2) < \cdots < x(k)$. Then

$$p(X) = \prod_{1 \leqslant i < j \leqslant k} \left(1 - \frac{\delta}{x(j) - x(i)}\right).$$

From the inequality $1 - x \leqslant e^{-x}$, it follows that

$$1 - \frac{\delta}{x(j) - x(i)} \leqslant e^{-\frac{\delta}{x(j)-x(i)}},$$

and so $p(X) \leqslant e^{-y}$ where

$$y = \sum_{1 \leqslant i < j \leqslant k} \frac{\delta}{x(j) - x(i)}.$$

Since $1 \leqslant x(i) \leqslant 2n$ for $1 \leqslant i \leqslant k$, it follows from 6.4 applied to the numbers $\frac{x(i)}{2n}$ $(1 \leqslant i \leqslant k)$ that

$$y \geqslant \delta(2n)^{-1}k^2 \ln(k/3).$$

This proves (1).

For $X \subseteq \{1, \dots, 2n\}$, let $P(X)$ denote the probability that $X$ is transitive in $G$.

(2) *For $X \subseteq V(G)$, $P(X) \leqslant (\frac{1}{1-2\delta})^{|X|} p(X)$.*

Let $X = \{x(1), \dots, x(k)\}$ say, where $x(1) < x(2) < \cdots < x(k)$. As before,

$$p(X) = \prod_{1 \leqslant i < j \leqslant k} \left(1 - \frac{\delta}{x(j) - x(i)}\right).$$

We say that $\sigma \in S(k)$ is *satisfied* if for all $i, j$ with $1 \leqslant i < j \leqslant k$, $x(i)$ is adjacent to $x(j)$ in $G$ if and only if $\sigma(i) < \sigma(j)$. Thus, $X$ is transitive if and only if some member of $S(k)$ is satisfied. Let $P(\sigma)$ denote the probability that $\sigma \in S(K)$ is satisfied. Then $P(X) \leqslant \sum_{\sigma \in S(k)} P(\sigma)$ (in fact, equality holds, since at most one member of $S(k)$ is satisfied). For $\sigma$ to be satisfied, we need that for all $i, j$ with $1 \leqslant i < j \leqslant k$,

- if $(i, j) \notin I(\sigma)$ then $x(i)x(j)$ is an edge (this has probability $1 - \frac{\delta}{x(j)-x(i)}$),
- if $(i, j) \in I(\sigma)$ then $x(j)x(i)$ is an edge (this has probability $\frac{\delta}{x(j)-x(i)}$, and hence at most $2\delta(1 - \frac{\delta}{x(j)-x(i)})$ since $\delta \leqslant 1/2$).

Thus

$$P(\sigma) \leqslant p(X)(2\delta)^{I(\sigma)}.$$

Summing over all $\sigma$, we deduce that

$$P(X) \leqslant p(X) \sum_{\sigma \in S(k)} (2\delta)^{I(\sigma)} = W_k(2\delta)p(X).$$

From 6.5, it follows that $P(X) \leqslant (\frac{1}{1-2\delta})^k p(X)$. This proves (2).

(3) *For any real number $t > 0$, if $t \leqslant \frac{1}{3}n^{1/3}$, and $6t\ln(4et) \leqslant \delta\ln(n)$, then the probability that $\alpha(G) \geqslant n/t$ is at most $e^{-1}$.*

We may assume (by replacing $t$ by $\frac{n}{\lceil n/t \rceil}$) that $n/t = g$ is an integer. Let $P$ be the probability that $\alpha(G) \geqslant g$. Then $P$ is at most the expected value of the number of transitive sets of cardinality $g$. Now there are $\binom{2n}{g}$ subsets of cardinality $g$, and hence at most $(2ne/g)^g = (2et)^{n/t}$, by Stirling's approximation. By (1) and (2) (summed over all choices of $X$ of cardinality $n/t$),

$$P \leqslant \left(\frac{2et}{1-2\delta}\right)^{n/t} e^{-\delta(n/t)^2 \ln(n/(3t))/(2n)}.$$

Since $1 - 2\delta \geqslant 1/2$ (because $\delta \leqslant 1/4$), it follows that

$$-n^{-1}\ln(P) \geqslant -\frac{\ln(4et)}{t} + \frac{\delta}{2t^2}\ln(n) - \frac{\delta}{2t^2}\ln(3t).$$

But from the hypotheses, we have

$$\frac{\delta}{6t^2}\ln(n) \geqslant \frac{\ln(4et)}{t},$$
$$\frac{\delta}{6t^2}\ln(n) \geqslant \frac{\delta}{2t^2}\ln(3t).$$

Moreover, since $\ln(4et) \geqslant 1 \geqslant t/n$, the first of these two inequalities implies that

$$\frac{\delta}{6t^2}\ln(n) \geqslant n^{-1}.$$

Summing these four inequalities implies that $-n^{-1}\ln(P) \geqslant n^{-1}$, that is, $P \leqslant e^{-1}$. This proves (3).

Let $B$ be the backedge graph of $G$.

(4) *For each $k \geqslant 1$, and each $v \in \{1, \dots, 2n\}$, the expected number of paths of $B$ with $k$ vertices and first vertex $v$ is at most $(4\delta\ln n)^{k-1}$.*

Let this expectation be $E_k(v)$. Certainly $E_1(v) = 1$, and we proceed by induction on $k$ and may assume that $k \geqslant 2$. Thus we may enumerate all possible $k$-vertex paths with first vertex $v$ by listing their possible second vertices $u$ say. For each choice of $u \neq v$, let $E'(u)$ be the expected number of paths in $B$ with $k-1$ vertices and first vertex $u$ that do not contain $v$, conditioned on $u, v$ being adjacent in $B$. The probability that $\{u, v\}$ is an edge of $B$ is $\frac{\delta}{|v-u|}$, and so

$$E_k(v) = \sum_{1 \leqslant u \leqslant 2n, u \neq v} E'(u)\frac{\delta}{|v-u|}.$$

But $E'(u) \leqslant E_{k-1}(u) \leqslant (4\delta \ln n)^{k-2}$ from the inductive hypothesis, and so

$$E_k(v) \leqslant (4\delta \ln n)^{k-2} \sum_{1 \leqslant u \leqslant 2n, u \neq v} \frac{\delta}{|v-u|}.$$

Now

$$\sum_{1 \leqslant u \leqslant 2n, u \neq v} \frac{\delta}{|v-u|} \leqslant 2 \sum_{1 \leqslant i \leqslant 2n} \frac{\delta}{i} \leqslant 2\delta\bigl(1+\ln(2n)\bigr) \leqslant 4\delta \ln n,$$

since $n \geqslant 6$, and on substitution This proves (4).

For each $v \in \{1, \dots, 2n\}$ and every integer $x \geqslant 1$, let $Z_v(x)$ be the sum, over all $u \neq v$ with $x \leqslant |u-v| \leqslant rx$, of the probability that $\{u, v\}$ is a backedge. (We recall that $r, s, t$ are in the statement of the theorem.)

(5) $Z_v(x) \leqslant 2\delta(1+\ln r)$ *for each* $v \in \{1, \dots, 2n\}$, *and every integer* $x \geqslant 1$.

For

$$Z_v(x) \leqslant \sum_{v+x \leqslant u \leqslant v+rx} \frac{\delta}{u-v} + \sum_{v-rx \leqslant u \leqslant v-x} \frac{\delta}{v-u} \leqslant 2 \sum_{x \leqslant i \leqslant rx} \frac{\delta}{i}.$$

But

$$\sum_{x \leqslant i \leqslant rx} \frac{1}{i} \leqslant \frac{1}{x} + \ln r \leqslant 1 + \ln r,$$

and so $Z_v(x) \leqslant 2\delta(1+\ln r)$. This proves (5).

Let $\phi(v) = v$ for $1 \leqslant v \leqslant 2n$. A path of the backedge graph $B$ with at least two edges, and with end-edges $e, f$, is *balanced* if $\phi(e) \leqslant r\phi(f)$ and $\phi(f) \leqslant r\phi(e)$. (We recall that for an edge $e = \{u, v\}$ of $B$, $\phi(e)$ means $|\phi(v) - \phi(u)|$.)

(6) *For each* $k \geqslant 2$, *the expected number of balanced paths in* $B$ *with* $k$ *edges is at most*

$$(4\delta)^k(1+\ln r)(\ln n)^{k-1} n.$$

The expected number of such paths is at most the sum (over all $v \in \{1, \dots, 2n\}$) of the expected number of pairs $(e, R)$, where

- $e$ is an edge of $B$ incident with $v$, with ends $u, v$ say,
- $R$ is a path of $B$ with one end $v$, not containing $u$, and with $k$ vertices,
- $\phi(f) \leqslant \phi(e) \leqslant r\phi(f)$, where $f \in E(R)$ is incident with the end of $R$ different from $v$.

(Note that one balanced path may correspond to two such pairs, if its end-edges have the same $\phi$-value.) Let us fix $v$ for the moment, and let $E_v$ denote the expected number of pairs $(e, R)$ as above. Let $M$ be the set of all sequences $(v_1, \dots, v_k)$ of distinct members of $\{1, \dots, n\}$ with $v_1 = v$. For $(v_1, \dots, v_k) \in M$, let $P(v_1, \dots, v_k)$ be the probability that $\{v_i, v_{i+1}\}$ $(1 \leqslant i \leqslant k-1)$ are all backedges, and let $Q(v_1, \dots, v_k)$ be the sum, over all $v_0$ different from $v_1, \dots, v_k$ with

$$|v_0 - v_1| \leqslant |v_{k-1} - v_k| \leqslant r|v_0 - v_1|,$$

of the probability that $\{v_0, v_1\}$ is a backedge. Then $E_v$ is the sum of $P(v_1, \dots, v_k)Q(v_1, \dots, v_k)$ over all $(v_1, \dots, v_k) \in M$. But for each $(v_1, \dots, v_k) \in M$,

$$Q(v_1, \dots, v_k) \leqslant Z_v\bigl(|v_{k-1} - v_k|\bigr) \leqslant 2\delta(1+\ln r)$$

by (5), and so $E_v$ is at most the sum, over all $(v_1, \dots, v_k) \in M$, of $2\delta(1 + \ln r)P(v_1, \dots, v_k)$. By (4), it follows that

$$E_v \leqslant 2\delta(1 + \ln r)(4\delta \ln n)^{k-1}.$$

By summing over the $2n$ values of $v$, This proves (6).

Now let us specify $\delta$. We take

$$\delta = \frac{1}{4}\bigl(2 + 2\ln(r)\bigr)^{-1/s}(\ln n)^{-\frac{s-1}{s}},$$

and note that $\delta \leqslant 1/4$.

(7) *The expected number of $(r, s)$-comparable pairs of edges in $B$ is at most $n/2$.*

From (6), this expected number is at most the sum of the expression from (6), summed for $2 \leqslant k \leqslant s$. Since

$$(4\delta)^k(\ln n)^{k-1} \leqslant (4\delta)^s(\ln n)^{s-1}$$

for $2 \leqslant k \leqslant s$ (because $n \geqslant (re)^2$ and hence $4\delta \ln(n) \geqslant 1$), this sum is at most

$$(1 + \ln r)(4\delta)^s(\ln n)^{s-1}n = \frac{1}{2},$$

from the choice of $\delta$. This proves (7).

Now to prove the theorem, let $\epsilon > 0$ be some real number, and let $t = (\ln(n))^{1/s-\epsilon}$. It follows that for all sufficiently large $n$, $t \leqslant \frac{1}{3}n^{1/3}$ and $6t\ln(4et) \leqslant \delta \ln(n)$. From (7) and Markov's inequality, the probability that there are at least $n$ $(r, s)$-comparable pairs is at most $1/2$. Also from (3), the probability that $\alpha(G) \geqslant n/t$ is at most $1/e$. Consequently, the probability that there are at most $n$ $(r, s)$-comparable pairs and $\alpha(G) < n/t$ is at least $1/2 - 1/e > 0$. Let $G$ be some tournament with these properties. Then there is a subset $X \subseteq V(G)$ with $|X| = n$, such that for every $(r, s)$-comparable pair of edges $f, g$, at least one of $f, g$ is incident with a vertex in $X$. It follows that the tournament $T$ induced on $V(G) \setminus X$ satisfies the theorem. This proves 6.6. $\square$

For the proof of 6.2 we need one more lemma, the following.

**6.7.** *Let $G$ be isomorphic to $\Delta(2, 2, 2)$, and let $\phi : V(G) \to \mathbb{Z}$ be injective. Then there is a $(2, 3)$-comparable pair of edges of the backedge graph.*

**Proof.** Let $B$ be the backedge graph. We assume (for a contradiction) that there is no $(2, 3)$-comparable pair of edges of $B$.

(1) *There is no cycle in $B$ of length at most five.*

Suppose that $C$ is such a cycle. For all distinct edges $e, f$ of $C$, there is a two- or three-edge path of $B$ containing them both, and since they are not $(2, 3)$-comparable, it follows that either $\phi(e) > 2\phi(f)$ or $\phi(f) > 2\phi(e)$. Let $E(C) = \{e_1, \dots, e_k\}$ say, where $\phi(e_i) > \phi(e_{i+1})$ for $1 \leqslant i < k$. Hence $\phi(e_i) > 2\phi(e_{i+1})$ for $1 \leqslant i < k$, and so

$$\phi(e_1) > \phi(e_2) + \phi(e_3) + \dots + \phi(e_k),$$

a contradiction since $e_2, e_3, \dots, e_k$ are the edges of a path of $B$ between the ends of $e_1$. This proves (1).

If $e \in E(B)$, we call $\phi(e)$ the *length* of $e$. Let $V(G) = \{a_1, a_2, b_1, b_2, c_1, c_2\}$, where

$$\bigl(\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}\bigr)$$

is a trisection of $G$. We may assume that $\phi(a_1) < \phi(a_2)$, and $\phi(b_1) < \phi(b_2)$, and $\phi(c_1) < \phi(c_2)$. Choose $x \in V(G)$ with $\phi(x)$ minimum, and $y \in V(G)$ with $\phi(y)$ maximum. We may assume that $x, y \notin \{a_1, a_2\}$, from the symmetry. Consequently $x$ is one of $b_1, c_1$, and $y$ is one of $b_2, c_2$. Let $\phi(y) - \phi(x) = n$ say.

(2) $x = c_1$.

Suppose not; then $x = b_1$. Consequently $\{a_1, b_1\}$, $\{a_2, b_1\}$ are both backedges. By (1) not both $\{a_1, c_2\}$, $\{a_2, c_2\}$ are backedges, and so $\phi(c_2) < \phi(a_2)$, and consequently $y = b_2$. Since $\{a_1, b_1\}$, $\{a_2, b_1\}$ have length at most $n$, and they are not $(2, 3)$-comparable, it follows that $\{a_1, b_1\}$ has length less than $n/2$, because of the path $a_1$-$b_1$-$a_2$. Since $\{b_2, c_1\}$, $\{b_2, c_2\}$ are both backedges it follows similarly that $\{b_2, c_2\}$ has length less than $n/2$. Consequently $\phi(a_1) < \phi(c_2)$, and so $\{a_1, c_2\}$ is a backedge. Then $b_1$-$a_1$-$c_2$-$b_2$ is a path of $B$, and the sum of the lengths of the three edges of this path equals $n$. Since no two of these three edges are $(2, 3)$-comparable, it follows that one of them has length at least $n/2$. We have already seen that the first and third edges of this path have length less than $n/2$, and so $\{a_1, c_2\}$ has length at least $n/2$. But then $\{a_2, b_1\}$ has length between $n/2$ and $n$, and so $\{a_1, c_2\}$, $\{a_2, b_1\}$ are $(2, 3)$-comparable (because of the path $c_2$-$a_1$-$b_1$-$a_2$), a contradiction. This proves (2).

From the symmetry under reversing the directions of all edges and exchanging $a_1$ with $a_2$ and $b_1$ with $c_2$ and $b_2$ with $c_1$, and replacing $\phi(v)$ by $-\phi(v)$ for each $v$, it follows from (2) that $y = b_2$. Since $\{b_1, c_1\}$, $\{b_2, c_2\}$ and $\{b_2, c_1\}$ are all backedges, forming a three-edge path of $B$, and the longest of them has length $n$, it follows that the other two both have length less than $n/2$, and one of them has length less than $n/4$. Consequently $\phi(c_2) - \phi(b_1) \geqslant n/4$. From (1), not both $\{a_1, b_1\}$, $\{a_1, c_2\}$ are backedges, so either $\phi(b_1)$, $\phi(c_2)$ are both less than $\phi(a_1)$, or they are both greater than $\phi(a_1)$. Similarly either $\phi(b_1)$, $\phi(c_2)$ are both less than $\phi(a_2)$, or they are both greater than $\phi(a_2)$. Thus there are three cases:

- $\phi(c_1) < \phi(b_1) < \phi(c_2) < \phi(a_1) < \phi(a_2) < \phi(b_2)$. In this case, $\{a_1, b_1\}$ and $\{a_2, b_1\}$ are backedges. Since $\phi(c_2) - \phi(b_1) \geqslant n/4$, it follows that $\{a_1, b_1\}$ has length at least $n/4$. Since $\{a_1, b_1\}$, $\{a_2, b_1\}$ are not $(2, 3)$-comparable, it follows that $\{a_2, b_1\}$ has length at least $n/2$. But then $\{a_2, b_1\}$, $\{b_2, c_1\}$ are $(2, 3)$-comparable, because of the path $a_2$-$b_1$-$c_1$-$b_2$, a contradiction.
- $\phi(c_1) < \phi(a_1) < \phi(b_1) < \phi(c_2) < \phi(a_2) < \phi(b_2)$. In this case, $\{a_2, b_1\}$ and $\{a_1, c_2\}$ are backedges. Since $\phi(c_2) - \phi(b_1) \geqslant n/4$, it follows that $\{a_2, b_1\}$ has length at least $n/4$; and since no two edges of the path $a_2$-$b_1$-$c_1$-$b_2$ are $(2, 3)$-comparable, it follows that $\{b_1, c_1\}$ has length less than $n/4$; and similarly $\{b_2, c_2\}$ has length less than $n/4$. It follows that $\phi(c_2) - \phi(b_1) \geqslant n/2$, and so $\{a_2, b_1\}$ has length at least $n/2$; but then $\{b_2, c_1\}$ and $\{a_2, b_1\}$ are $(2, 3)$-comparable, because of the path $a_2$-$b_1$-$c_1$-$b_2$, a contradiction.
- $\phi(c_1) < \phi(a_1) < \phi(a_2) < \phi(b_1) < \phi(c_2) < \phi(b_2)$; this is equivalent to the first case under the symmetry of reversing all edges, and therefore is impossible.

This proves that some pair of edges of $B$ are $(2, 3)$-comparable, and therefore proves 6.7. □

**Proof of 6.3 and 6.2.** Given $\epsilon > 0$, let $n$ be large enough that there is a tournament $T$ with $n$ vertices satisfying the conclusion of 6.6, taking $(r, s) = (2, 3)$. It follows from 6.7 that $T$ does not contain $\Delta(2, 2, 2)$. This proves 6.3 and hence 6.2. □

Thus, $H = \Delta(2, 2, 2)$ is not a celebrity; for each $c > 0$ there is an $H$-free tournament $G$ with $\alpha(G) < c|V(G)|$. Three of us will prove in another paper [2] that for every $\epsilon > 0$ there exists $c > 0$ such that every $H$-free tournament $G$ satisfies $\alpha(G) \geqslant c|V(G)|^{1-\epsilon}$.

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