Radial symmetry and monotonicity for an integral equation

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Abstract

In this paper we study radial symmetry and monotonicity of positive solutions of an integral equation arising from some higher-order semilinear elliptic equations in the whole space \( \mathbb{R}^n \). Instead of the usual method of moving planes, we use a new Hardy–Littlewood–Sobolev (HLS) type inequality for the Bessel potentials to establish the radial symmetry and monotonicity results. © 2007 Elsevier Inc. All rights reserved.

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1. Introduction

The radial symmetry and monotonicity of nonnegative solutions of elliptic equations in the unit ball or the whole Euclidean space have been studied by Gidas, Ni and Nirenberg using the technique of moving planes [7,8]. However, their method of moving planes is based on the maximum principle, and hence one is not able to apply it in the absence of the maximum principle. Recently, it was first observed by Chen, Li and Ou [4,5,12] that instead of the maximum principle, one can use a Hardy–Littlewood–Sobolev (HLS) type inequality to obtain the radial symmetry and monotonicity of nonnegative solutions of (systems of) elliptic equations of Yamabe type.

Motivated by the works mentioned above, we consider in this paper the positive solutions of the following semilinear partial differential equation in \( \mathbb{R}^n \):

\[
(I - \Delta) \frac{\alpha}{2} (u) = u^\beta,
\]

where \( \alpha > 0 \), \( \beta > 1 \), and \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) denotes the usual Laplace operator in \( \mathbb{R}^n \). We note that under some appropriate decay assumptions of the solutions at infinity, (1) is equivalent to the following integral equation

\[
u = g_\alpha \ast u^\beta,
\]
where $\ast$ denotes the convolution of functions and $g_\alpha$ is the Bessel kernel whose precise definition will be given in Section 2.

An important case of (1) corresponding to $\alpha = 2$ appears in the study of standing waves of the nonlinear Klein–Gordon equations
\[-\Delta u + u = u|u|^{\beta-1}, \quad \text{in} \ \mathbb{R}^n.\] (3)

Such a kind of equation also arises from the ground states of the Schrödinger equation [2]. Here we just want to mention that it has been shown that under some decay assumptions at the infinity, the smooth positive solutions of (3) are unique and radially symmetric. For more details, we refer to the paper [9] and the references therein.

Our main theorem is

**Theorem 1.** For any $q > \max\{\beta, n(\beta - 1)\alpha\}$, if $u \in L^q(\mathbb{R}^n)$ is a positive solution of (2), then $u$ must be radially symmetric and monotonely decrease about some point.

The proof of the main theorem is based on the method of moving planes introduced by Chen, Li and Ou. Here the new ingredient is a HLS type inequality for the Bessel potentials, which is established in Theorem 9.

### 2. Preliminaries on Bessel potentials

In this section, we recall some basic properties of the Bessel potentials. For more details, one may refer to [14,17].

**Definition 2.** The Bessel kernel $g_\alpha, \alpha \geq 0$, is given by
\[g_\alpha(x) = \frac{1}{\gamma(\alpha)} \int_0^\infty \exp\left(-\frac{\pi}{\delta}|x|^2\right) \exp\left(-\frac{\delta}{4\pi}\right)^{\frac{n-1}{2}} d\delta,\]
where $\gamma(\alpha) = (4\pi)^\frac{n}{2} \Gamma\left(\frac{n}{2}\right)$.

There are some elementary facts about the Bessel kernel $g_\alpha$.

**Proposition 3.**

1. **The Fourier transform of $g_\alpha$ is given by**
\[\hat{g}_\alpha(x) = \frac{1}{(1 + 4\pi^2|x|^2)^\frac{n}{2}},\]
where the Fourier transform of $f$ is defined by
\[\hat{f}(x) = \int_{\mathbb{R}^n} f(t) \exp(-2\pi i xt) dt.\]

2. For each $\alpha > 0$, $g_\alpha(x) \in L^1(\mathbb{R}^n)$.

3. **The following Bessel composition formula holds**
\[g_\alpha \ast g_\beta = g_{\alpha+\beta}, \quad \alpha, \beta \geq 0.\] (4)

**Definition 4.** The Bessel potentials $B_\alpha(f), \ f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, is defined by
\[B_\alpha(f) = \begin{cases} g_\alpha \ast f, & \alpha > 0, \\ f, & \alpha = 0. \end{cases}\]

For later use, we mention the following two properties of the Bessel potentials.
Proposition 5.

(1) \[ \|B_\alpha(f)\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p \leq \infty. \]

(2) \[ B_\alpha \circ B_\beta = B_{\alpha+\beta}, \quad \alpha, \beta \geq 0. \]

Proof. (1) Follows from the Young inequality and the fact that \( \|g_\alpha\|_{L^1(\mathbb{R}^n)} = 1 \).

(2) Follows immediately from the Bessel composition formula (4).

The most interesting fact concerning Bessel potentials is that they can be employed to characterize the Sobolev spaces \( W^{k,p}(\mathbb{R}^n) \). This is expressed in the following theorem where we employ the notation \( L^{\alpha,p}(\mathbb{R}^n) \), \( \alpha > 0 \) and \( 1 \leq p \leq \infty \), to denote all functions \( u \) such that \( u = g_\alpha * f \) for some \( f \in L^p(\mathbb{R}^n) \). We may define a norm on \( L^{\alpha,p}(\mathbb{R}^n) \) as follows

\[ \|u\|_{L^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}. \]

With respect to this norm, \( L^{\alpha,p}(\mathbb{R}^n) \) becomes a Banach space.

Theorem 6. If \( k \) is a nonnegative integer and \( 1 < p < \infty \), then \( L^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n) \). Moreover, if \( u \in L^{k,p}(\mathbb{R}^n) \) with \( u = g_\alpha * f \), then

\[ C^{-1} \|f\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{W^{k,p}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \]

where \( C = C(\alpha, n, p) \).

Before ending this section, we would like to point out that on Sobolev spaces \( W^{k,p}(\mathbb{R}^n) \), \( B_\alpha = (I - \Delta)^{-\frac{\alpha}{2}} \).

3. A HLS type inequality

In this section, we prove a new HLS type inequality which will play an important role in the proof of Theorem 1. To that purpose, we recall the classical Sobolev embedding theorem of the spaces \( W^{k,p}(\mathbb{R}^n) \) [1].

Theorem 7 (The Sobolev embedding theorem). For \( 1 \leq p < \infty \), there exist the following embeddings:

\[ W^{k,p}(\mathbb{R}^n) \hookrightarrow \begin{cases} L^{q}(\mathbb{R}^n), & kp < n \text{ and } q \leq \frac{np}{n-kp}, \\ L^{q}(\mathbb{R}^n), & kp = n \text{ and } p \leq q < \infty, \\ C_b(\mathbb{R}^n), & kp > n, \end{cases} \]

where \( C_b(\mathbb{R}^n) = \{u \in C(\mathbb{R}^n) \mid u \text{ is bounded on } \mathbb{R}^n\} \).

The following lemma comes from [1, Theorem 7.63, (d) and (e)].

Lemma 8.

(1) If \( t \leq s \) and \( 1 < p \leq q \leq \frac{np}{n-(s-1)p} < \infty \), then \( L^{s,p}(\mathbb{R}^n) \hookrightarrow L^{1,q}(\mathbb{R}^n) \).

(2) If \( 0 \leq \mu \leq s - n/p < 1 \), then \( L^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n) \).

Now we are ready to prove the following HLS type inequality.

Theorem 9. Let \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \). If \( f \in L^{\frac{q}{q}}(\mathbb{R}^n) \), then \( B_\alpha(f) \in L^{q}(\mathbb{R}^n) \). Moreover, we have the estimate

\[ \|B_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^{\frac{q}{q}}(\mathbb{R}^n)}, \quad \text{where } C = C(\alpha, \beta, n, q). \]

Proof. By definition, \( B_\alpha(f) \in L^{\alpha,q}(\mathbb{R}^n) \) provided that \( f \in L^{\frac{q}{q}}(\mathbb{R}^n) \). We divide the proof into several cases.

Case 1. \( \alpha \) is an integer.
It follows from Theorem 6 that $B_\alpha(f) \in W^{\alpha, \frac{q}{p}}(\mathbb{R}^n)$. Note that

$$\frac{n \cdot \frac{q}{p}}{n - \alpha \cdot \frac{q}{p}} = \frac{n}{n \beta - \alpha q} \cdot q > \frac{n}{n \beta - \alpha \cdot n (\beta - 1)} \cdot q = q > \frac{q}{\beta}. $$

So if $\alpha \cdot \frac{q}{p} \leq n$, then by the Sobolev embedding theorem, we have $B_\alpha(f) \in L^q(\mathbb{R}^n)$. Now it follows again from Theorem 6 that

$$\|B_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C \|B_\alpha(f)\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)} \leq C \|B_\alpha(f)\|_{L^\alpha, \frac{q}{p}(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}. $$

If $\alpha \cdot \frac{q}{p} > n$, then it follows from the Sobolev embedding theorem that $B_\alpha(f) \in C_b(\mathbb{R}^n)$ and $\|B_\alpha(f)\|_{C_b(\mathbb{R}^n)} \leq C \|B_\alpha(f)\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)} \leq C \|f\|_{L^\alpha, \frac{q}{p}(\mathbb{R}^n)}. $ Now a straightforward computation shows that

$$\|B_\alpha(f)\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |B_\alpha(f)|^q dx\right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n} |B_\alpha(f)|^\frac{q}{\beta} \cdot \|B_\alpha(f)\|_{C_b(\mathbb{R}^n)}\right)^{\frac{1}{q}} = \|B_\alpha(f)\|_{L^\alpha, \frac{q}{p}(\mathbb{R}^n)} \cdot \|B_\alpha(f)\|_{C_b(\mathbb{R}^n)} \leq \|f\|_{L^\alpha, \frac{q}{p}(\mathbb{R}^n)} \cdot C \|f\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)} = C \|f\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)}. $$

**Case 2.** $\alpha$ is a fraction.

In this case $\alpha - [\alpha] > 0$, where $[\alpha]$ denotes the integer satisfying $[\alpha] \leq \alpha < [\alpha] + 1$. We divide this case into two subcases, i.e. $q < \frac{n \beta}{\alpha - [\alpha]}$ and $q \geq \frac{n \beta}{\alpha - [\alpha]}$.

**Case 2.1.** $q < \frac{n \beta}{\alpha - [\alpha]}$.

Let $r_0 = \frac{n q}{n - (\alpha - [\alpha]) \frac{q}{p}}$. Since $[\alpha] \leq \alpha$ and $\frac{q}{p} > 1$, it follows from Lemma 8(1), that for any $r$ satisfying $\frac{q}{p} \leq r \leq r_0$, we have $L^{\alpha, \frac{q}{p}}(\mathbb{R}^n) \hookrightarrow L^{[\alpha], r}(\mathbb{R}^n)$. Note that $B_\alpha(f) \in L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)$ and $[\alpha]$ is an integer. Thus $B_\alpha(f) \in W^{[\alpha], r}(\mathbb{R}^n)$. There are two possibilities.

**Case 2.1.1.** $q \leq r_0$.

This is equivalent to $q \geq \frac{n (\beta - 1)}{\alpha - [\alpha]}$. In this situation, $q$ lies in the interval $[\frac{q}{p}, r_0]$. Hence $B_\alpha(f) \in W^{[\alpha], q}(\mathbb{R}^n)$. Moreover,

$$\|B_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq \|B_\alpha(f)\|_{W^{[\alpha], q}(\mathbb{R}^n)} \leq C \|B_\alpha(f)\|_{L^{[\alpha], q}(\mathbb{R}^n)} \leq C \|B_\alpha(f)\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)} = C \|f\|_{L^{\alpha, \frac{q}{p}}(\mathbb{R}^n)}. $$

**Case 2.1.2.** $q > r_0$.

This is equivalent to $q < \frac{n (\beta - 1)}{\alpha - [\alpha]}$. In this situation, we have $B_\alpha(f) \in W^{[\alpha], r_0}(\mathbb{R}^n)$.

**Case 2.1.2.1.** $q < \frac{n \beta}{\alpha - [\alpha]}$. 

We have $[\alpha]_0 < n$. By assumption, $q \geq \frac{n(\beta - 1)}{\alpha}$. Hence $q$ actually lies in the interval $[r_0, \frac{n r_0}{n - [\alpha]_0}]$. Now it follows from the Sobolev embedding theorem that $B_\alpha(f) \in L^q(\mathbb{R}^n)$.

**Case 2.1.2.2.** $q = \frac{n \beta}{\alpha}$.

We have $[\alpha]_0 = n$. Again it follows from the Sobolev embedding theorem that $B_\alpha(f) \in L^q(\mathbb{R}^n)$.

**Case 2.1.2.3.** $q > \frac{n \beta}{\alpha}$.

We have $[\alpha]_0 > n$. Again it follows from the Sobolev embedding theorem that $B_\alpha(f) \in C_b(\mathbb{R}^n)$. A straightforward calculation shows that
\[
\left\| B_\alpha^\beta(f) \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| B_\alpha(f) \right\|_{L^{\frac{q}{\beta}}(\mathbb{R}^n)}.
\]

**Case 2.2.** $q \geq \frac{n \beta}{\alpha - [\alpha]}$.

First, we observe that $B_{\alpha - [\alpha]}(f) \in L^{\alpha - [\alpha], \frac{q}{\beta}}(\mathbb{R}^n)$. Note that $q \geq \frac{n \beta}{\alpha - [\alpha]} \iff 0 \leq \mu_0 < 1$, where $\mu_0 = \alpha - [\alpha] - \frac{n \beta}{q}$.

It follows from Lemma 8(2), that $B_{\alpha - [\alpha]}(f) \in C^{0, \mu_0}(\mathbb{R}^n)$. Then as before, we obtain the following estimate.
\[
\left\| B_\alpha(f) \right\|_{L^q(\mathbb{R}^n)} = \left\| B_\alpha \circ B_{\alpha - [\alpha]}(f) \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| B_{\alpha - [\alpha]}(f) \right\|_{L^{\frac{q}{\beta}}(\mathbb{R}^n)}.
\]

This completes the proof.

4. **Proof of Theorem 1**

For a given real number $\lambda$, we may define
\[
\Sigma_\lambda = \left\{ x = (x_1, \ldots, x_n) \mid x_1 \geq \lambda \right\}.
\]

Let $x^\lambda = (2\lambda - x_1, \ldots, x_n)$ and define $u_\lambda(x) = u(x^\lambda)$.

**Lemma 10.** For any solution $u(x)$ of (2), we have
\[
u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) (u(y) - u_\lambda(y))^\beta \, dy.
\]
**Proof.** Let $\Sigma^c_\lambda = \{x = (x_1, \ldots, x_n) \mid x_1 < \lambda\}$. It is easy to see that

$$
u(x) = \int_{\Sigma_\lambda} g_\alpha(x - y)u(y)^\beta \, dy + \int_{\Sigma^c_\lambda} g_\alpha(x - y)u(y)^\beta \, dy$$

$$= \int_{\Sigma_\lambda} g_\alpha(x - y)u(y)^\beta \, dy + \int_{\Sigma^c_\lambda} g_\alpha(x - y^\lambda)u(y^\lambda)^\beta \, dy$$

$$= \int_{\Sigma_\lambda} g_\alpha(x - y)u(y)^\beta \, dy + \int_{\Sigma^c_\lambda} g_\alpha(x^\lambda - y)u_\lambda(y)^\beta \, dy.$$ 

Here we have used the fact that $|x - y^\lambda| = |x^\lambda - y|$. Substituting $x$ by $x^\lambda$, we get

$$\nu(x^\lambda) = \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y)u(y)^\beta \, dy + \int_{\Sigma^c_\lambda} g_\alpha(x^\lambda - y)u_\lambda(y)^\beta \, dy.$$ 

Thus

$$\nu(x) - \nu(x^\lambda) = \int_{\Sigma_\lambda} g_\alpha(x - y)(u(y)^\beta - u_\lambda(y)^\beta) \, dy - \int_{\Sigma^c_\lambda} g_\alpha(x^\lambda - y)(u(y)^\beta - u_\lambda(y)^\beta) \, dy$$

$$= \int_{\Sigma_\lambda} (g_\alpha(x - y) - g_\alpha(x^\lambda - y))(u(y)^\beta - u_\lambda(y)^\beta) \, dy.$$ 

This completes the proof. \(\square\)

We may define $\Sigma^-_\lambda = \{x \mid x \in \Sigma_\lambda, \nu(x) < u_\lambda(x)\}$.

**Lemma 11.** For $\lambda \ll 0$, $\Sigma^-_\lambda$ must be measure zero.

**Proof.** Whenever $x, y \in \Sigma_\lambda$, we have $|x - y| \leq |x^\lambda - y|$. Moreover, since $\lambda < 0$, $|y^\lambda| \geq |y|$ for any $y \in \Sigma_\lambda$. Then it follows from Lemma 10 that for any $x \in \Sigma^-_\lambda$,

$$u_\lambda(x) - \nu(x) \leq \int_{\Sigma^-_\lambda} g_\alpha(x - y)(u_\lambda(y)^\beta - u(y)^\beta) \, dy$$

$$\leq \beta \int_{\Sigma^-_\lambda} g_\alpha(x - y)[u_\lambda^{\beta - 1}(u_\lambda - u)](y) \, dy.$$ 

It follows first from the HLS type inequality (Theorem 9) and then the Hölder inequality that for $q$ as in Theorem 1, we have

$$\|u_\lambda - \nu\|_{L^q(\Sigma^-_\lambda)} \leq \beta \|B_\alpha[u_\lambda^{\beta - 1}(u_\lambda - u)]\|_{L^q(\Sigma^-_\lambda)}$$

$$\leq C \|u_\lambda^{\beta - 1}(u_\lambda - u)\|_{L^q(\Sigma^-_\lambda)}$$

$$\leq C \|u_\lambda\|_{L^q(\Sigma^-_\lambda)}^{\beta - 1} \|u_\lambda - u\|_{L^q(\Sigma^-_\lambda)}$$

$$\leq C \|u_\lambda\|_{L^q(\Sigma^-_\lambda)}^{\beta - 1} \|u_\lambda - u\|_{L^q(\Sigma^-_\lambda)}$$

$$= C \|u\|_{L^q(\Sigma^-_\lambda)}^{\beta - 1} \|u_\lambda - u\|_{L^q(\Sigma^-_\lambda)}.$$ 

Since $u \in L^q(\mathbb{R}^n)$, we can choose $N \gg 0$, s.t. for $\lambda \leq -N$, $C \|u\|_{L^q(\Sigma^-_\lambda)}^{\beta - 1} \leq \frac{1}{2}$. Then it follows from the last inequality that $\|u_\lambda - \nu\|_{L^q(\Sigma^-_\lambda)} = 0$, and therefore $\Sigma^-_\lambda$ must be measure zero. \(\square\)
Now we fix an $N \gg 0$ s.t. for $\lambda \leq -N$,
\[ u(x) \geq u_{\lambda_j}(x), \quad \forall x \in \Sigma_{\lambda_j}. \] (5)

We start moving the planes continuously from $\lambda \leq -N$ to the right as long as (5) holds. Suppose that at a $\lambda_0 < 0$, we have $u(x) \geq u_{\lambda_0}(x)$ and $m(\{x \in \Sigma_{\lambda_0} \mid u(x) > u_{\lambda_0}(x)\}) > 0$. By Lemma 10, we know that $u(x) > u_{\lambda_0}(x)$ in the interior of $\Sigma_{\lambda_0}$. Let $\bar{\Sigma}_{\lambda_0} = \{x \in \Sigma_{\lambda_0} \mid u(x) \leq u_{\lambda_0}(x)\}$. It is easy to see that $\bar{\Sigma}_{\lambda_0}$ has measure zero and $\lim_{\lambda \to \lambda_0} \Sigma_{\lambda}^{-} \subset \bar{\Sigma}_{\lambda_0}^{-}$. Let $(\Sigma_{\lambda}^{-})^\ast$ be the reflection of $\Sigma_{\lambda}^{-}$ about the planes $x_1 = \lambda$. It follows from the above arguments that
\[ \|u_\lambda - u\|_{L^q(\Sigma_{\lambda}^\ast)} \leq C \|u\|_{L^q(\Sigma_{\lambda}^{-})}^{\frac{1}{q}} \|u_\lambda - u\|_{L^q(\Sigma_{\lambda}^\ast)}^{\frac{1}{q}}. \] (6)

As before, the integrability condition on $u$ guarantees that there exists an $\epsilon \ll 0$, s.t. $\forall \lambda \in [\lambda_0, \lambda_0 + \epsilon)$, $C \|u\|_{L^q(\Sigma_{\lambda}^\ast)}^{\frac{1}{q}} \leq \frac{1}{2}$. Now by (6), we have $\|u_\lambda - u\|_{L^q(\Sigma_{\lambda}^\ast)} = 0$, and hence $\Sigma_{\lambda}^{-}$ must be measure zero. Therefore the planes can be moved further to the right, i.e., there exist an $\epsilon$ depending on $n$, $\alpha$, $\beta$, $q$ and the solution $u$, such that $u(x) \geq u_{\lambda_j}(x)$ on $\Sigma_\lambda$, $\forall \lambda \in [\lambda_0, \lambda_0 + \epsilon)$.

The planes either stops at some $\lambda < 0$ or can be moved until $\lambda = 0$. In the former case, it turns out that $u_{\lambda_j}(x) = u(x)$, $\forall x \in \Sigma_\lambda$. In the latter case, we have $u_{\lambda_j}(x) \leq u(x)$, $\forall x \in \Sigma_0$. However, if we move the planes from the right to the left, then we may see that $u_{\lambda_j}(x) \geq u(x)$, $\forall x \in \Sigma_0$. Thus $u_{\lambda_j}(x) = u(x)$, $\forall x \in \Sigma_0$. Since we can choose any direction to start the process, we conclude that $u$ is actually radial symmetric and monotone decreases about some point in $\mathbb{R}^n$. This completes the proof of Theorem 1.

We point out that there are some interesting works related to ours. See the references [3,6,10,11,13,15,16].

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