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J. Math. Anal. Appl. 332 (2007) 570–576

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Analytic discs in the boundary and compactness of Hankel operators with essentially bounded symbols [☆]

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Received 5 May 2006

Available online 27 November 2006

Submitted by K.-T. Kim

Abstract

We derive conditions for compactness of Hankel operators $H_f: A^2(\Omega) \to L^2(\Omega)$ $(H_f(g) := (I-P)(\bar{f}g))$ with bounded, holomorphic symbols f for a large class of convex and bounded domains Ω with $\Omega \subseteqq \mathbb{D}^k$.

Keywords: Hankel operator; Bergman spaces and Bergman kernel; Bergman projection; Canonical solution operator to $\bar\partial$

1. Introduction

The Bergman space $A^2(\Omega)$ of the bounded and convex domain Ω in $\mathbb{C}^k$ is defined as

$$A^2(\Omega) := \left\{ f\colon \text{holomorphic in } \Omega \text{ and } \int_\Omega |f(z)|^2 \, dv(z) < \infty \right\},$$

where v denotes the Lebesgue-measure in $\mathbb{C}^k$. Remember, that the Hankel operator with symbol g is given by

$$H_g(f): A^2(\Omega) \to L^2(\Omega)\colon\ H_g(f) = (I-P)(\bar{g}f),$$

☆ This work was supported by FWF (P 15279).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.10.015

where

$$L^2(\Omega) := \left\{ f\colon \text{ measurable in } \Omega \text{ and } \|f\|^2 := \int_\Omega |f(z)|^2\, dv(z) < \infty \right\}$$

and P is the orthogonal projection onto $A^2(\Omega)$ (the Bergman projection). It is obvious that the Hankel operator H_g is bounded if g is essentially bounded. The aim of this paper is to investigate compactness of Hankel operators with (essentially-) bounded holomorphic symbols g. Such a symbol can be written as $g(z) = \sum_l a_l z^l$, where the summation is over all possible (positive) multi-indices $l = (l_1, \dots, l_k)$. Furthermore, we will assume that the set $\{\frac{z^n}{c_n}\colon n \in \mathbb{N}^k\}$ constitutes a complete orthonormal system, where $c_n^2 = \int_\Omega |z^n|^2\, dv(z)$ are the so-called moments. It is obvious that convex Reinhardt-domains satisfy this property.

In the last years there have been several results concerning Hankel operators on different spaces of holomorphic functions. We only mention some of them (for an introduction to the theory of Hankel operators see [7]). In [11] it is shown, that there are no Hilbert–Schmidt Hankel operators with holomorphic symbols on the Bergman space of the unit disc. Furthermore [12] investigates Schatten-class membership of Hankel operators on the Bergman space, whereas [13] considers weighted Bergman spaces. Also some Hankel operators with certain L^2-symbols (that is, not necessarily holomorphic ones) have been studied (see for example [9,10]).

There is some work on the connection of boundary conditions of domains and operator theoretic properties of Hankel operators (see [6]). Additional results can be found in [1,2].

Further results deal with spaces of entire functions—and especially the Fock-space. For more on this see [5,8].

2. Non-compactness of Hankel operators with bounded, holomorphic symbols

First, we want to show that an affine variety in the boundary (of complex dimension greater or equal to 1) implies that the operators $H_{z_i^n}$ cannot be compact for all $n \in \mathbb{N}$. The proof is similar to the one found in [3], where it is shown that the canonical solution operator to $\bar\partial$ cannot be compact if there is an affine variety in the boundary. However, we include it for the convenience of the reader.

Proposition 1. *If Ω has an affine variety in the boundary then for each $n \in \mathbb{N}$ the Hankel operator $H_{z_i^n}$ is not compact.*

Proof. We only carry out the proof for $i = 1$. The proof for general i is analogous. We can assume without loss of generality that $\{(z_1, 0) \in \mathbb{C}^k\colon |z_1| < 2\} \subseteq \partial\Omega$. Let $z'' = (z_2, \dots, z_k)$. Let $\Omega_1 = \{z'' \in \mathbb{C}^{k-1}\colon (0, z'') \in \Omega\}$. It follows from the convexity of Ω that Ω_1 is a convex domain in $\mathbb{C}^{k-1}(z'')$. Let $\Omega_2 = \{z'' \in \mathbb{C}^{k-1}\colon 2z'' \in \Omega_1\}$.

Let $p_0 \in \Omega_2$, $p_j = p_0/j$, $j \in \mathbb{N}$, and

$$f_j(z'') = \frac{K_{\Omega_1}(z'', p_j)}{\sqrt{K_{\Omega_1}(p_j, p_j)}}.$$

Here K_{Ω_1} denotes the Bergman kernel of $A^2(\Omega_1)$. Then it is shown as in [3] that $\|f_j\|_{\Omega_1} = 1$ and

$$\|f_j\|_{\Omega_2}^2 \geqslant C_1 > 0$$

for j large enough. Furthermore $f_j \to 0$ locally uniformly on Ω_1. That is, the sequence $(f_j)_{j\in\mathbb{N}}$ converges uniformly to 0 on every compact subset K of Ω_1. We can conclude, that f_j has no subsequence that is a Cauchy sequence in $L^2(\Omega_2)$. This can be seen as follows. Let to the contrary (without loss of generality) $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence. Then the functions f_j converge in L^2 to some function f and there is a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ that converges to f almost everywhere (with respect to ν). Since the sequence $(f_j)_{j\in\mathbb{N}}$ converges to 0 locally uniformly, we have $f\equiv 0$ (almost everywhere), which is a contradiction to

$$\|f_j\|^2_{\Omega_2}\geqslant C_1>0$$

for j large enough.

According to the Ohsawa–Takegoshi extension theorem [4] there exist holomorphic functions $F_j(z_1,z'')$ on Ω such that $F_j(0,z'')=f_j(z'')$ and $\|F_j\|_\Omega\leqslant C_2$. Furthermore let $g_j(z_1,z'')=\tilde{H}_{z_1}F_j(z_1,z'')$ and $\chi\in C_0^\infty(-\infty,\infty)$ a cutoff function with $0\leqslant\chi\leqslant 1$ with $\chi(t)=1$ if $t\leqslant\frac{1}{2}$ and $\chi(t)=0$ if $t\geqslant\frac{3}{4}$. Here $\tilde{H}_{z_1}$ is the Hankel operator with symbol z_1 corresponding to the projection P', where P' denotes the projection onto the kernel of $\frac{\partial^n}{\partial^n\bar{z}_1}$. Note that the Hankel operator $\tilde{H}_{z_1^n}=(I-P')z_1^n$ is the canonical solution operator to $\frac{\partial^n}{\partial^n\bar{z}_1}$ (restricted to holomorphic functions). Then we have

$$\begin{aligned}\left|f_j(z'')-f_l(z'')\right| &= C\left|\int\limits_{|z_1|<1}\left(F_j(z_1,z'')-F_l(z_1,z'')\right)\chi\left(|z_1|\right)d\nu(z_1)\right|\\ &= C\left|\int\limits_{|z_1|<1}\frac{\partial^n}{\partial^n\bar{z}_1}\tilde{H}_{z_1^n}\left(F_j(z_1,z'')-F_l(z_1,z'')\right)\chi\left(|z_1|\right)d\nu(z_1)\right|\\ &= C\left|\int\limits_{|z_1|<1}\left(g_j(z_1,z'')-g_l(z_1,z'')\right)\frac{\partial^n}{\partial^n z_1}\chi\left(|z_1|\right)d\nu(z_1)\right|\\ &\leqslant C'\left\{\int\limits_{|z_1|<1}\left|g_j(z_1,z'')-g(z_1,z'')\right|^2 d\,\nu(z_1)\right\}^{1/2}.\end{aligned}$$

Integrating with respect to z'' yields

$$\|f_j-f_l\|_{\Omega_2}\leqslant C'\|g_j-g_l\|_\Omega.$$

Since $(f_j)_{j\in\mathbb{N}}$ has no subsequence, that is a Cauchy sequence, g_j cannot have a convergent subsequence (in the $L^2(\Omega)$-topology) and therefore the operator $H_{z_1^n}$ is not compact (because even $\tilde{H}_{z_1^n}$ is not compact). □

Remark 2. Assume that Ω contains an affine variety of dimension $k-1$ in the boundary. Then the above proposition also holds if one replaces the operators $H_{z_i{}^n}$ by H_{z^n}, where $n=(n_1,\ldots,n_k)$ is a multi-index with at least one $n_i=0$.

Remark 3. The relation of the above proposition to [6] should be pointed out. There Hankel operators on the Bergman space of ellipsoids $D(m_1,m_2)$ are considered. Here,

$$D(m_1,m_2)=\left\{(z_1,z_2)\in\mathbb{C}^2\colon\ |z_1|^{2m_1}+|z_2|^{2m_2}<1\right\}$$

and $m_1\leqslant m_2$. Let $Y_p(D(m_1,m_2))$ be the collection of all functions $f\in L^2(D(m_1,m_2))$ with

$$\|f\|^p_{Y_p}=\int\limits_{D(m_1,m_2)}\left(\int\limits_{D(m_1,m_2)}\left|f(w)-\tilde{f}(z)\right|^2 d\nu(w)\right)^{p/2}d\lambda(z)<\infty,$$

where

$$d\lambda(z) = K(z,z)\,d\nu(z),$$

$K(z,w)$ is the reproducing kernel of $A^2(D(m_1,m_2))$ and $\tilde{f}$ is the Berezin transform of f. For $f \in A^2(D(m_1,m_2))$ it is shown in [6] that if H_f is in the Schatten-class $S_p(A^2(D(m_1,m_2)))$ then

(1) for $0 \leqslant p \leqslant \max\{2m_1, 4\}$ we have $f \in Y_p(D(m_1,m_2))$ and $\frac{\partial}{\partial z_1} f(0,z_2) = \frac{\partial}{\partial z_2} f(z_1,0) = 0$ for all $z_1, z_2 \in D(m_1,m_2)$, and
(2) for $0 \leqslant p \leqslant \max\{2m_2, 4\}$ we have $f \in Y_p(D(m_1,m_2))$ and $\frac{\partial}{\partial z_2} f(z_1,0) = 0$ for all $z_1 \in D(m_1,m_2)$.

In addition the limit $m_1, m_2 \to \infty$ of the ellipsoids $D(m_1,m_2)$ is considered. This is the polydisc in $\mathbb{C}^2$. It is shown for this domain and $f \in A^2(\mathbb{D}^2)$ that H_f on $A^2(\mathbb{D}^2)$ is compact if and only if f is constant. For $\mathbb{D}^2$ this would imply the conclusion of Proposition 1 and the remark following Proposition 1. In addition, this result is generalized to $\mathbb{D}^k$ in [6].

Remark 4. Let Ω be convex. It is well known that if the boundary of Ω contains an analytic variety then it contains an affine variety. (See [3].)

We will only formulate the following propositions for the case one complex dimension. However, analogous results are valid if one replaces indices by multi-indices.

Proposition 5. *Let the Hankel operator H_g be compact, where g is an holomorphic symbol. Then we must have*

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\| \to 0$$

as $n \to \infty$.

Proof. We note that $\frac{z^n}{c_n}$ converges weakly to 0 as $n \to \infty$. (It is an orthonormal basis of $A^2(\Omega)$.) Since compact operators map weakly convergent sequences to norm-convergent sequences it follows that

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\| \to 0$$

as $n \to \infty$. □

Remark 6. The preceding proposition could have been directly verified if we would have restricted our attention to polynomial symbols g

$$g = \sum_{l=0}^{N} a_l z^l.$$

In this case we have

$$\left(H_g \frac{z^n}{c_n} \,\middle|\, H_g \frac{z^m}{c_m} \right) = 0$$

if $|n-m|$ is large enough. Consequently, it is not difficult to show that if the condition of the above proposition is fulfilled for the symbol g, the corresponding Hankel operator is compact.

Let Ω' be a convex Reinhardt-domain such that $\overline{\Omega} \subset \Omega'$. The following proposition shows that the necessary conditions from Proposition 1 are sufficient as well if $g \in A^2(\Omega')$. All norms in the following are as before the ones of $L^2(\Omega)$ unless stated otherwise.

Proposition 7. *Let $g = \sum a_l z^l \in A^2(\Omega')$. If we have*

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\| \to 0 \tag{1}$$

as $n \to \infty$, the Hankel operator H_g is compact on $A^2(\Omega)$.

Proof. Since convergence in $A^2(\Omega')$ implies uniform convergence on compact subsets of Ω' it is clear that g is the uniform limit of the functions $\sum_{l=0}^N a_l z^l$ as $N \to \infty$ on $\overline{\Omega}$. The Hankel operators satisfy

$$\left\| H_{z^l}\left(\frac{z^n}{c_n}\right)\right\|^2 = \frac{c_{n+l}^2}{c_n^2} - \frac{c_n^2}{c_{n-l}^2} > 0$$

if $l \leqslant n$ and

$$\left\| H_{z^l}\left(\frac{z^n}{c_n}\right)\right\|^2 = \frac{c_{n+l}^2}{c_n^2}$$

otherwise. One can derive the following identity as in [5]:

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_{l \leqslant n} |a_l|^2 \left[\frac{c_{n+l}^2}{c_n^2} - \frac{c_n^2}{c_{n-l}^2}\right] + \sum_{l>n} |a_l|^2 \frac{c_{n+l}^2}{c_n^2}.$$

Consequently the (diagonal) operators H_{z^l} are compact if assumption (1) is satisfied and $a_l \neq 0$. Since the operator H_g is the limit (in the operator norm) of a sequence Hankel-operators with polynomial symbols $(\sum_{l=0}^N a_l z^l)$, which are compact by the above argument, the operator H_g is compact. This finishes the proof. □

Remark 8. It follows from the fact that H_{z^n} is compact for all n that $H_{z^n \bar{z}^l}$ is compact for all $l, n \in \mathbb{N}$. To see this we note that

$$\left\| H_{z^n \bar{z}^l}\left(\frac{z^m}{c_m}\right)\right\| = \frac{c_{l+m}}{c_m} \left\| H_{z^n}\left(\frac{z^{m+l}}{c_{m+l}}\right)\right\|.$$

Since

$$0 < C_{l,1} \leqslant \frac{c_{l+m}}{c_m} \leqslant 1 \quad \forall l, m$$

$(\Omega \subseteq \mathbb{D}^k)$ we know that

$$\left\| H_{z^n \bar{z}^l}\left(\frac{z^m}{c_m}\right)\right\| \to 0 \tag{2}$$

as $m \to \infty$ if and only if

$$\left\| H_{z^n}\left(\frac{z^{m+l}}{c_{m+l}}\right)\right\| \to 0 \tag{3}$$

as $m \to \infty$. Finally, we note that $H_{z^n \bar{z}^l}(\frac{z^m}{c_m})$ is orthogonal to $H_{z^n \bar{z}^l}(\frac{z^j}{c_j})$ if $j \neq m$.

Remark 9. It follows from above, that if the operator H_{z^n} is compact for some $n \in \mathbb{N}$, then also the operators $H_{z^n \bar{z}^l}$ are compact for all $l \in \mathbb{N}$. Moreover, if H_g is compact for $g = \sum a_k z^l$, then the operators H_{z^l} are compact for all l with $|a_l| \neq 0$. We will see in the following section that analogous results hold in the case of several complex dimensions.

3. Applications in several complex dimensions

As mentioned above, all the results of the preceding section also hold true, if one replaces indices by multi-indices in a suitable way. Concretely, it is easily shown that

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_{l \leqslant n} |a_l|^2 \left[\frac{c_{n+l}^2}{c_n^2} - \frac{c_n^2}{c_{n-l}^2}\right] + \sum_{l \nleqslant n} |a_l|^2 \frac{c_{n+l}^2}{c_n^2}.$$

for $g = \sum_l a_l z^l$. Here $l = (l_1, \ldots, l_k) \in \mathbb{N}^k$ and $l \leqslant n$ if and only if $l_j \leqslant n_j$ for all j. Furthermore,

$$z^l := z_1^{l_1} \ldots z_k^{l_k}.$$

The following proposition is an application of the characterization of compactness of the Hankel operators with anti-holomorphic and essentially bounded symbols and of Proposition 1.

Proposition 10. *Let $g = \sum a_l z^l \in A^2(\Omega)$ be a holomorphic symbol. Here $l = (l_1, \ldots, l_k) \in \mathbb{N}^k$ is a multi-index. Assume furthermore that H_g is compact and that Ω has an affine variety of dimension $k-1$ in the boundary. Then we must have $a_l = 0$ for all $l \in \mathbb{N}^k$ with $l_i = 0$ for at least one $i \in \{1, \ldots, k\}$.*

Proof. It follows as in the previous section from the equation

$$\left\| H_g\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_{l \leqslant n} |a_l|^2 \left[\frac{c_{n+l}^2}{c_n^2} - \frac{c_n^2}{c_{n-l}^2}\right] + \sum_{l \nleqslant n} |a_l|^2 \frac{c_{n+l}^2}{c_n^2}$$

that if $H_{\bar{g}}$ is compact and $g = \sum a_l z^l$ then the operators H_{z^l} need to be compact if $a_l \neq 0$. Now the result follows from Proposition 1. □

Acknowledgment

We thank an anonymous referee for many helpful suggestions that have helped to simplify some of the proofs and to improve the manuscript as a whole.

References

[1] F. Beatrous, S.-Y. Li, On the boundedness and compactness of operators of Hankel type, J. Funct. Anal. 111 (1993) 350–379.
[2] F. Beatrous, S.-Y. Li, On the trace ideals of operators of Hankel type in a domain in C^n, Illinois J. Math. 39 (1995) 723–754.
[3] S. Fu, E.J. Straube, Compactness of the $\bar{\partial}$-Neumann problem on convex domains, J. Funct. Anal. 159 (1998) 629–641.
[4] T. Ohsawa, K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. 195 (1987) 197–204.
[5] W. Knirsch, G. Schneider, Continuity and Schatten–von Neumann p-class membership of Hankel operators with anti-holomorphic symbols on (generalized) Fock spaces, J. Math. Anal. Appl. 320 (2006) 403–414.

[6] S. Krantz, S.-Y. Li, R. Rochberg, Effect of the boundary geometry on the Hankel operators belonging to trace ideals of Bergman spaces, Integral Equations Operator Theory 28 (1997) 196–213.
[7] V. Peller, Hankel Operators and Their Applications, Springer Monogr. Math., vol. 15, 2003.
[8] G. Schneider, Hankel-operators with anti-holomorphic symbols on the Fock-space, Proc. Amer. Math. Soc. 132 (2004) 2399–2409.
[9] K. Stroethoff, Compact Hankel-operators on the Bergman space, Illinois J. Math. 34 (1990) 159–174.
[10] K. Stroethoff, Compact Hankel-operators on the Bergman spaces of the unit ball and polydisc in $\mathbb{C}^n$, J. Operator Theory 23 (1990) 153–170.
[11] K.H. Zhu, Hilbert–Schmidt Hankel-operators on the Bergman space, Proc. Amer. Math. Soc. 109 (1990) 721–730.
[12] K.H. Zhu, Schatten class Hankel operators on the Bergman space of the unit ball, Amer. J. Math. 113 (1991) 147–167.
[13] K.H. Zhu, BMO and Hankel operators on Bergman spaces, Pacific J. Math. 155 (1992) 377–395.