A cubic-order variant of Newton’s method for finding multiple roots of nonlinear equations

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A B S T R A C T

A second-derivative-free iteration method is proposed below for finding a root of a nonlinear equation \( f(x) = 0 \) with integer multiplicity \( m \geq 1 \):

\[
\begin{align*}
x_{n+1} &= x_n - \frac{f(x_n) - \mu f(x_n)/f'(x_n)) + \gamma f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

We obtain the cubic order of convergence and the corresponding asymptotic error constant in terms of multiplicity \( m \) and parameters \( \mu \) and \( \gamma \). Various numerical examples are presented to confirm the validity of the proposed scheme.

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1. Introduction

Newton’s method is commonly used to find an approximate simple root of a given nonlinear equation. Many researchers [1–6] have improved Newton’s method and designed its higher-order variants. Kalantari [7,8] established a general and elegant approach for a \( k \)-point family of iterative methods, using the ideas of the divided-difference matrix, confluent divided differences and the determinantal Taylor theorem. In addition, high-order algebraic methods [9] for approximating square roots have been extensively investigated; visualization via polynomiography [9] in the complex plane has enabled us to observe the fascinating beauty of fractals which attracts some readers to this area.

Suppose that a function \( f: \mathbb{C} \to \mathbb{C} \) has a multiple root \( \alpha \) with integer multiplicity \( m \geq 1 \) and is analytic in a small neighborhood of \( \alpha \). In this paper, a new iteration method free of second derivatives is proposed below for finding an approximate root \( \alpha \), given an initial guess \( x_0 \) sufficiently close to \( \alpha \):

\[
x_{n+1} = x_n - \frac{f(x_n) - \mu f(x_n)/f'(x_n)) + \gamma f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots
\]

where \( \mu \) and \( \gamma \) are parameters to be chosen for maximal order of convergence [10,11]. Observe that the pair \((\mu = 0, \gamma = 0)\) in (1.1) yields the classical Newton method for a simple root. For when \( m = 1 \) for a function \( f: \mathbb{R} \to \mathbb{R} \) having \( \alpha \in \mathbb{R} \), Kou [12] and Potra [5] investigated the cases with \((\mu = -1, \gamma = 1)\) and \((\mu = 1, \gamma = -1)\), respectively. A general extension of their studies with root multiplicity taken into account is the main objective of this paper. Observe that (1.1) is free of second derivatives, unlike the modified Halley method [3] and the Euler–Chebyshev method [6] requiring the second
derivatives shown below respectively:

\[ x_{n+1} = x_n - \frac{2}{\left(1 + \frac{1}{m}\right) \left( f'(x_n) \right) - \frac{m}{(m+1)} f'(x_n)} f(x_n), \quad n = 0, 1, 2, \ldots, \]  

\[ x_{n+1} = x_n - \frac{mf(x_n)}{2f'(x_n)} \left( 3 - m + \frac{mf''(x_n)f(x_n)}{f'(x_n)^2} \right), \quad n = 0, 1, 2, \ldots. \]  

We will show that iterative scheme (1.1) has cubic convergence and the asymptotic error constant [10,11] is expressed in terms of \( m, \mu \) and \( \gamma \). We rewrite the given equation \( f(x) = 0 \) in the form \( x - g(x) = 0 \), where \( g : \mathbb{C} \to \mathbb{C} \) is analytic in a sufficiently small neighborhood of \( \alpha \). Thus we can obtain an approximate \( \alpha \) by computing \( x_n \) repeatedly with following scheme:

\[ x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots. \]

for a given \( x_0 \in \mathbb{C} \). Let \( p \in \mathbb{N} \) be given and \( g(x) \) satisfy the relation below:

\[
\begin{align*}
\left| \frac{d^p}{dx^p} g(x) \right|_{x=\alpha} &= \left| g^{(p)}(\alpha) \right| < 1, & \text{if } p = 1. \\
\left| g^{(i)}(\alpha) \right| &= 0 & \text{for } 1 \leq i \leq p - 1 \text{ and } g^{(p)}(\alpha) \neq 0, & \text{if } p \geq 2.
\end{align*}
\]

Then it can be shown, by extending the similar analysis [2] for \( \mathbb{C} \), that the asymptotic error constant \( \eta \) with order of convergence \( p \) is

\[ \eta = \lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n} \right| = \frac{|g^p(\alpha)|}{p!}. \]  

Let us define

\[ h(\alpha) = \begin{cases} f(x)/f'(x), & \text{if } x \neq \alpha \\ \lim_{x \to \alpha} f(x)/f'(x), & \text{if } x = \alpha \end{cases} \]  

and

\[ g(\alpha) = \begin{cases} x - G(\alpha), & \text{if } x \neq \alpha \\ x - \lim_{x \to \alpha} G(\alpha), & \text{if } x = \alpha, \end{cases} \]

where \( G(\alpha) = \frac{f(x-h(\alpha)) + y(\alpha)}{f'(\alpha)} \). Our aim is to obtain the maximal order of convergence \( p \) using (1.5) as well as to derive the asymptotic error constant associated with \( p \) in terms of \( m \) and properly chosen \( \mu \) and \( \gamma \). To this end, we need to investigate some local properties of \( g(x) \) in a small neighborhood of \( \alpha \). From \( g(x) \) as defined in (1.8), we obtain

\[ (x - \alpha) \cdot f' = -f(x) + yf, \]  

where \( f = f(x), f' = f'(x), z = g(x) = x - \mu h(x) \) and \( g = g(x) \) are used for brevity and the symbol ‘’ denotes the derivative with respect to \( x \). With the aid of (1.9), some relationships between \( \mu, \gamma, g'(\alpha), g''(\alpha) \) and \( g'''(\alpha) \) are sought for maximum order of convergence in Section 2.

By Lemmas 1.1 and 1.2, we have \( [f(z)]_{z=x}^{(m)} = 0, \ 0 \leq k \leq m-1 \) and \( f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0, \ f^{(m)} \neq 0 \). Using L'Hospital's rule [13] repeatedly, we get

\[ \lim_{x \to \alpha} G(x) = \frac{[f(z)]_{z=x}^{(m-1)} + \gamma f^{(m-1)}(\alpha)}{f^{(m)}(\alpha)} = 0. \]  

To effectively compute \( g'(\alpha), g''(\alpha) \) and \( g'''(\alpha) \) for cubic order of convergence, some local properties of \( h(x) \) and \( f(x) \) are stated in the following lemmas that can be proved by repeated applications of L'Hospital's rule and Leibnitz's rule [13] for differentiation.

**Lemma 1.1.** Let \( f : \mathbb{C} \to \mathbb{C} \) have a root \( \alpha \) with a given integer multiplicity \( m \geq 1 \) and be analytic in a small neighborhood of \( \alpha \). Then the function \( h(x) \) and its derivatives up to order 3 evaluated at \( \alpha \) have the following properties with \( \theta_j = \frac{f^{(j)(\alpha)}}{f^{(m)(\alpha)}} \) for \( j \in \mathbb{N} \):

1. \( h(\alpha) = 0 \).
2. \( h'(\alpha) = \frac{1}{m} \).
3. \( h''(\alpha) = -\frac{2}{m(m+1)} \theta_1 \).
4. \( h^{(3)}(\alpha) = \frac{6}{m^2(m+1)} \left[ \theta_1^2 - \frac{2m}{m+2} \theta_2 \right] \).
Lemma 1.2. Let \( z(x) = x - \mu h(x) \in \mathbb{C} \) and \( h(x) \) be described as at the beginning of Section 1. Let \( \alpha \) be a zero of multiplicity \( m \) and \( t = z'(\alpha) = 1 - \frac{\mu}{m} \) with \( m \in \mathbb{N} \). Let \( \theta_k \) for \( k \in \mathbb{N} \) be as described in Lemma 1.1. Let \( F_k = \frac{d^k}{dx^k}f(z) \big|_{x=\alpha} \) for \( k = 0, 1, 2, \ldots \). Then the following hold:

1. \( F_k = 0 \) for \( 0 \leq k \leq m - 1 \).
2. \( F_m = f^{(m)}(\alpha) \theta_1 \) for \( k = m \).
3. \( F_{m+1} = f^{(m)}(\alpha) \cdot \theta_1 - \theta_2 \cdot t - t^2 \) with \( t^0 = 1 \), for any \( t \in \mathbb{C} \).
4. \( F_{m+2} = f^{(m)}(\alpha) \cdot t - t^2 \cdot \theta_1 \cdot \theta_2 - q_1(t) \theta_1^2 + q_2(t) \theta_2 \), with \( q_1(t) = -\frac{(m+2)}{2} (t - 1)^2 - (m-1)^2 t \), \( q_2(t) = t(t^3 - 2t + 2) \) and \( t^0 \equiv 1 \) for any \( t \in \mathbb{C} \).

2. Convergence analysis

In this section, we develop the order of convergence and the asymptotic error constant for iteration scheme (1.1) in terms of parameters \( \mu \) and \( \gamma \). We differentiate both sides of (1.7) with respect to \( x \) to obtain

\[
(g' - 1) \cdot f' + (g - x) \cdot f'' = -(\gamma f').
\]

Since \( g' \) is continuous at \( \alpha \), we have

\[
g'(\alpha) = \begin{cases} G_1(\alpha), & \text{if } x \neq \alpha \\ \lim_{x \to \alpha} G_1(x), & \text{if } x = \alpha, \end{cases}
\]

where \( G_1(\alpha) = -f(\alpha) \cdot f''(\alpha) \). To evaluate \( G_1(\alpha) \), we repeatedly apply L'Hospital's rule with Lemma 1.2 and the fact that \( \alpha = g(x) \). Since \( \alpha = g(x) \), we have

\[
\begin{align*}
\left[(g - x)^k\right]_{x=\alpha}^{(k)} &= \sum_{j=0}^{k} \binom{k}{j} (g - x)^{k-2j} f^{(k-2j)}(\alpha) \\
\left[(\gamma f')\right]_{x=\alpha}^{(k)} &= \begin{cases} 0, & \text{if } 0 \leq k \leq m - 2, m \geq 2 \\
(m - 1)(g - \alpha) f^{(m)}(\alpha)(t^m + \gamma), & \text{if } k = m - 1, \end{cases}
\end{align*}
\]

with \( t = 1 - \frac{\mu}{m} \), it follows that

\[
g'(\alpha) - 1 = G_1(\alpha) = \frac{-(m - 1)(g' - 1)f(\alpha) - f(\alpha)(t^m + \gamma)}{f^{(m)}(\alpha)}.
\]

Letting \( g'(\alpha) = 0 \) in the above relation, we get

\[
m = t^m + \gamma.
\]

We differentiate both sides of (2.1) with respect to \( x \) to obtain

\[
g'' \cdot f' + 2(g' - 1) \cdot f'' + (g - x) \cdot f''' = -(\gamma f'').
\]

Hence, we have

\[
g''(\alpha) = \begin{cases} G_2(\alpha), & \text{if } x \neq \alpha \\ \lim_{x \to \alpha} G_2(x), & \text{if } x = \alpha, \end{cases}
\]

where \( G_2(\alpha) = -2g(\alpha)f''(x) - (g - x)f'''(x) \). We can obtain \( G_2(\alpha) \) by computation similar to that done in \( G_1(\alpha) \). Using Lemma 1.2, we have

\[
\begin{align*}
\left[-2(g' - 1)f'' - (g - x)f''' - f(z')\right]_{x=\alpha}^{(k)} &= \begin{cases} 0, & \text{if } 0 \leq k \leq m - 3 \\
f^{(m)}(\alpha) t^m + \gamma, & \text{if } k = m - 2 \\
f^{(m)}(\alpha) \left(\theta_1 (m + 1 - t^m + t^{m-1} + t^m - \gamma) - g''(\alpha)^2 \right), & \text{if } k = m - 1. \end{cases}
\end{align*}
\]
Repeated applications of L'Hospital's rule yield

\[ G_2(\alpha) = g''(\alpha) = \frac{2}{m(m+1)} \theta_1(m + 1 - t^{m+1} + t^m - t^{m-1} - \gamma). \tag{2.10} \]

Using \( \gamma = m - t^m \) from (2.6), we finally get

\[ g''(\alpha) = -\frac{2}{m(m+1)} \theta_1 \cdot \rho(t), \tag{2.11} \]

where \( \rho(t) = t^{m+1} - 2t^m + t^{m-1} - 1 \) with \( t^0 \equiv 1 \) for any \( t \in \mathbb{C} \). Observe that \( \rho(t) = t^2 - 2t \) when \( m = 1 \). We seek \( t \) such that \( g''(\alpha) = 0 \) as \( m \) varies to obtain possible third-order convergence. To this end, we find a root \( t \) of the polynomial \( \rho(t) \) whose graph is sketched in Fig. 1. Such a root \( t \) is easily found to be an algebraic number \([14]\) since all the coefficients are integers. The next theorem guarantees the existence of real roots \( t \) of \( \rho(t) \).

**Theorem 2.1.** Let \( \rho(t) \) be defined as in (2.11) with \( m \geq 1 \) as the multiplicity of root \( \alpha \) of the given nonlinear equation \( f(x) = 0 \). Then \( \rho \) has only two real roots \( t_1^* \in (-1, 0) \) and \( t_2^* \in (1, 2) \) for any odd \( m \), while it has a unique real root \( t^* \in (1, 2) \) for any even \( m \).

**Proof.** If \( m = 1 \), then \( \rho(t) = t^2 - 2t \) clearly reflects the assertion. In fact, two roots \( t_1^* = 0 \) and \( t_2^* = 2 \) exist only when \( m = 1 \). If \( m = 2 \), then \( \rho(t) = t^3 - 2t^2 + t - 1 \) and \( \rho(1) \cdot \rho(2) < 0 \) guarantees the existence of a root \( t^* \in (1, 2) \). Let \( \rho(t) = t^3 + a_1 t^2 + a_2 t + a_3 \), \( \Delta = (3a_2 - a_1^2)/9 \) and \( \mathcal{R} = (9a_1 a_2 - 27a_3 - 2a_1^2)/54 \) with \( a_1 = -2 \), \( a_2 = 1 \), \( a_3 = -1 \). Then the discriminant \([15]\) \( \mathcal{D} = \Delta^2 + \mathcal{R}^2 = 23/108 > 0 \), showing that \( \rho \) has only one real root \( t^* \).

Let us consider the case when \( m \geq 3 \). Since \( \rho'(t) = (m + 1) t^{m-2} (t - \frac{m+1}{m}) (t -1) \), \( \rho \) has local minima \([16]\) at \( t = 0 \) and \( t = 1 \), while it has local maximum at \( t = \frac{m+1}{m} \); furthermore, it is monotone increasing in \([0, \frac{m+1}{m}] \) and monotone decreasing in \([\frac{m-1}{m+1}, 1] \). By direct computation, \( \rho \) has the local maximum \( 4 (m+1)^{m-3} (m-1) \) at \( t = \frac{m-1}{m+1} \). Combining this with \( \rho(0) = \rho(1) = -1 < 0 \), we find that no positive real root exists in \([0, 1] \). In view of the fact that \( \rho(1) \cdot \rho(2) = 1 - 2^{m-1} < 0 \), \( \rho \) has a positive real root \( t^* \in (1, 2) \) which is also unique due to the monotonicity of \( \rho \) in \((1, \infty) \).

Hence it suffices to prove the assertion for negative real roots. When \( m \geq 3 \) is odd, \( \rho(-1) \cdot \rho(0) < 0 \) means the existence of a root \( t^* \in (-1, 0) \). Since \( \rho(-t) = t^{m+1} + 2t^m + t^{m-1} - 1 \) has one sign change in its coefficients, \( \rho \) has at most one negative real root according to the Descartes sign rule \([2]\); as a result, only one negative real root \( t^* \in (-1, 0) \) exists. When \( m \geq 4 \) is even, \( \rho(-t) = -t^{m+1} - 2t^m - t^{m-1} - 1 \) has no sign change in its coefficients, and the Descartes sign rule assures that no negative real root of \( \rho \) exists. This completes the proof. \( \square \)

Table 1 shows the typical real roots of \( \rho(t) \) with \( 1 \leq m \leq 8 \) to ensure Theorems 2.1 and 2.2. Complex roots of \( \rho(t) \) are not selected, to make iteration scheme \((1.1) \) simpler.

We differentiate both sides of (2.7) with respect to \( x \) to obtain

\[ g^{(3)} \cdot f' + 3g'' \cdot f'' + (g' - 1) \cdot f^{(3)} + (g - x) \cdot f^{(4)} = -(f(x))^{(3)} + \gamma f^{(3)}. \tag{2.12} \]

From (2.12), we get

\[ g^{(3)}(x) = \begin{cases} G_3(x), & \text{if } x \neq \alpha \\ \lim_{x \to \alpha} G_3(x), & \text{if } x = \alpha, \end{cases} \tag{2.13} \]
3. The algorithm, numerical results and discussion

On the basis of the description stated in Sections 1 and 2, we develop a zero-finding algorithm to be implemented with high-precision Mathematica [17] programming.

### Table 1
Typical real roots of \( \rho(t) \) with \( 1 \leq m \leq 8 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \rho(t) )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( t^2 - 2t )</td>
<td>( t_{11} = 0, t_{12} = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>( t^3 - 2t^2 + t - 1 )</td>
<td>( t_3 = \frac{2 \pm \sqrt{3}}{2} \approx 1.75487766624 )</td>
</tr>
<tr>
<td>3</td>
<td>( (t^2 - t + 1)(t^2 - t - 1) )</td>
<td>( t_{13} = \frac{1 \pm \sqrt{5}}{2} \approx -0.618033988794, t_{14} = \frac{1 \pm \sqrt{5}}{2} \approx 1.61803398874 )</td>
</tr>
<tr>
<td>4</td>
<td>( t^4 - 2t^3 + t^2 - 1 )</td>
<td>( t_4 = 1.5289463545197057 )</td>
</tr>
<tr>
<td>5</td>
<td>( (t^2 - t + 1)(t^2 - t - 1) )</td>
<td>( t_5 = \frac{1}{2}(1 - \rho_1 - \rho_2), t_{15} = \frac{1}{2}(1 + \rho_3 + \rho_4) )</td>
</tr>
<tr>
<td>6</td>
<td>( t^6 - 2t^5 + t^4 - 1 )</td>
<td>( t_6 = 1.4177967508060623 )</td>
</tr>
<tr>
<td>7</td>
<td>( (t^4 - t^3 + 1)(t^4 - t^3 - 1) )</td>
<td>( t_7 = \frac{1}{2} \sigma \left(1 - \sqrt{\frac{3}{\sigma}}\right), t_{17} = \frac{1}{2} \sigma \left(1 + \sqrt{\frac{3}{\sigma}}\right) )</td>
</tr>
<tr>
<td>8</td>
<td>( t^8 - 2t^7 + t^6 - 1 )</td>
<td>( t_8 = 1.3499001972014136 )</td>
</tr>
</tbody>
</table>

where \( \sigma_3(x) = \frac{-3g'' \cdot f'' - 3g'(\alpha)(g' - 1) \cdot f'(x) - (g - x) \cdot f'(4)(x)}{f' \cdot f''} \), Using Lemma 1.2 and the fact that \( g(\alpha) = \alpha, g'(\alpha) = 0, g''(\alpha) = 0 \) for cubic order of convergence, we find the relation below:

\[
-3g'' \cdot f'' \left|_{x=\alpha} \right. = 3(g' - 1) \cdot f'(3) \left|_{x=\alpha} \right. - (g - x) \cdot f'(4) \left|_{x=\alpha} \right. - [f(z)'(3) + \gamma f(z)] \left|_{x=\alpha} \right.
\]

\[
\phi_1 = \begin{cases} 
0, & \text{if } 0 \leq k \leq m - 4 \\
\frac{f^{(m)}}{f''}(\alpha)(m - t^m - \gamma), & \text{if } k = m - 3 \\
\frac{\theta f^{(m)}}{f''}(\alpha)(m + 1 - t^{m+1} + t^m - t^{m-1} - \gamma), & \text{if } k = m - 2 \\
\phi_1(\mbox{if } k = m - 3) \end{cases}
\]

(2.14)

From (2.13) and (2.14) with repeated applications of L' Hospital's rule, we finally get

\[
g^{(3)}(\alpha) = \frac{6}{m(m+1)(m+2)}(\phi_1 \phi_2^2 + \phi_2 \phi_2). \tag{2.15}
\]

Consequently, we obtain the asymptotic error constant \( \eta \) described in the theorem below.

**Theorem 2.2.** Let \( f : \mathbb{C} \to \mathbb{C} \) have a zero \( \alpha \) with integer multiplicity \( m \geq 1 \) and be analytic in a small neighborhood of \( \alpha \). Let \( \phi_1, \phi_2 \) be defined as in Lemma 1.1 and \( f_1, f_2 \) be defined as in (2.14). Let \( t \) be a root of \( \rho(t) \) defined in (2.11). Let \( x_0 \) be an initial guess chosen in a sufficiently small neighborhood of \( \alpha \). Then the iteration method (1.1) with \( \mu = m(1 - t), \gamma = m - t^m \) has order 3 and its asymptotic error constant \( \eta \) is as follows:

\[
\eta = \frac{1}{6} \left| g^{(3)}(\alpha) \right| = \frac{1}{m(m+1)(m+2)}(\phi_1 \phi_2^2 + \phi_2 \phi_2), \tag{2.16}
\]

provided that \( \phi_1 \phi_2^2 + \phi_2 \phi_2 \neq 0 \).

**Remark.** Two pairs \( (m = 1, t = 0) \) and \( (m = 1, t = 2) \) in (2.16) yield the asymptotic error constants given by Kou [12] and Potra [5], respectively.

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Table 2
Asymptotic error constants for \( f(x) = \frac{\alpha^2 - x + \gamma^2}{\pi \cos x} \) with \( m = 2, \alpha = \frac{1 + \sqrt{3}}{2} \) \((t, \mu, \gamma) = (1.75487766624, -1.50975533249, -1.0795962349)\).

| \( n \) | \( x_n \) | \( |x_n - \alpha| \) | \( \varepsilon_{n+1}/\varepsilon_n \) | \( \eta \) |
|---|---|---|---|---|
| 0 | 0.396 + 2.387i | 0.253285 | | 0.9449259108 |
| 1 | 0.499577476751567 + 2.57388126341839i | 0.0241986 | | 1.489236779 |
| 2 | 0.500012395983446 + 2.59807060633620i | 0.000136043 | | 0.9600703765 |
| 3 | 0.499999999999999 + 2.59807621135332i | 2.37911 \times 10^{-15} | | 0.9494013940 |
| 4 | 0.500000000000000 + 2.59807621135332i | 1.27245 \times 10^{-14} | | 0.949259108 |
| 5 | 0.500000000000000 + 2.59807621135332i | 1.94677 \times 10^{-12} | | 0.949259108 |
| 6 | 0.500000000000000 + 2.59807621135332i | 0.00 \times 10^{-240} | | |

Table 3
Convergence for \( f(x) = (e^{1-x} + \log[1 + x - \pi])^2(x - \pi)^2 \sin^2 x \) with \( m = 7, \alpha = \pi \) \((t, \mu, \gamma) = (-0.8191725133961718, 12.7342075937732, 7.247529862496331)\).

| \( n \) | \( x_n \) | \( |x_n - \alpha| \) | \( \varepsilon_{n+1}/\varepsilon_n \) | \( \eta \) |
|---|---|---|---|---|
| 0 | 2.900000000000000 | 0.241593 | | 0.2826722582 |
| 1 | 3.13818475241933 | 0.00340790 | | 0.2416772773 |
| 2 | 3.14159264244215 | 1.11476 \times 10^{-9} | | 0.2816579509 |
| 3 | 3.1415926538979 | 3.91590 \times 10^{-25} | | 0.2826722549 |
| 4 | 3.1415926538979 | 1.69738 \times 10^{-74} | | 0.2826722582 |
| 5 | 3.1415926538979 | 1.38236 \times 10^{-222} | | 0.2826722582 |
| 6 | 3.1415926538979 | 0.00 \times 10^{-240} | | |

Algorithm 3.1 (Zero-Finding Algorithm).

Step 1. Construct iteration scheme (1.1) with the given function \( f \) having a multiple zero \( \alpha \).

Step 2. Set the minimum number of precision digits. With the exact or most accurate zero \( \alpha \), provide the asymptotic error constant \( \eta \) and order of convergence \( p \), as well as \( \theta_1, \theta_2, t, \phi_1 \) and \( \phi_2 \) stated in Section 2. Compute \( \mu = m(1 - t) \) and \( \gamma = m - t^m \). Set the error range \( \varepsilon \), the maximum iteration number \( n_{\text{max}} \) and the initial guess \( x_0 \). Compute \( f(x_0) \) and \( |x_0 - \alpha| \).

Step 3. Display, in accordance with (1.1), \( n, x_n, e_n = |x_n - \alpha|, e_{n+1}/e_n \) and \( \eta \).

A variety of examples have been experimented on, with error bound \( \varepsilon = 0.5 \times 10^{-235} \) and minimum number of precision digits 250. Mathematica command \( \text{FindRoot} \) is used to find real roots of \( \psi(t) \). With such real roots \( t \) chosen, relation \( t = 1 - \mu/m \) allows us to compute \( \mu = m(1 - t) \) and \( \gamma = m - t^m \) for complete iteration scheme (1.1). Symbol \( i \) is used to denote \( \sqrt{-1} \). The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is actually rounded to be accurate up to the 235 significant digits, but displayed only up to 15 significant digits. The residual errors of \( f(x) \) are nearly zero, although they are not displayed here due to the limited space.

Iterative scheme (1.1) applied to test functions \( (x^2 + 7x^4 + 3x^7) \) \((x^2 + \cos x)\), \((e^{-x} \sin x + \log[1 + x - \pi])^2(x - \pi)^2 \sin^2 x \) and \( 3x^7 - 37x^4 + 208 \) clearly shows successful asymptotic error constants with cubic convergence. Tables 2–4 list \( n, x_n, e_n = |x_n - \alpha| \) and the computational asymptotic error constants \( e_{n+1}/e_n \) as well as the theoretical asymptotic error constant \( \eta \).

Further test functions with \( t \)-values shown in Table 1 are listed below:

\[
f_1(x) = x^8 - x^4 + 73, \quad \alpha = -1.24943225052 - 1.0410353493i, \quad m = 1, \quad x_0 = -1.57 - 0.78i
\]

\[
f_2(x) = (x - 2) \cos \left( \frac{\pi}{x} \right), \quad \alpha = 2, \quad m = 2, \quad x_0 = 1.97
\]

\[
f_3(x) = (x^2 + 16) \log^2(x^2 + 17), \quad \alpha = -4i, \quad m = 3, \quad x_0 = -3.92i
\]

\[
f_4(x) = \frac{(3 - x + x^2)^4}{x^4 + \sin x}, \quad \alpha = 1 - \sqrt{1/3}, \quad m = 4, \quad x_0 = 0.37 - 1.89i
\]
Table 5 shows successful convergence for a list of other test functions with $m$, $t$, $\mu$, $\gamma$, the least iteration number $\nu$ for convergence and the asymptotic error constant $\eta$.

Let $p$ denote the order of convergence and $d$ the number of new evaluations of $f(x)$ or its derivatives per iteration. Taking into account the computational cost, an efficiency of the given iteration function is measured by the efficiency index $^{\ast}\text{EFF} = p^{1/d}$ introduced in [6]. A bigger efficiency index indicates a more efficient and less expensive iteration scheme. For our proposed iteration scheme, we find $p = 3$ and $d = 3$ to get $^{\ast}\text{EFF} = 3^{1.5} \approx 1.44224957$ which is better than $\sqrt{2}$, the efficiency index of the modified Newton method.

In this paper, we have proposed a cubic-order iteration scheme (1.1) free of second derivatives. The current approach can be extended to other iterative methods including the Halley method, requiring higher-order derivatives.

References