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Neighborhoods of a Certain Family of Multivalent Functions with Negative Coefficients

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Abstract—By making use of the familiar concept of neighborhoods of analytic and p-valent functions, the authors prove coefficient bounds and distortion inequalities, and associated inclusion relations for the (n, δ)-neighborhoods of a family of multivalent functions with negative coefficients, which is defined by means of a certain nonhomogeneous Cauchy-Euler differential equation. Relevant connections of the various function classes investigated in this paper with those considered by earlier workers on the subject are also mentioned. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{T}(n,p)$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ p,n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

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Following the earlier investigations by Goodman [1] and Ruscheweyh [2] (see also [3,4]), we define the (n,δ)-neighborhood of a function $f(z) \in \mathcal{T}(n,p)$ by

$$N_{n,\delta}(f;g) := \left\{ g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}, \tag{1.2}$$

so that, obviously,

$$N_{n,\delta}(h;g) := \left\{ g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k\, z^k \text{ and } \sum_{k=n+p}^{\infty} k\,|b_k| \leqq \delta \right\}, \tag{1.3}$$

where, *and in what follows*,

$$h(z) = z^p \qquad (p \in \mathbb{N}). \tag{1.4}$$

We denote by $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ the classes of *p-valently starlike functions of order* α in $\mathbb{U}$ $(0 \leqq \alpha < p)$ and *p-valently convex functions of order* α in $\mathbb{U}$ $(0 \leqq \alpha < p)$, respectively. Thus, by definition, we have

$$\mathcal{S}_n^*(p,\alpha) := \left\{ f \in \mathcal{T}(n,p) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \;\; (z \in \mathbb{U};\; 0 \leqq \alpha < p) \right\} \tag{1.5}$$

and

$$\mathcal{C}_n(p,\alpha) := \left\{ f \in \mathcal{T}(n,p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \;\; (z \in \mathbb{U};\; 0 \leqq \alpha < p) \right\}. \tag{1.6}$$

An interesting unification of the function classes $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ is provided by the class $\mathcal{T}_n(p,\alpha,\lambda)$ of functions $f \in \mathcal{T}(n,p)$, which also satisfy the following inequality:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\; 0 \leqq \alpha < p;\; 0 \leqq \lambda \leqq 1). \tag{1.7}$$

The class $\mathcal{T}_n(p,\alpha,\lambda)$ was investigated by Altintaş *et al.* [5] and (subsequently) by Irmak *et al.* [6]. In particular, the class $\mathcal{T}_n(1,\alpha,\lambda)$ was considered earlier by Altintaş [7]. Clearly, we have

$$\mathcal{T}_n(p,\alpha,0) = \mathcal{S}_n^*(p,\alpha) \qquad \text{and} \qquad \mathcal{T}_n(p,\alpha,1) = \mathcal{C}_n(p,\alpha) \tag{1.8}$$

in terms of the *simpler* classes $\mathcal{S}_n^*(p,\alpha)$ and $\mathcal{C}_n(p,\alpha)$ defined by (1.5) and (1.6), respectively (see also [8,9]).

The main object of the present sequel to the aforementioned recent works is to derive several coefficient bounds and distortion inequalities, and associated inclusion relations for the (n,δ)-neighborhood of functions in the subclass $\mathcal{K}_n(p,\alpha,\lambda,\mu)$ of the class $\mathcal{T}(n,p)$, which consists of functions $f \in \mathcal{T}(n,p)$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$\begin{gathered} z^2 \frac{d^2 w}{dz^2} + 2(\mu+1) z \frac{dw}{dz} + \mu(\mu+1) w = (p+\mu)(p+\mu+1) g(z) \\ (w = f(z) \in \mathcal{T}(n,p);\; g \in \mathcal{T}_n(p,\alpha,\lambda);\; \mu > -p \;\; (\mu \in \mathbb{R})). \end{gathered} \tag{1.9}$$

2. COEFFICIENT BOUNDS AND DISTORTION INEQUALITIES

In our present investigation of the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, we shall require Lemmas 1 and 2 below.

LEMMA 1. *(See [5, p. 10, Theorem 1].) Let the function $f \in \mathcal{T}(n,p)$ be defined by (1.1). Then the function $f(z)$ is in the class $\mathcal{T}_n(p,\alpha,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} (k-\alpha)\left[\lambda(k-1)+1\right] a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right] \tag{2.1}$$
$$(0 \leqq \alpha < p;\ 0 \leqq \lambda \leqq 1;\ n,p \in \mathbb{N}).$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{(p-\alpha)\left[\lambda(p-1)+1\right]}{(n+p-\alpha)\left[\lambda(n+p-1)+1\right]} z^{n+p} \qquad (n,p \in \mathbb{N}). \tag{2.2}$$

LEMMA 2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{T}_n(p,\alpha,\lambda)$. Then*

$$\sum_{k=n+p}^{\infty} a_k \leqq \frac{(p-\alpha)\left[\lambda(p-1)+1\right]}{(n+p-\alpha)\left[\lambda(n+p-1)+1\right]} \tag{2.3}$$

and

$$\sum_{k=n+p}^{\infty} k a_k \leqq \frac{(n+p)(p-\alpha)\left[\lambda(p-1)+1\right]}{(n+p-\alpha)\left[\lambda(n+p-1)+1\right]}. \tag{2.4}$$

PROOF. By using Lemma 1, we find from (2.1) that

$$(n+p-\alpha)\left[\lambda(n+p-1)+1\right] \sum_{k=n+p}^{\infty} a_k$$
$$\leqq \sum_{k=n+p}^{\infty} (k-\alpha)\left[\lambda(k-1)+1\right] a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right],$$

which immediately yields the first assertion (2.3) of Lemma 2.

By appealing to (2.1), we also have

$$\left[\lambda(n+p-1)+1\right] \sum_{k=n+p}^{\infty} (k-\alpha) a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right],$$

that is,

$$\left[\lambda(n+p-1)+1\right] \sum_{k=n+p}^{\infty} k a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right] + \alpha\left[\lambda(n+p-1)+1\right] \sum_{k=n+p}^{\infty} a_k,$$

which, in view of the coefficient inequality (2.3), can be put in the following form:

$$\left[\lambda(n+p-1)+1\right] \sum_{k=n+p}^{\infty} k a_k \leqq (p-\alpha)\left[\lambda(p-1)+1\right] + \frac{\alpha(p-\alpha)\left[\lambda(p-1)+1\right]}{n+p-\alpha}. \tag{2.5}$$

Upon simplifying the right-hand side of (2.5), we are led eventually to the second assertion (2.4) of Lemma 2.

Our *main* distortion inequalities for functions in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$ are given by Theorem 1 below.

THEOREM 1. *If $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, then*

$$|f(z)| \leqq |z|^p + \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}|z|^{n+p} \qquad (z \in \mathbb{U}) \tag{2.6}$$

and

$$|f(z)| \geqq |z|^p - \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}|z|^{n+p} \qquad (z \in \mathbb{U}). \tag{2.7}$$

PROOF. Suppose that $f \in \mathcal{T}(n,p)$ is given by (1.1). Also let the function $g \in \mathcal{T}_n(p,\alpha,\lambda)$, occurring in the nonhomogeneous Cauchy-Euler differential equation (1.9), be given as in the definitions (1.2) and (1.3) *with*, of course,

$$b_k \geqq 0 \qquad (k = n+p, n+p+1, n+p+2, \dots).$$

Then we readily find from (1.9) that

$$a_k = \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k \qquad (k = n+p, n+p+1, n+p+2, \dots), \tag{2.8}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k z^k, \tag{2.9}$$

and

$$|f(z)| \leqq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k \qquad (z \in \mathbb{U}). \tag{2.10}$$

Next, since $g \in \mathcal{T}_n(p,\alpha,\lambda)$, the first assertion (2.3) of Lemma 2 yields the following coefficient inequality:

$$b_k \leqq \frac{(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \qquad (k = n+p, n+p+1, n+p+3, \dots), \tag{2.11}$$

which, in conjunction with (2.10), yields

$$|f(z)| \leqq |z|^p + \frac{(p-\alpha)[\lambda(p-1)+1](p+\mu)(p+\mu+1)}{(n+p-\alpha)[\lambda(n+p-1)+1]}|z|^{n+p} \cdot \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} \qquad (z \in \mathbb{U}). \tag{2.12}$$

Finally, in view of the following *telescopic* sum:

$$\sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} = \sum_{k=n+p}^{\infty} \left(\frac{1}{k+\mu} - \frac{1}{k+\mu+1}\right) = \frac{1}{n+p+\mu} \qquad (\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, -n-p-2, \dots\}), \tag{2.13}$$

the first assertion (2.6) of Theorem 1 follows at once from (2.12).

The second assertion (2.7) of Theorem 1 can be proven by similarly applying (2.9), (2.11), and (2.13).

By setting $\lambda = 0$ and $\lambda = 1$ in Theorem 1, and using the relationships in (1.8), we arrive at Corollaries 1 and 2, respectively.

COROLLARY 1. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation (1.9) with $g \in \mathcal{S}_n^*(p,\alpha)$, then*

$$|z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \leqq |f(z)| \leqq |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \quad (z \in \mathbb{U}). \tag{2.14}$$

COROLLARY 2. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation (1.9) with $g \in \mathcal{C}_n(p,\alpha)$, then*

$$|z|^p - \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \leqq |f(z)| \leqq |z|^p + \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}|z|^{n+p} \quad (z \in \mathbb{U}). \tag{2.15}$$

3. NEIGHBORHOODS FOR THE CLASSES $\mathcal{T}_n(p,\alpha,\lambda)$ AND $\mathcal{K}_n(p,\alpha,\lambda,\mu)$

In this section, we determine inclusion relations for the classes $\mathcal{T}_n(p,\alpha,\lambda)$ and $\mathcal{K}_n(p,\alpha,\lambda,\mu)$ involving the (n,δ)-neighborhoods defined by (1.2) and (1.3).

THEOREM 2. *If $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{T}_n(p,\alpha,\lambda)$, then*

$$\mathcal{T}_n(p,\alpha,\lambda) \subset N_{n,\delta}(h;f), \tag{3.1}$$

where $h(z)$ is given by (1.4) and

$$\delta := \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}. \tag{3.2}$$

PROOF. Assertion (3.1) would follow easily from the definition of $N_{n,\delta}(h;f)$, which is given by (1.3) with $g(z)$ replaced by $f(z)$, and the second assertion (2.4) of Lemma 2.

THEOREM 3. *If $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{K}_n(p,\alpha,\lambda,\mu)$, then*

$$\mathcal{K}_n(p,\alpha,\lambda,\mu) \subset N_{n,\delta}(g;f), \tag{3.3}$$

where $g(z)$ is given by (1.9) and

$$\delta := \frac{(n+p)(p-\alpha)[\lambda(p-1)+1][n+(p+\mu)(p+\mu+2)]}{(n+p-\alpha)[\lambda(n+p-1)+1](n+p+\mu)}. \tag{3.4}$$

PROOF. Suppose that $f \in \mathcal{K}_n(p,\alpha,\lambda,\mu)$. Then, upon substituting from (2.8) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leqq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k \qquad (a_k \geqq 0;\ b_k \geqq 0), \tag{3.5}$$

we obtain

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leqq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}kb_k. \tag{3.6}$$

Next, since $g \in \mathcal{T}_n(p, \alpha, \lambda)$, the second assertion (2.4) of Lemma 2 yields

$$kb_k \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \qquad (k = n+p, n+p+1, n+p+2, \dots). \tag{3.7}$$

Finally, by making use of (2.4) as well as (3.7) on the right-hand side of (3.6), we find that

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \left(1 + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\right), \tag{3.8}$$

which, by virtue of the *telescopic* sum (2.13), immediately yields

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leqq \frac{(n+p)(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} \left(\frac{n+(p+\mu)(p+\mu+2)}{n+p+\mu}\right) =: \delta. \tag{3.9}$$

Thus, by definition (1.2) with $g(z)$ interchanged by $f(z)$, $f \in N_{n,\delta}(g; f)$. This evidently completes the proof of Theorem 2.

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