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# On Optimal Control Problems with Discontinuities

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## INTRODUCTION

The contributions of this paper are twofold. First, a new necessary condition is developed for optimal control problems with discontinuities. Second, a numerical algorithm employing the second variation is obtained for such problems. Some preliminary results were reported in references [11] and [12].**

For the purpose of this paper "control problems with discontinuities" include those where there are jump discontinuities in the derivative of the state variables. Such discontinuities arise in problems with bang-bang control.

If such discontinuities are present, it is desirable to determine whether they are optimally located. As shall be shown it is not sufficient to apply Pontryagin's Maximum Principle (Weierstrass's $E$ condition) and the classical corner conditions alone. A further condition on the second variation is required. An attempt to derive this condition was made by Reid [1]. However, as will be shown later by means of an example, Reid's result is incorrect. In this paper the correct second-order condition is developed. By including this condition with the other well-known conditions a sufficiency proof for a local maximum could be obtained.

Most of the numerical algorithms for problems with discontinuities have been based upon an extension of the gradient method (see [2]-[4], for example). Some special algorithm's have been developed for bang-bang control problems (e.g., Neustadt [5] and Eaton [6]). A Newton-Raphson algorithm based upon Reid's work was developed by Schmarack [7].

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** *Note added in proof*: It has been called to the authors' attention that similar results have been obtained independently by Jacobson [14].

## Extension of the Successive Sweep Method

For this paper a Newton-Raphson algorithm will be developed by extending the successive sweep method [13]. This method was obtained by expanding the dynamic programming equations to second order around a nominal solution. The successive sweep algorithm is equivalent to many of the Newton-Raphson techniques in the literature (e.g., [8] and [9]). The sweep method avoids, as do most second variation methods, discontinuities because the standard technique of weak variations breaks down in the region containing the discontinuity. In order to extend Newton-Raphson methods to such problems, the technique of strong variations must be employed. Heretofore, the application of strong variations to Newton-Raphson methods has been restricted to the region of the final time for free terminal time problems such as in Ref. [9]. Here the technique of strong variations is used to join together the solutions obtained by weak variations at either side of the discontinuity. In the main body of the paper this is done by expanding the dynamic programming equations. In Appendix A an alternate approach is given in terms of the classical variational equations.

The final results consist of jump conditions for the second partials of the return function, and, in the case of nonextremal solutions, a jump condition for the first partials. In the process, the second derivatives of the performance index with respect to the switching time is derived and hence the new necessary condition is developed.

## The Problem

The following optimal control problem shall be considered. Find the control function $u(t) = (u_1(t),..., u_m(t))$ which maximizes the performance of some system. The performance of the system is measured by the performance index, $J$, where

$$J = \Phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt. \tag{1}$$

The system dynamics are described by a set of first-order differential equations,

$$\dot{x} = f(x, u, t), \tag{2}$$

where $x(t) = (x_1(t),..., x_n(t))$ denotes the state of the system.

## The Dynamic Programming Solution

Standard dynamic programming theory characterizes the solution of this problem in terms of the following boundary value problem.

$$0 = \max_u \left[L + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t}\right]; \qquad V(x, T) = \Phi(x, T) \tag{3}$$

where $V(x(t_0), t_0)$ is the optimized value of $J$. It will be assumed that $V$ has has all of the derivatives required below. At points where the control is discontinuous neither $\dot{x}(t)$ nor the derivatives of $V$ will necessarily be continuous. However from physical considerations $x(t)$ must be continuous. If $t_i$ $(i = 1,..., p)$ denote the time of discontinuity

$$x(t_i^-) = x(t_i^+) \tag{4}$$

where the, $-$, and, $+$, refer to the left- and right-hand sides of the discontinuity.

Since the integral in (1) is a continuous function of $t_0$, it follows that $V(x, t_0)$ must also be continuous, i.e.,

$$V(x(t_i^-), t_i^-) = V(x(t_i^+), t_i^+). \tag{5}$$

The equations (2)-(5) must be satisfied by the optimal solution.

## Partial Derivatives of $V$

To facilitate the analysis in later sections certain derivatives of $V$ are required. The notation will be simplified by considering a single discontinuity at time $t^*$. $V^+(x, t)$ is used to denote the optimal return function to the right of $t^*$ as well as its analytic extension obtained by using a continuous control law. $V^+(x, t)$ satisfies the partial differential equations (3), viz.,

$$V_t^+ + V_x^+ f^+ + L^+ = 0. \tag{6}$$

The partial derivative of this identity with respect to, $x$, yields

$$V_{tx}^+ = - V_{xx}^+ f^+ - V_x^+ f_x^+ - L_x^+ \tag{7}$$

and the partial derivative with respect to, $t$,

$$V_{tt}^+ = - V_{tx}^+ f^+ - V_x^+ f_t^+ - L_t^+. \tag{8}$$

Eliminating $V^+_{xt}$ from (8), one obtains

$$V^+_{tt} = f\, V^+_{xx} f - (V^+_x f^+_x + L_x) f^- - V^+_x f^+_t - L^+_t . \tag{9}$$

It is the equations (6), (7), and (9) that are required in the following analysis.

## The First and Second Variations with Respect to the Switching Time

Suppose that a constant control $u^-$ is used to the left of the discontinuity. Let $V^-(x_0, t_0, t_1)$ denote the return function when the initial time and state are $t_0$ and $x_0$, and the discontinuity is at $t_1 > t_0$. The following relation holds,

$$V^-(x_0, t_0, t_1) = V^+(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u^-, t)\, dt. \tag{10}$$

The right-hand side is now expanded about the nominal solution $t_1 = t^*$, $x(t_1) = x(t^*)$, viz.,

$$\begin{aligned} V^- = V^+(x(t^*), t^*) + \int_{t_0}^{t^*} L\, dt + (L^- + V^+_t)\, dt + V^+_x\, dx \\ + \tfrac{1}{2}\left[V^+_{xx}\, dx^2 + 2V^+_{xt}\, dx\, dt + (L^-_t + L^-_x f^- + V^+_{tt})\, dt^2\right] \\ + \cdots, \end{aligned} \tag{11}$$

where $dx = x(t_1) - x(t^*)$, $dt = t_1 - t^*$, and the partials of $V^+$ are evaluated at $x(t^*)$, $t^*$. Now

$$dx = f^-\,|_{x^*, t^*}\, dt + \tfrac{1}{2}\,(f^-_t + f^-_x f^-)\,|_{x^*, t^*}\, dt^2 + \cdots . \tag{12}$$

Substituting in Eq. (11) for $dx$ gives,

$$\begin{aligned} V^- = V^+ + (L^- + V^+_t + V^+_x f^-)\, dt \\ + \tfrac{1}{2}\left[V^+_x(f^-_t + f^-_x f^-) + f^- V^+_{xx} f^- + 2V^+_{tx} f^- + L^-_x f^- \right. \\ \left. + L^-_t + V^+_{tt}\right] dt^2 + \cdots . \end{aligned} \tag{13}$$

Now substituting Eq. (6), (7), and (9) into (13) one obtains

$$V^- = V^+ + \int_{t_0}^{t^*} L^-\, dt + \dot{V}\, dt + \tfrac{1}{2}\, \ddot{V}\, dt^2 + \cdots , \tag{14}$$

where

$$\dot{V} = L^- - L^+ + V_x^+(f^- - f^+) \tag{15}$$

and

$$\begin{aligned}\ddot{V} = (f^- - f^+)^T V_{xx}^+(f^- - f^+) - (V_x^+ f_x^+ + L_x^+)(f^- - f^+)\\ + (V_x^+(f_x^- - f_x^+) + L_x^- - L_x^+) f^-\\ + V_x^+(f_t^- - f_t^+) + L_t^- - L_t^+.\end{aligned} \tag{16}$$

If $u^-$ is not a constant, but $V_x^- f_u^- + L_u^- = 0$, the above results are also valid.

## Necessary Condition for a Maximum

For $V$ to be a maximum with respect to $t^*$

$$\dot{V} = 0 \qquad \text{at} \qquad t = t^*. \tag{17}$$

This condition implies the continuity of the Hamiltonian. Secondly $\ddot{V}$ must be nonpositive i.e.,

$$\ddot{V} \leqslant 0. \tag{18}$$

The second condition, Eq. (18) is believed to be a new necessary condition for a maximum.

## Jump Conditions for the Partial Derivatives $V_x$, $V_{xx}$

Spacial variations will now be considered in order to relate $V_x^+$ and $V_x^-$, $V_{xx}^+$ and $V_{xx}^-$. In Eq. (14) let $t_0$ approach $t^*$ so that it may be written

$$V^-(x(t^*), t^*) = V^+(x(t^*), t^*) + \dot{V}\, dt + \tfrac{1}{2}\, \ddot{V}\, dt^2 + \cdots. \tag{19}$$

Expanding the right-hand side of (19) about $x_{\text{opt}}(t^*)$ one obtains, to second order,

$$\begin{aligned}V^-(x(t^*), t^*) = V^+(x_{\text{opt}}(t^*), t^*) + V_x^+ \,\delta x + \dot{V}\, dt + \tfrac{1}{2}\delta x^T V_{xx}^+ \,\delta x\\ + \dot{V}_x \,\delta x\, dt + \tfrac{1}{2}\, \ddot{V}\, dt^2 + \cdots,\end{aligned} \tag{20}$$

where

$$\dot{V}_x = V_{xx}^+(f^- - f^+) + V_x^+(f_x^- - f_x) + L_x^- - L_x^+ \tag{21}$$

Choosing $dt$ to maximize this expression gives

$$dt = -\ddot{V}^{-1}(\dot{V} + \dot{V}_x \,\delta x). \tag{22}$$

Replacing $dt$ in Eq. (20) by (22) leads to

$$V^- = V^+ - \dot{V}^2\ddot{V}^{-1} + [V_x^+ - \dot{V}\ddot{V}^{-1}\dot{V}_x]\,\delta x + \tfrac{1}{2}\,\delta x^T[V_{xx}^+ - \dot{V}_x\ddot{V}^{-1}\dot{V}_x]\,\delta x. \tag{23}$$

Hence because of the continuity of $V$, Eq. (5),

$$V_x^- = V_x^+ - \dot{V}\ddot{V}^{-1}\dot{V}_x \tag{24}$$

and

$$V_{xx}^- = V_{xx}^+ - \dot{V}_x^T\ddot{V}^{-1}\dot{V}_x\,. \tag{25}$$

Equation (24) determines the discontinuity in $V_x$, which can be seen to be zero for an optimum as $\dot{V} = 0$. There will, however, generally be a discontinuity in $V_{xx}$ at the switching time. Equations (22), (24), and (25) are required to extend the sweep method to include discontinuities.

## A Modification

It is relevant to derive one further relation. Suppose the above linearization was made about some suboptimal trajectory, then it is possible that Eq. (22) would be invalid (such is the case when $\ddot{V} \geqslant 0$). In these circumstances a more suitable choice (gradient) of $dt$ would be

$$dt = \epsilon\dot{V}, \tag{26}$$

where the parameter $\epsilon$ is chosen to ensure the validity of the expansions. The jump discontinuities become

$$V_x^- = V_x^+ - \epsilon\dot{V}\dot{V}_x\,, \qquad V_{xx}^- = V_{xx}^+\,. \tag{27}$$

## An Example

Next an example is given to demonstrate several points. The first point is that the above formulas for $\ddot{V}$ and $\dot{V}_x$ are correct. This is done by checking the formula with an independent calculation. The second point is that Reid's condition is incorrect. This is done by finding an example that satisfies all the classical conditions for optimality, including Reid's condition, but not the new condition, and demonstrating that it is not a locally optimal solution.

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + u \qquad & x_1(0) &= x_1^0 \\ \dot{x}_2 &= -u \qquad & x_2(0) &= x_2^0 \end{aligned} \tag{28}$$

and the maximization of $J$ where

$$J = \tfrac{1}{2}\, x_1(T)^2 + \tfrac{1}{2}\, x_2(T)^2, \tag{29}$$

and where the control, $u$, is bounded i.e.,

$$| u | \leqslant 1.$$

The Hamiltonian, $H$, is given by

$$H = (\lambda_1 - \lambda_2)\, u + \lambda_1 x_2\,, \tag{30}$$

which is a maximum when

$$u = \operatorname{sgn}(\lambda_1 - \lambda_2)$$

where the adjoint variables $\lambda_1$ and $\lambda_2$ are given by

$$\begin{aligned} \lambda_1(t) &= x_1(T) \\ \lambda_2(t) &= x_2(T) - x_1(T)\,(t - T). \end{aligned} \tag{31}$$

Since the switching function is linear in $t$, it is clear that no more than a single switch may exist on an optimal solution.

Because of the simplicity of the equations it is possible to express the terminal state in terms of the initial state and the single switching time, $t^*$,

$$\begin{aligned} x_1(T) &= x_1{}^0 + x_2{}^0 T + (T - 2t^*) + t^{*2} - \left(\frac{T - 2t^*}{2}\right)^2 \\ x_2(T) &= x_2{}^0 - (T - 2t^*) \end{aligned} \tag{32}$$

($u^+$ has been taken as $+ 1$).

Hence the expression for (29) becomes

$$J = \left[\frac{1}{2}\left\{x_1{}^0 + x_2{}^0 T + (T - 2t^*) + t^{*2} - \left(\frac{T - 2t^*}{2}\right)^2\right\}^2 \right. \\ \left. + \frac{1}{2}\{x_2{}^0 - (T - 2t^*)\}^2\right]. \tag{33}$$

The derivative of $J$ with respect to $t^*$ is given by

$$J_{t^*} = - 2\left\{x_1{}^0(1 - T) + x_2{}^0(- 1 + T - T^2) + 2T - \frac{3T^2}{1} + \frac{T^3}{2} \right. \\ + t^*[x_1{}^0 + x_2{}^0 T - 4 + 5T^2 - \tfrac{5}{2}\, T^2] \\ \left. - 3(1 - T)\, t^{*2} - t^{*3}\right\}. \tag{34}$$

The second derivative of $J$ with respect to $t^*$ is then

$$J_{t^*t^*} = \ddot{V} = -2\{x_1^0 + x_2^0 T - 4 + 5T - \tfrac{5}{2} T^2 - 6(1 - T)\, t^* - 3t^{*2}\} \tag{35}$$

and $J_{t^*x^0}$ is

$$J_{t^*x^0} = -2(1 - T + t^*, -1 + T - T^2 + t^*T). \tag{36}$$

However in the following analysis $J_{t^*x(t^*)}$ is required (as opposed to $J_{t^*x^0}$). But

$$J_{t^*x(t^*)} = CJ_{t^*x(0)},$$

where

$$C = \begin{bmatrix} 1 & -t^* \\ 0 & 1 \end{bmatrix}.$$

Hence

$$J_{t^*x(t)^*} = -2[1 - T + t^*, -1 + T - t^* - (T - t^*)^2] = \dot{V}_x . \tag{37}$$

The equations (35) and (37) will now be used to verify the formulas for $\dot{V}$ and $\dot{V}_x$ (Eqs. (16) and (21)).

The Ricatti variable $P = V_{xx}$ is given by

$$P = -\frac{\partial f^T}{\partial x} P - P \frac{\partial f}{\partial x} \qquad \text{with} \qquad P(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and where

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$P(t) = \begin{bmatrix} 1 & T - t \\ T - t & 1 + (T - t)^2 \end{bmatrix}$$

also

$$f^- - f^+ = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$H_x^+ - H_x^- = 0, \qquad H_t^+ - H_t^- = 0.$$

Hence

$$\dot{V}_x = 2\begin{bmatrix}1 & T - t^* \\ T - t & 1 + (T - t^*)^2\end{bmatrix}\begin{bmatrix}-1 \\ 1\end{bmatrix}$$

$$= -2[1 - T + t^*, -1 + T - t^* - (T - t^*)^2],$$

which agrees with (37). Similarly

$$\dot{V} = (f^- - f^+)^T P^+(f^- - f^+) - H_x^T(f^- - f^+)$$

$$= 4[-1, \ +1]\begin{bmatrix}1 & T - t^* \\ T - t^* & 1 + (T - t^*)^2\end{bmatrix}\begin{bmatrix}-1 \\ +1\end{bmatrix}$$

$$-2[T - 1 - t^*, 1 - T + t^* + (T - t^*)^2]\begin{bmatrix}-1 \\ +1\end{bmatrix}.$$

And

$$\dot{V} = -2[x_1^0 + x_2^0 T - 4 + 5T - 5/2T^2 - 6(1 - T)t^* - 3t^{*2}], \quad (38)$$

which again checks.

Now consider the particular case when $x_1(T) = 1$, $x_2(T) = 0$, and $T = 2$. Proceeding via Pontryagin's principle the control, $u$, is chosen such that

$$u = \mathrm{sgn}(\lambda_1 - \lambda_2),$$

i.e., $u^+ = +1$.

A switch occurs when $p_1 = p_2$, i.e., when $t = t^*$. In this case $t^* = 1$ and $x_1^0 = x_2^0 = 0$. This trajectory completely satisfies Pontryagin's principle and the Hamiltonian and the costate variables are continuous. Furthermore Reid's condition

$$\dot{x}^+\dot{\lambda}^- - \dot{x}^-\dot{\lambda}^+ > 0$$

gives

$$[x_2^+ + u^+, -u^+]\begin{bmatrix}0 \\ -1\end{bmatrix} - [x_2 + u \ , -u^-]\begin{bmatrix}0 \\ -1\end{bmatrix} > 0,$$

and as $u^+ = +1$, $u^- = -1$ the equation becomes

$$[+1 + 1, -1 - 1]\begin{bmatrix}0 \\ -1\end{bmatrix} = +2.$$

Hence $\dot{x}^+\dot{\lambda}^- - \dot{x}^-\dot{\lambda}^+ = 2$ and Reid's condition is satisfied.

Now evaluating the second derivative of $J$ (Eq. (38) we have

$$J'' = -2[-4 + 10 - 10 + 6 - 3] = +2,$$

which is positive, indicating that the solution is *not* a maximum. This result may be confirmed by evaluation of the return function (33).

For the nominal, $t^*$, $J$ is given by

$$J = \tfrac{1}{2}\,[-(2 - 2t^*) - t^{*2} + (2 - 2t^*)^2]^2 + \tfrac{1}{2}\,[2 - 2t^*]^2,$$

substituting $t^* = 1$ gives $J = 0.5$.

Now considering a small perturbation in $t^*$, $\Delta t = \pm .05$ gives

$$J = \tfrac{1}{2}\left[\pm .1 - \begin{pmatrix}1.05\\ .95\end{pmatrix}^2 + .005\right]^2 + \tfrac{1}{2}\,[.1]^2$$

$$= \tfrac{1}{2}\begin{bmatrix}.9975\\ .9975\end{bmatrix}^2 + \tfrac{1}{2}\,[.1]^2$$

$$= 0.502603125,$$

i.e., $J(t^* \pm \Delta t) > J(t^*)$, and clearly there is not a maximum. Hence the new necessary condition is shown to be valid while Reid's condition is in error.

## A Numerical Example

The design of a minimum fuel attitude control system for the rigid body in orbit is an ideal problem with which to illustrate the application of the second variations approach. The system equations are nonlinear and control is known to be of the bang-bang type. For analytic solutions of greatly simplified versions of this problem see Athans and Falb [10], who include an extensive list of references. But, in general, a numerical technique must be used to solve the usual two-point boundary value problem. This problem has been approached by Flugge-Lotz [4] using a gradient technique. Consequently it will be possible to compare these results with the second variations results.

The system equations together with the detailed expressions required for the iterative procedure are given in Appendix B and only the results will be presented here. It must be remarked, however, that the choice of the priming trajectory is important. The initial trajectories used by Flügge-Lotz (4) were found to be unsuitable, in fact, even the optimized trajectories presented there were unsuitable. A more profitable approach, in this case, was to select one switching time in each control variable, the initial polarities being determined by the initial angular velocities. These switches were then optimized and extra switches were added as indicated by the switching function. Only *then* was rapid convergence obtained.

The optimized trajectories are shown in Fig. 1 compared with the trajectories obtained by Flügge-Lotz. The minimum fuel was .13 as comapred to .16. The convergence is shown in Table 1. Although the state trajectories in Fig. 1 differ only slightly, the control is significantly different.

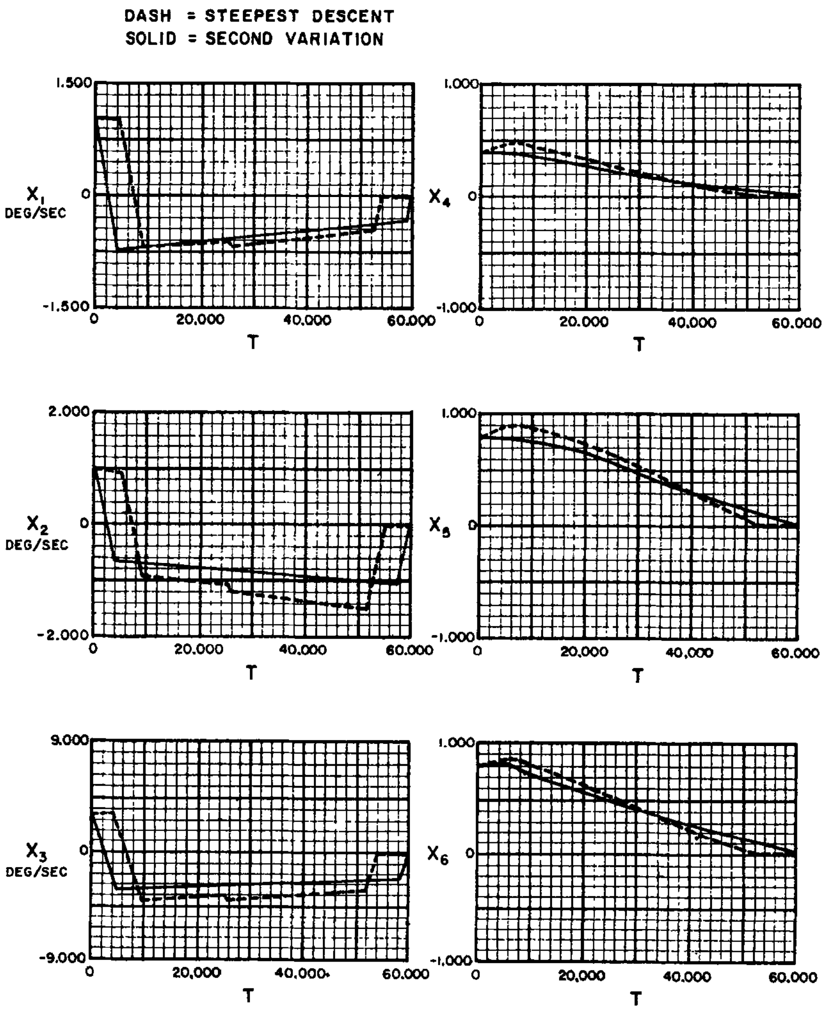


FIG. 1. Comparison of response curves.

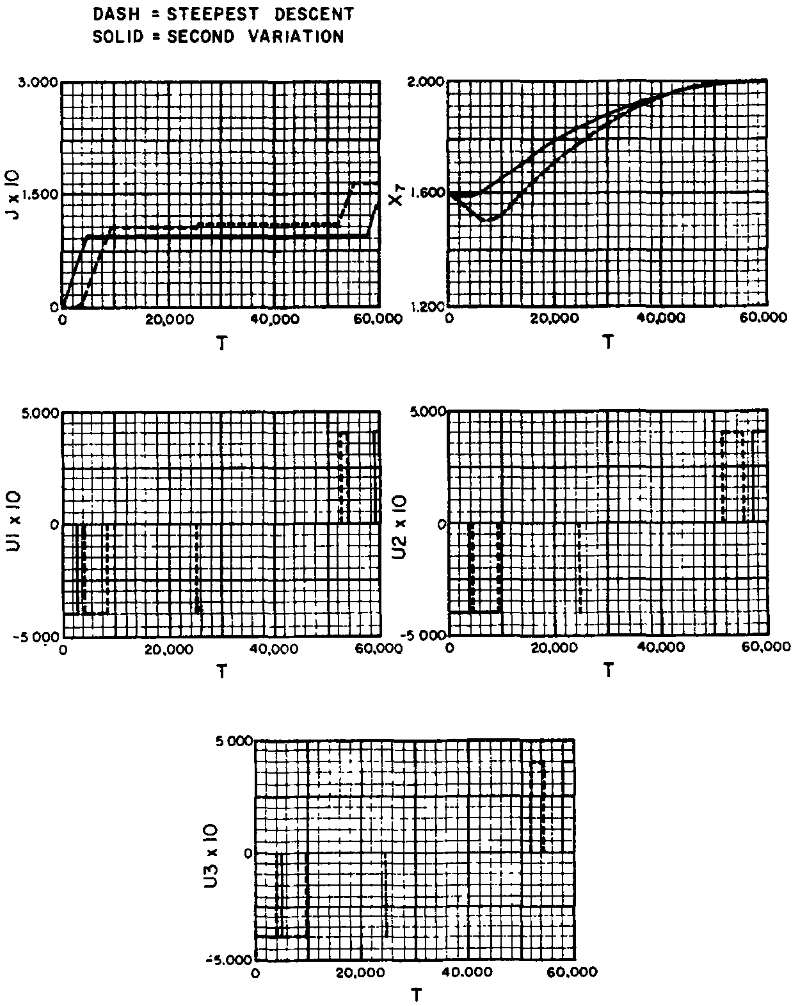


FIG. 1b. Comparison of response curves.

The control obtained using the second variation method satisfied all of the conditions for a minimum including the new necessary condition and hence is at least locally optimal.

TABLE 1

CONVERGENCE OF THE SECOND VARIATIONS METHOD

| Iteration No. | Cost | Switching Times $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ |
|---|---|---|---|---|---|---|---|
| Prime | .4101 | 3.500 | 4.000 | 5.000 | 57.500 | 58.000 | 59.000 |
| 1 | .2726 | 3.535[a] | 4.035[a] | 4.912[a] | 57.625[a] | 58.000[a] | 59.127 |
| 2 | .2047 | 3.564[a] | 4.073[a] | 4.927 | 57.750 | 58.010 | 59.255 |
| 3 | .1612 | 3.606[a] | 4.146[a] | 4.962 | 57.618 | 58.009 | 59.382 |
| 4 | .1399 | 3.672[a] | 4.145 | 5.011 | 57.562 | 58.003 | 59.446 |
| 5 | .1318 | 3.738 | 4.127 | 5.038 | 57.524 | 58.018 | 59.384 |
| 6 | .1303 | 3.780 | 4.116 | 5.055 | 57.499 | 58.028 | 59.344 |
| 7 | .1303 | 3.780 | 4.117 | 5.055 | 57.499 | 58.029 | 59.344 |
| | | Final Values of $\dot{V}$ | | | | | |
| | | 1.635 | 1.254 | 1.376 | .854 | .854 | .849 |

[a] Gradient Steps. $\Delta t = \epsilon \dot{V}$.

## APPENDIX A

*Derivation via Hamiltonian Theory*

In this appendix the jump conditions for the second derivatives of the return function will be derived by classical variational theory.

The system equations are assumed to be given by

$$\dot{x} = f(x, t). \tag{A-1}$$

The explicit appearance of $u$ will be ignored. The counterpart of the first derivative of the return with respect to $x$, usually denoted by $\lambda$, is referred to in the variational theory as the adjoint vector and is defined by the differential equation

$$\dot{\lambda} = -H_x{}^T, \tag{A-2}$$

where

$$H = \lambda^T f + L \tag{A-3}$$

is the standard variational Hamiltonian.

At discontinuities it is necessary that $\lambda$ and $H$ be continuous,

$$\lambda^- = \lambda^+ \tag{A-4}$$

and

$$H^- = H^+. \tag{A-5}$$

The second derivative of the return, which is denoted by $P$, satisfied the standard matrix Ricatti differential equation. The relation

$$\delta\lambda(t) = P(t)\,\delta x(t) \tag{A-6}$$

holds.

Now as before, let $t^*$ denote a time at which a discontinuity in $\dot{x}$ occurs. To obtain the jump condition for $P$, the technique of strong variations must be applied. If on a neighboring path the time of the discontinuity in $\dot{x}$ is changed to $t^* + dt$, then

$$x(t^* + dt) - x(t^*) = \delta x^- + f^-(x(t^*), t^*)\,dt. \tag{A-7}$$

This relationship, together with the definition of $\delta x^-$, is pictured in Fig. (A-1). Also one obtains

$$x(t^* + dt) - x(t^*) = \delta x^+ + f^+(x(t^*), t^*)\,dt. \tag{A-8}$$

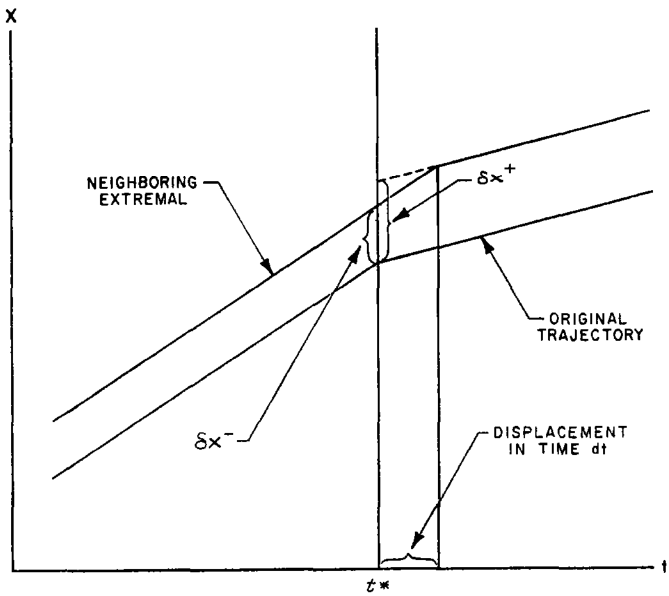


FIG. A.1. Neighboring extremals near a discontinuity.

Thus substracting (A-7) from (A-8)

$$\delta x^- + f^-\,dt = \delta x^+ + f^+\,dt. \tag{A-9}$$

A similar strong perturbation may be employed on Eq. (A-4) to give

$$\delta\lambda^- - H_x^-\,dt = \delta\lambda^+ - H_x^+\,dt. \tag{A.10}$$

Similarily a strong variation of Eq. (A-5) yields

$$f^- \delta\lambda^- + H_x^- \delta\lambda^- + H_t^- dt = f^+ \delta\lambda^+ + H_x^+ \delta x^+ + H_t^+ dt. \tag{A.11}$$

Now since $P$ is integrated backwards the relation

$$\delta\lambda^+ = P^+ \delta x^+ \tag{A-12}$$

may be obtained. Employing it in Eq. (A-10) yields

$$\delta\lambda^- = P^+ \delta x^+ - (H_x^+ - H_x^-)\, dt \tag{A.13}$$

and in (A-11) gives

$$f^- \delta\lambda^- + H_x^- \delta x^- = (f^+ P^+ + H_x^+)\, \delta x^+ + (H_t^+ - H_t^-)\, dt. \tag{A.14}$$

Now $\delta\lambda^-$ may be eliminated from Eq. (A-14) by using (A-13),

$$H_x^- \delta x^- = ((f^+ - f^-) P^+ + H_x^+)\, \delta x^+ + (H_x^+ - H_x^-) f^- + (H_t^+ - H_t^-)\, dt. \tag{A.15}$$

Now Eq. (A-9) will be used to eliminate $\delta x^+$ giving

$$0 = \dot{V}_x \, \delta x^- + \ddot{V}\, dt, \tag{A-16}$$

where

$$\dot{V}_x = P^+(f^- - f^+) + H_x^- - H_x^+ \tag{A.17}$$

and

$$\ddot{V} = (f^- - f^+)\, P^+(f^- - f^+) - H_x^+(f^- - f^+) + (H_x^- - H_x^+) f^- + H_t^- - H_t^+, \tag{A.18}$$

cf., Eqs. (21) and (16).

Solving for $dt$ gives, cf. Eq. (22),

$$dt = -\, \ddot{V}^{-1} \dot{V}_x \, \delta x. \tag{A-19}$$

And finally if $dt$ is eliminated from Eq. (A-13) the jump condition for $P$ results, viz.,

$$P^- = P^+ - \dot{V}_x \ddot{V}^{-1} \dot{V}_x \, . \tag{A-20}$$

## Appendix B

*Minimum Fuel Control of a Rigid Body in Orbit*

The purpose of this control system will be to correct for gross errors in orientation and angular velocities. Hence, a penalty function approach is used to ensure that the final state is within an acceptable region.

The equations describing the motion of a rigid body are well known and to facilitate a comparison with [4] the four parameter system was used to describe position. The system constants (moments of inertia) are also taken from Ref. [4].

The system equations are:

$$\begin{aligned}
\dot{x}_1 &= u_1 - K_x x_2 x_3 \\
\dot{x}_2 &= u_2 - K_y x_1 x_3 \\
\dot{x}_3 &= u_3 - K_z x_1 x_2 \\
\dot{x}_4 &= \tfrac{1}{2}(x_1 x_7 - x_2 x_6 + x_3 x_5) \\
\dot{x}_5 &= \tfrac{1}{2}(x_1 x_6 + x_2 x_7 - x_3 x_4) \\
\dot{x}_6 &= \tfrac{1}{2}(- x_1 x_5 + x_2 x_4 + x_3 x_7) \\
x_7 &= - \tfrac{1}{2}(x_1 x_4 + x_2 x_5 + x_3 x_6),
\end{aligned} \tag{B.1}$$

where the states $x_1$, $x_2$, and $x_3$ are angular velocities and $x_4$, $x_5$, $x_6$, and $x_7$ are parameters describing position. The parameters $K_x$, $K_y$, $K_z$ are $-.35125$, $.86058$, and $-.73000$, respectively. For this example the initial state is taken to be

$$\begin{aligned}
x_1 &= x_2 = x_3 = \frac{1}{57.3} \text{ rads/sec} \\
x_4 &= .4 \\
x_5 &= x_6 = 0.8 \\
x_7 &= 1.6.
\end{aligned} \tag{B.2}$$

The desired final state, $x_i$, is the origin, i.e,

$$\begin{aligned}
x_i &= 0 \qquad i = 1, 2, \ldots, 6 \\
x_7 &= 2.
\end{aligned} \tag{B-3}$$

The control is constrained so that $|u_i| \leqslant .412/57.3$ rads/sec². The performance index is

$$J = \int_0^{60} |u_1| + |u_2| + |u_3| \, dt + (x(60) - x_f)^T A(x(60) - x_f), \tag{B.4}$$

where the weighting matrix, $\Lambda$, is chosen such that

$$\left[\sum_{i=1}^{6} x_i(60)^2\right]^{1/2} \leqslant 10^{-2} \tag{B.5}$$

for the optimized trajectory. Here suitable values were $\Lambda_{ii} = 2.5 \times 57.3^2$, $i = 1, 2, 3$ and $\Lambda_{ii} = 5$, $i = 4, 5, 6, 7$, and $\Lambda_{ij} = 0$, $i \neq j$.

The Riccati equation, in reverse time, is

$$\dot{P} = B^T P + PB + H_{xx}; \qquad P(T) = 2\Lambda, \tag{B-6}$$

where

$$B = \begin{bmatrix} 0 & -K_x x_3 & -K_x x_2 & 0 & 0 & 0 & 0 \\ -K_y x_3 & 0 & -K_y x_1 & 0 & 0 & 0 & 0 \\ -K_z x_2 & -K_z x_1 & 0 & 0 & 0 & 0 & 0 \\ 0.5x_7 & -0.5x_6 & 0.5x_5 & 0 & 0.5x_3 & -0.5x_2 & 0.5x_1 \\ 0.5x_6 & 0.5x_7 & -0.5x_4 & -0.5x_3 & 0 & 0.5x_1 & 0.5x_2 \\ -0.5x_5 & 0.5x_4 & 0.5x_7 & 0.5x_2 & -0.5x_1 & 0 & 0.5x_3 \\ -0.5x_4 & -0.5x_5 & -0.5x_6 & -0.5x_1 & -0.5x_2 & -0.5x_3 & 0 \end{bmatrix}$$

$$H_{xx} = \begin{bmatrix} 0 & -\lambda_3 K_z & -\lambda_2 K_y & -0.5\lambda_7 & -0.5\lambda_6 & 0.5\lambda_5 & 0.5\lambda_4 \\ & 0 & -\lambda_1 K_z & 0.5\lambda_6 & -0.5\lambda_7 & -0.5\lambda_4 & 0.5\lambda_5 \\ & & 0 & -0.5\lambda_5 & 0.5\lambda_4 & -0.5\lambda_7 & 0.5\lambda_6 \\ & & & 0 & 0 & 0 & 0 \\ & \text{Symmetric} & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix},$$

and where the equation for $\lambda = V_x$ is

$$\dot{\lambda} = -B^T \lambda$$

$$\lambda(T) = 2\Lambda(x(60) - x_f). \tag{B-7}$$

In this problem $H_x^+ = H_x^-$ and hence $\dot{V}_x$ becomes

$$\dot{V}_x^T = \Delta f^T P,$$

where

$$\Delta f = [u_1^- - u_1^+, u_2^- - u_2^-, u_3^- - u_3^+, 0, 0, 0, 0]^T.$$

$\dot{V}$ is given by

$$\ddot{V} = \Delta f^T P \,\Delta f - H_x \Delta f, \tag{B-8}$$

where

$$H_x = B^T \lambda. \tag{B-9}$$

The equations (24) and (25), which are used to update the value of $P^+$ and $P^-$ and $\lambda^+$ and $\lambda^-$ at each switching time, complete the relations needed for the iterative procedure.

This procedure may now be summarized as follows:

1. Choose a suitable priming trajectory.
2. Integrate Eqs. ($B$-1) forward and store $x$.
3. Integrate Eqs. ($B$-6) and (B-7) in reverse time as far as the first switch.
4. Compute and store $\dot{V}$, $\dot{V}_x$ and $\ddot{V}$. Compute $\lambda^-$ and $P^-$.
5. Continue the integration to the next switch.
6. Repeat steps 4 and 5 until the initial time.
7. Integrate the state equations forward to the first switch. Compute $\Delta t = -(\ddot{V})^{-1}(\dot{V}_x\,\delta x + \dot{V})$ and form $t_{\text{new}} = t_{\text{old}} + \Delta t$. (N.B. at the first switch $\delta x = 0$ always)
8. Continue the integration storing the state $x$.
9. Compute the performance index.
10. Repeat steps 3-9 until no further improvement is made.
11. Check that the switching function is satisfied.

Here

$$u_i = -\operatorname{sgn}(\lambda_i) \qquad \text{for} \quad |\lambda_i| \geq 1 \qquad i = 1, 2, 3$$
$$u_i = 0 \qquad \text{for} \quad |\lambda_i| < 1.$$

It should be noted that in general several gradient steps would have to be taken before the full Newton-Raphson step could be used. In these cases, in step (7), $\Delta t$ would be chosen as

$$\Delta t = -\epsilon\dot{V}$$

and appropriate equations should be used to form $\lambda^-$ and $P^-$, viz., Eq. (27).

As has been mentioned the choice of priming trajectory is fairly important. Clearly any number of priming trajectories might be suitable. One technique, suggested by Flügge-Lotz [4], was to start with a series of narrow pulses. This was found to be impracticable when using the second variations scheme as the system quickly converged on incorrect extrema. In this regard it is interesting to compare the control histories obtained by the two methods Fig. 1. The authors' method of choosing one switching time and then increasing the number of switches was found to be more promising.

A fixed integration step length of 0.5 sec was used with a Runge-Kutta integration scheme. The switching time was chosen within the limits of single precision arithmetic although double precision was used in the integration scheme. One iteration (forward or backward integration) took about 23 sec on the IBM 7090 Computer.

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