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Stability properties of numerical methods for solving delay differential equations

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Abstract: Stability properties of numerical methods for delay differential equations are considered. Some suitable definitions for the stability of the numerical methods are included and Runge–Kutta type methods satisfying these properties are tested on a numerical example.

Keywords: Delay differential equations, absolutely stable, A-stable, DA-stable, P-stable, Q-stable, stability regions

1. Introduction

Consider a delay differential equation (DDE) of the type

$$u'(t)=f(t, u(t), u(t-d(t, u(t)))), \quad t_0 \leqslant T \leqslant T,$$
$$u(t)=g(t), \quad \min(t^*, t_0) \leqslant t \leqslant t_0, \tag{1.1}$$

where $t^* = \min t - d(t)$, $t \in [t_0, T]$, $g(t)$ is the initial function and $d(t, u(t)) \geqslant 0$ is the delay term.

The stability of numerical methods for DDEs has previously been considered by Brayton and Willoughby [7], Cryer [8] and Widerholt [9,10]. In this paper we first discuss the asymptotic properties of the solution of linear DDEs, then consider suitable definitions for the stability of the numerical methods and finally some Runge–Kutta methods satisfying these properties are tested on some numerical example.

Consider a system of linear DDEs of the form

$$\boldsymbol{u}'(t)=A\boldsymbol{u}(t)+B\boldsymbol{u}(t-d), \quad t \geqslant t_0,$$
$$\boldsymbol{u}(t)=\boldsymbol{g}(t), \quad -d \leqslant t \leqslant t_0, \tag{1.2}$$

where $d \geqslant 0$ is the delay, A and B are constant $n \times n$ real matrices, $\boldsymbol{u}$ is an n-dimensional vector and $\boldsymbol{g}(t)$ a continuous function.

One of the fundamental methods for finding the solution of (1.2) is to build up the solution as a sum of simple exponential terms. Assuming the solution of the form $\boldsymbol{u}(t)=\boldsymbol{c}e^{st}$, where s is constant and $\boldsymbol{c}$ an n-dimensional constant vector, then this solution will be a solution of (1.2) if and only if the number s is a zero of the transcendental function

$$H(s)=\det(Is-A-Be^{-ds}). \tag{1.3}$$

$H(s)=0$ is called the characteristic equation of (1.2) and s_r a characteristic root if it is a zero of this equation. These results have been summarized by a theorem given in [6] which states that there are in general infinitely many characteristic roots of (1.3) and therefore, infinitely many exponential solutions of (1.2). To have a good idea of the location of zeros of $H(s)$, we discuss first the distribution of the zeros of the characteristic equation of a single linear DDE of the form

$$u'(t)=au(t)+bu(t-d), \quad t \geqslant t_0,$$
$$u(t)=g(t), \quad -d \leqslant t \leqslant 0. \tag{1.4}$$

The characteristic equation has the form

$$h(s) = s - a - be^{-ds} = 0, \tag{1.5}$$

which can be written

$$h(s) = s\{1 + \epsilon(s)\} - be^{-sd} = 0, \tag{1.6}$$

where $\epsilon(s) \to 0$ as $|s| \to \infty$. It is reasonable to suppose the zeros of $h(s)$ and the zeros of

$$\bar{h}(s) = s - be^{-sd} = 0 \tag{1.7}$$

are close together for s_r large and that the zeros of (1.7) satisfy

$$|se^{sd}| = |b|, \quad b = 0 \tag{1.8}$$

or

$$\mathrm{Re}(s) + \frac{1}{d} \ln|s| = \frac{1}{d} \ln|b|. \tag{1.9}$$

Hence zeros of $h(s)$ lie asymptotically along the curve defined in (1.9). It is shown in [6] that the roots s_r of (1.5) are infinite, complex conjugate and that all lie in the left half plane $\mathrm{Re}(s) < c$, for some constant c. This last property is a characteristic of DDEs with constant delay.

Before we consider the concept of stability, we give some definitions.

Definition 1.1. The DDE (1.2) is called *stable* if for any sufficiently small initial function the solution $\boldsymbol{u}(t)$ approaches zero as t approaches infinity, that is, for a small constant $\delta > 0$, $\lim_{t\to\infty} \|u(t)\| = 0$, for $\|u(t)\| < \delta$, $-d \leqslant t \leqslant t_0$.

This type of stability is commonly referred to as asymptotic stability. To find conditions for DDE (1.2) to be stable, we have the following result.

Theorem 1.1. *A necessary and sufficient condition for all continuous solutions of* (1.2) *to approach zero as* $t \to \infty$ *is that all the characteristic roots have negative real parts.*

So the best model for studying stability is DDE (1.4) if a and b, in general, are complex numbers.

We now give results which impose conditions on a and b in (1.5) for the roots of $h(s) = 0$ to have negative real parts.

Case 1. a and b are real. This case is discussed by Bellman and Cook [6] and their result is:

Theorem 1.2. *All roots of equation* (1.5) *have negative real parts if and only if*

(i) $a < 1$,

(ii) $a < -b < \sqrt{\theta^2 + a^2}$,

where θ *is the root of* $\theta = a\tan(\theta d)$ *such that* $0 < \theta d < \pi$, *if* $a = 0$ *we take* $\theta = \frac{1}{2}\pi/d$.

Case 2. $a = 0$, and b complex. This case has been considered by Barwell [4] and his result is:

Theorem 1.3. *Let* $b = re^{i\phi}$, *then a sufficient condition that all the roots of* (1.5) *have negative real parts is*

(i) $\mathrm{Re}(b) < 0$ $(\frac{1}{2}\pi < \phi < \frac{3}{2}\pi)$,

(ii) $0 < rd < \min(\frac{3}{2}\pi - \phi, \phi - \frac{1}{2}\pi)$.

Case 3. a and b are complex. This is also considered by Barwell [4] and his result is:

Theorem 1.4. *A sufficient condition that all the roots of* (1.5) *have negative real parts is*

$$\mathrm{Re}(a) < -|b|.$$

2. Definitions and comparison of different approaches

Assume that a numerical method is applied to the DDE (1.1) with a fixed stepsize h, then the global error is defined by

$$e_n = y(t_n) - u(t_n), \tag{2.1}$$

where $t_n = t_0 + nh$, $y(t)$ is the numerical solution and $u(t)$ the exact solution at t.

In a stability analysis of a numerical method one is concerned not with the source of the error but only with the behaviour of the global error as $t_n \to \infty$ after some error has been introduced. Since the behaviour of the global error (2.1) depends on the behaviour of the solution of the DDE (1.1), we adopt the model DDE (1.4) as

$$\begin{aligned} u'(t) &= au(t) + bu(t-1), \quad t \geqslant t_0, \\ u(t) &= g(t), \quad -1 \leqslant t \leqslant t_0, \end{aligned} \tag{2.2}$$

where a, b, in general, are complex and $g(t)$ is a continuous function.

Since the definition of absolute stability is only concerned with the case where the solution $u(t)$ satisfies

$$u(t) \to 0 \quad \text{as } t \to \infty, \tag{2.3}$$

we need to know the asymptotic stability properties of the solution (2.2) which have been discussed in an earlier section with some conditions imposed on a and b so that the solution satisfies (2.3).

For a numerical method for solving (2.2) we expect the global error $e_n \to 0$ as $n \to \infty$ if the solution satisfies (2.3), which leads us to adopt the following definition.

Definition 2.1. A numerical method applied to DDE (2.2) is said to be *absolutely stable* for the stepsize h, if for any problem whose solution satisfies (2.3), the numerical solution at step h satisfies $y(t_n) \to 0$ as $t_n \to \infty$.

If the absolute stability of a method is independent of h, then we get the following definition similar to the A-stability definition of ordinary differential equation (ODE).

Definition 2.2. A numerical method is said to be *DA-stable* if for any solution of (2.2) which satisfies (2.3), the numerical solution $y(t_n) \to 0$ as $t_n \to \infty$, for any $h > 0$.

The definition of DA-stability depends on knowing the necessary and sufficient conditions on a and b such that the solution satisfies (2.3). By assuming $mh = 1$, $m \in I^+$ (set of positive integers), Cryer [8] considers a definition of DA-stability for linear multistep methods using DDE (2.2) with $a = 0$ and b real. Later, Barwell [5] generalizes Cryer's definition by considering the DDE (2.2) with $a = 0$ and b complex. His definition is adopted here for one-step methods.

Definition 2.3. Let $b = re^{i\phi}$ and $a = 0$ in (2.2). A numerical method is said to be *Q-stable* if under the conditions provided in Theorem 1.3, the numerical solution $y(t_n) \to 0$ as $t \to \infty$ for all h satisfying $mh = 1$, $m \in I^+$.

Barwell [5], after getting a sufficient condition on a and b, as in (2.2), such that $u(t) \to 0$ as $t \to \infty$, considers the following definition.

Definition 2.4. A numerical method, applied to (2.2) is said to be *P-stable* if under the condition $\mathrm{Re}(a) < -|b|$, the numerical solution $y(t_n) \to 0$ as $t_n \to \infty$ for all h satisfying $mh = 1$, $m \in I^+$.

It is clear from Definitions 2.3 and 2.4 that if the method is P-stable then it is A-stable, but if it is Q-stable then it is not necessarily A-stable.

For a definition of an absolute stability region, we introduce the following.

Definition 2.5. For the stepsize h:

(1) If a and b are real in (2.2), the region RP(a, b) in the a, b plane is called the *P-stability region* if for any a, $b \in$ RP(a, b) the numerical solution of (2.2) satisfies $y(t_n) \to 0$ as $t_n \to \infty$.

(2) If $a = 0$ and b is complex in (2.2), the region RQ(b) in the b-plane is called the *Q-stability region* if for any $b \in$ RQ(b) the numerical solution $y(t_n) \to 0$ as $t_n \to \infty$.

3. Stability properties of some numerical methods

We now consider the stability properties of the methods given in [2,3] Assume that for each method, the numerical solution of (2.2) is calculated up to point t_n with a fixed stepsize h such that

$$t_n = t_0 + nh \quad \text{and} \quad mh = 1, \quad m \in I^+. \tag{3.1}$$

Since the purpose of introducing P-stability and Q-stability, in Definitions 2.3 and 2.4, is to find methods which can be used in practice with no restriction on stepsize because of stability, and since $h = 1/m$, m increases as h decreases. Hence, the important case is to show that the method is Q-stable or P-stable for small values of m. For each of the following methods for solving DDE (also given in [1]), we give the results for $m = 1, 2, 3, 4$. Let $z(s)$ be the approximation of the delay term at s.

3.1. Kutta–Merson method for solving DDE

To advance the numerical solution of the DDE (2.2) to the point t_{n+1}, the Kutta–Merson method of [2] yields intermediate values y_1, y_2, y_3, y_4 and y_5, thus we have

$$\begin{aligned} y(t_{n+1}) \simeq y_5 = {} & \left(1 + ha + \tfrac{1}{2}h^2a^2 + \tfrac{1}{6}h^3a^3 + \tfrac{1}{24}h^4a^4 + \tfrac{1}{144}h^5a^5\right) y(t_n) \\ & + \tfrac{1}{6}hb\left(1 + ha + \tfrac{1}{4}h^2a^2 + \tfrac{1}{8}h^3a^3 + \tfrac{1}{24}h^4a^4\right) z(t_n - 1) \\ & + \tfrac{1}{8}h^3ba^2\left(1 + \tfrac{1}{6}ha\right) z\left(t_n + \tfrac{1}{3}h - 1\right) + \tfrac{2}{3}hb\left(1 + \tfrac{1}{2}ha\right) z\left(t_n + \tfrac{1}{2}h - 1\right) \\ & + \tfrac{1}{6}hbz(t_n + h - 1). \end{aligned} \tag{3.2}$$

Using condition (3.1), and assuming that the values of the solution and its derivative are stored at earlier mesh points, then using Hermite interpolation for evaluating the delay term, we get

$$\begin{aligned} z\left(t_n + \tfrac{1}{3}h - 1\right) &= z\left(t_{n-m} + \tfrac{1}{3}h\right) \\ &= \left(\tfrac{20}{27} + \tfrac{4}{27}ha\right) y(t_{n-m}) + \left(\tfrac{7}{27} - \tfrac{2}{27}ha\right) y(t_{n-m+1}) \\ &\quad + \tfrac{4}{27}hby(t_{n-2m}) - \tfrac{2}{27}hby(t_{n-2m+1}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} z\left(t_n + \tfrac{1}{2}h - 1\right) &= z\left(t_{n-m} + \tfrac{1}{2}h\right) \\ &= \left(\tfrac{1}{2} + \tfrac{1}{8}ha\right) y(t_{n-m}) + \left(\tfrac{1}{2} - \tfrac{1}{8}ha\right) y(t_{n-m+1}) + \tfrac{1}{8}hby(t_{n-2m}) - \tfrac{1}{8}hby(t_{n-2m+1}). \end{aligned} \tag{3.4}$$

On replacing values of the function z in (3.2), we get a difference equation whose solutions tend to zero as $n \to \infty$, provided that all the roots of the following characteristic equation are in the unit circle:

$$\begin{aligned} & \xi^{2m+1} - \left(1 + ha + \tfrac{1}{2}h^2a^2 + \tfrac{1}{6}h^3a^3 + \tfrac{1}{24}h^4a^4 + \tfrac{1}{144}h^5a^5\right) \xi^{2m} \\ & - \tfrac{1}{2}hb\left(1 + \tfrac{1}{6}ha - \tfrac{1}{54}h^2a^2 - \tfrac{5}{648}h^3a^3 - \tfrac{1}{324}h^4a^4\right) \xi^{m+1} \\ & - \tfrac{1}{2}hb\left(1 + \tfrac{5}{6}ha + \tfrac{19}{54}h^2a^2 + \tfrac{71}{648}h^3a^3 + \tfrac{13}{648}h^4a^4\right) \xi^{m} \\ & + \tfrac{1}{12}h^2b^2\left(1 + \tfrac{1}{2}ha + \tfrac{1}{9}h^2a^2 + \tfrac{1}{54}h^3a^3\right) \xi - \tfrac{1}{12}h^2a^2\left(1 + \tfrac{1}{2}ha + \tfrac{2}{9}h^2a^2 + \tfrac{1}{27}h^3a^3\right) = 0. \end{aligned} \tag{3.5}$$

When a and b are real we give in Fig. 1, the P-stability region for $m = 1, 2, 3, 4$ and compare it with the

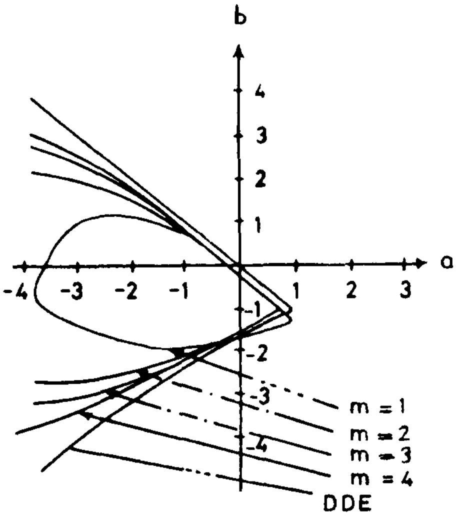

Fig. 1. P-stability regions of the Kutta–Merson method for solving DDE, with the stability region of the DDE (2.2), a and b real.

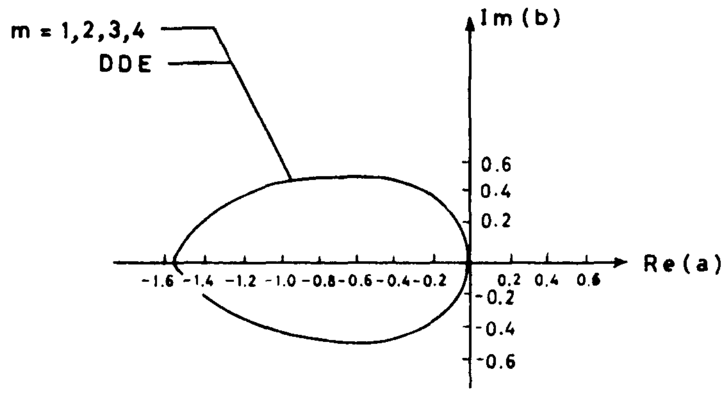

Fig. 2. Q-stability regions of the Kutta–Merson method for solving DDE with the stability region of the DDE (2.2), $a=0$ and b complex.

stability region of the DDE (2.2) in the (a, b) plane. All the P-stability regions are closed regions, they intersect the a-axis at the point $a = 3.54m$. The interval on the a-axis gives the absolute stability interval for ODE according to Definition 2.1.

If $a = 0$ and b is complex, we give in Fig. 2 the Q-stability region for $m = 1, 2, 3, 4$ and compare it with the stability region of the DDE (2.2) in the b-plane.

3.2. The trapezium method for solving DDE

To advance the solution from the point t_n to the point t_{n+1}, the trapezium method of [3] yields

$$y(t_{n+1}) = y(t_n) + \tfrac{1}{2}ha\big(y(t_n) + y(t_{n+1})\big) + \tfrac{1}{2}hb\big(z(t_n - 1) + z(t_n + h - 1)\big). \tag{3.6}$$

Using condition (3.1), it is clear that we get the same solution at t_{n+1} in (3.6) whether we use linear or Hermite interpolation for approximating the delay term, therefore,

$$y(t_{n+1}) = y(t_n) + \tfrac{1}{2}ha\big(y(t_n) + y(t_{n+1})\big) + \tfrac{1}{2}hb\big(y(t_{n-m}) + y(t_{n-m+1})\big). \tag{3.7}$$

The characteristic polynomial is

$$\left(1 - \frac{a}{2m}\right)\xi^{m+1} - \left(1 + \frac{a}{2m}\right)\xi^{m} - \frac{b}{2m}\xi^{m} - \frac{b}{2m} = 0. \tag{3.8}$$

For a and b real, we give in Fig. 3 the P-stability region for $m = 1, 2, 3, 4$ and compare it with the stability region of DDE (2.2) in the (a, b)-plane. When $a = 0$ and b is complex, we give in Fig. 4 the Q-stability region for $m = 1, 2, 3, 4$ and compare it with the stability region of the DDE (2.2) in the b-plane. We mention here that Cryer [8] proved that the method is Q-stable for b real.

3.3. The implicit Runge–Kutta method for solving DDE

By applying the fourth order implicit Runge–Kutta method of [3] to advance the solution of the linear DDE (2.2) from the point t_n to t_{n+1}, we get

$$y(t_{n+1}) = \frac{1 + \frac{1}{2}ha + \frac{1}{12}h^2a^2}{1 - \frac{1}{2}ha + \frac{1}{12}h^2a^2}y(t_n) + \frac{\frac{1}{6}hb\left(1 + \frac{1}{2}ha\right)}{1 - \frac{1}{2}ha + \frac{1}{12}h^2a^2}z(t_n - 1)$$

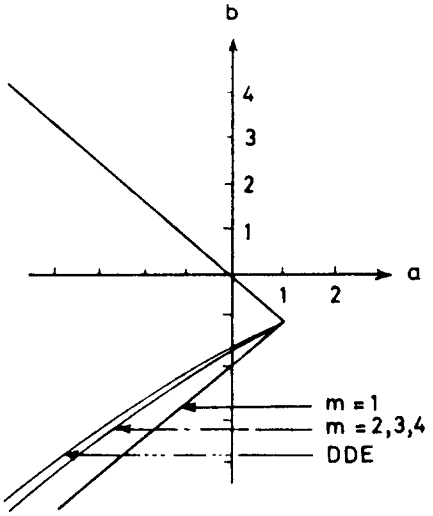

Fig. 3. P-stability regions of the trapezium method, with the stability region of the DDE (2.2), a and b real.

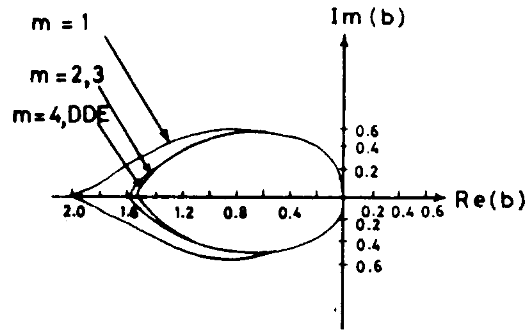

Fig. 4. Q-stability region of the trapezium method, with the stability region of the DDE (2.2), $a = 0$ and b complex.

$$+\frac{\frac{2}{3}hb}{1-\frac{1}{2}ha+\frac{1}{12}h^2a^2}z\left(t_n+\tfrac{1}{2}h-1\right)+\frac{\frac{1}{6}hb\left(1-\frac{1}{2}ha\right)}{1-\frac{1}{2}ha+\frac{1}{12}h^2a^2}z\left(t_n+h-1\right). \tag{3.9}$$

Using (3.1) and Hermite interpolation of the third degree for approximating the delay term, $z(t_n + \frac{1}{2}h - 1)$ has the same form as in (3.4), then (3.9) becomes

$$y(t_{n+1})=\frac{1+\frac{1}{2}ha+\frac{1}{12}h^2a^2}{1-\frac{1}{2}ha+\frac{1}{12}h^2a^2}y(t_n)+\frac{\frac{1}{6}hb}{1-\frac{1}{2}ha+\frac{1}{12}h^2a^2}$$
$$\times\left[(3+ha)y(t_{n-m})+(3-ha)y(t_{n-m+1})+\tfrac{1}{2}hby(t_{n-2m})-\tfrac{1}{2}hby(t_{n-2m+1})\right]. \tag{3.10}$$

Then the characteristic polynomial is

$$\left(1-\frac{a}{2m}+\frac{a^2}{12m^2}\right)\xi^{2m+1}-\left(1+\frac{a}{2m}+\frac{a^2}{12m^2}\right)\xi^{2m}-\frac{b}{2m}\left(1-\frac{a}{3m}\right)\xi^{m+1}$$
$$-\frac{b}{2m}\left(1+\frac{a}{2m}\right)\xi^{m}+\frac{b^2}{12m^2}\xi-\frac{b^2}{12m^2}=0. \tag{3.11}$$

For a and b real, we give in Fig. 5 the P-stability regions for $m = 1, 2, 3, 4$ and the stability region of the DDE (2.2).

For $a = 0$ and b complex, the Q-stability characteristic polynomial of this method is the same as that of Kutta–Merson method namely, equation (3.5), hence the Q-stability region is the same as for the Kutta–Merson method for solving DDE (2.2) for $m = 1, 2, 3, 4$.

Remark 3.1. It is clear from Figs. 2 and 4 that the regions of Q-stability for $m = 1, 2, 3, 4$ are all greater than or equal to the stability region of the DDE (2.2) in the b-plane for all the methods considered. Also in Figs. 3 and 5 the P-stability regions for the trapezium method and the implicit Runge–Kutta method are greater than the stability region of the DDE (2.2) in the a, b-plane, and so the stepsize is not restricted by the stability properties of the method. For the Kutta–Merson method, Fig. 1 shows the effect of choosing a certain stepsize on the P-stability region. We will give numerical results in the next section to show the effect of the stability properties of the method on the choice of the stepsize.

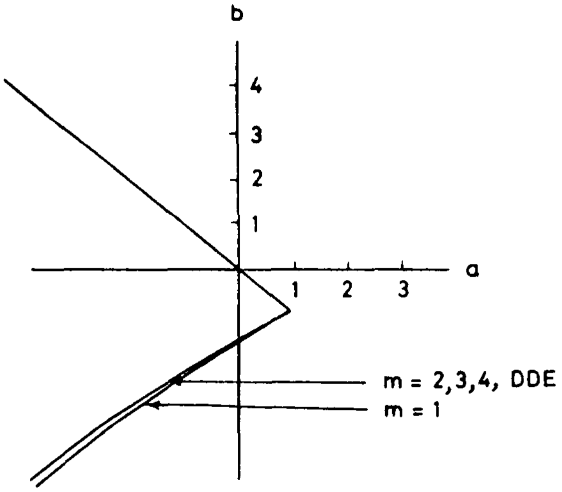

Fig. 5. P-stability regions of the implicit Runge–Kutta method, with the stability region of the DDE (2.2), a and b real.

Remark 3.2. To calculate the stability region, for example Fig. 1, we take different values of (a, b) along the co-ordinate axes, and find the roots of the stability polynomial using the NAG library routine CO2ADA. If all the roots have magnitude less than one then we accept the value of (a, b) as part of the stability region. For the P-stability regions we change the values of a and b by 0.25 each time, and for the Q-stability region by 0.1 each time. If it appears in some of the figures that the curves are identical, this is not exactly so, but they are the same for the accuracy we are using.

Remark 3.3. It is not easy to prove P-stability or Q-stability results for general m. Our conjecture is that all the methods we have considered are Q-stable, and the trapezium method and the implicit Runge–Kutta method for solving DDE are also P-stable.

4. Numerical example

The following example is chosen to show the advantage of methods which have no restriction on the stepsize because of stability properties of the methods. We present the maximum global discretization error on the range of integration as a measure of the reliability of the method and the number of derivative evaluations on this interval as a measure of the efficiency of the method. We use the starting stepsize $h = 0.1$ and use an absolute error test unless otherwise stated. The calculations are performed on the CDC 7600 computer at Victoria University, Manchester, U.K.. The following notations are used in the tables: ϵ = the required error tolerance, ND = number of derivative evaluations, GE = the maximum global discretization error on the interval of integration.

Problem 4.1.

$$u'(t) = au(t) + bu(t-d), \quad 0 < t \leqslant T, \tag{4.1}$$

$$u(t) = e^{s_i t}, \quad t \in [-d, 0], \tag{4.2}$$

where s_i are some of the real roots of the characteristic equation,

$$h(s) = s - a - be^{-ds} = 0.$$

Equation (4.1) has a smooth solution $u(t) = \Sigma e^{s_i t}$, $t \geqslant 0$. As a special case we take $a = 0$, $b = -1$, $d = 10^{-3}$, $s_1 = -1.001001502672$ and $s_2 = -9118.006470403$.

Table 1
Using the trapezium method for solving DDE with linear interpolation

ϵ	Interval					
	0, 10		0, 20		0, 40	
	GE	ND	GE	ND	GE	ND
10^{-2}	2.0227×10^{-2}	84	2.0227×10^{-2}	87	2.0227×10^{-2}	90
10^{-4}	1.5528×10^{-3}	172	1.5528×10^{-3}	178	1.5528×10^{-3}	181
10^{-6}	7.4131×10^{-5}	589	7.4131×10^{-5}	604	7.4131×10^{-5}	610
10^{-8}	3.4522×10^{-6}	2648	3.4522×10^{-6}	2629	3.4522×10^{-6}	2705
10^{-10}	*					

Table 2
Using the trapezium method for solving DDE with Hermite interpolation

ϵ	Interval					
	0, 10		0, 20		0, 40	
	GE	ND	GE	ND	GE	ND
10^{-2}	2.0223×10^{-2}	84	2.0223×10^{-2}	87	2.0223×10^{-2}	90
10^{-4}	1.5528×10^{-3}	172	1.5528×10^{-3}	178	1.5528×10^{-3}	181
10^{-6}	7.4122×10^{-5}	582	7.4122×10^{-5}	598	7.4122×10^{-5}	604
10^{-8}	3.4517×10^{-6}	2555	3.4517×10^{-6}	2603	3.4517×10^{-6}	2609
10^{-10}	*					

Table 3
Using the implicit Runge–Kutta for solving DDE

ϵ	Interval					
	0, 10		0, 20		0, 40	
	GE	ND	GE	ND	GE	ND
10^{-2}	4.0057×10^{-2}	125	4.0057×10^{-2}	131	4.0057×10^{-2}	137
10^{-4}	2.2729×10^{-4}	163	2.2729×10^{-4}	169	2.2729×10^{-4}	175
10^{-6}	2.8902×10^{-6}	255	2.8902×10^{-6}	273	2.8902×10^{-6}	285
10^{-8}	1.0867×10^{-7}	475	1.0867×10^{-7}	505	1.0867×10^{-7}	517
10^{-10}	3.1965×10^{-9}	1059	3.1965×10^{-9}	1131	3.1965×10^{-9}	1155

Table 4
Using the Kutta–Merson method for solving DDE

ϵ	Interval					
	0, 10		0, 20		0, 40	
	GE	ND	GE	ND	GE	ND
10^{-2}	3.3632×10^{-3}	171	3.3632×10^{-3}	183	6.2623×10^{-3}	225
10^{-4}	2.2622×10^{-5}	261	2.2622×10^{-5}	279	7.6883×10^{-5}	321
10^{-6}	1.0314×10^{-6}	532	1.0314×10^{-6}	562	1.0314×10^{-6}	598
10^{-8}	1.2216×10^{-8}	1431	1.2216×10^{-8}	1497	1.2216×10^{-8}	1533
10^{-10}	1.2181×10^{-10}	4459	1.2181×10^{-10}	4621	1.2181×10^{-10}	4657

Since all the methods considered are Q-stable, there should be no stability problem in solving this problem. In Tables 1 and 2 we give the results for the trapezium method with linear interpolation and for the trapezium method with Hermite interpolation respectively. In Table 3 we give the results when using the implicit Runge–Kutta method with Hermite interpolation, and in Table 4 we give the results when using the Kutta–Merson method of [2] for solving DDE with Hermite interpolation. For high accuracy requirements the Kutta–Merson method and the implicit Runge–Kutta achieve the required accuracy, but the trapezium method does not because of the low order of the method. Also, Tables 1 and 2 show that there is no significant improvement in using higher order interpolation formula for approximating the delay term with the trapezium method.

5. Concluding remarks

All three methods are discussed, being Q-stable, face no stability problem. Howver, the stability properties of numerical methods for solving DDE need further investigation. It would be interesting to know necessary and sufficient conditions on a linear DDE with constant delay and complex coefficients such that the solution is asymptotically stable, and then one can use the more general definition of DA-stability suggested in Section 2. It would also be interesting to know the relation between the roots of the characteristic equation of the linear DDE and the roots of the stability polynomial of the numerical methods for solving DDE.

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