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Vertex-transitive Graphs of Valency 3

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A regular graph of valency 1 is necessarily a disjoint union of paths of length 2 and one of valency 2 must be a disjoint union of cycles. However, regular graphs of valency 3 are so many and varied that it seems to be impossible to describe them all. Among regular graphs are the vertex-transitive ones, those on which a group of automorphisms acts transitively on the vertices. Every regular graph of valency 1 is vertex transitive and those of valency 2 are vertex transitive if and only if they are a disjoint union of cycles of the same length. This paper is intended to be a starting point for a possible description of all vertex transitive graphs of valency 3. It describes them in terms of graphs which can be built up from graphs of smaller valency, bipartite graphs and graphs on which simple groups act as groups of automorphisms. In that simple groups have now been classified, this suggests a program for classifying the graphs.

My interest in these graphs was stimulated by the conjecture of L. Lovász [1, page 249] that every connected vertex-transitive graph has a Hamiltonian path. A likely place to look for counterexamples to this conjecture is among graphs of low valency, perhaps among those of valency 3. I am indebted to C. C. Chen for conversations on this problem.

In a group acting on a graph of valency 3, the stabilizer of any vertex acts as a group of permutations on the vertices adjacent to the one it fixes and as such either has three orbits of length one, one of length one and one of length two, or one of length three. These three possibilities are dealt with separately in sections 2, 3 and 4. The first is the easiest and the third is part of a more general result in another of my papers [3]; the middle one requires the most space here.

The outcome of this paper can be summarized in the following statement:

THEOREM. *Let G be a vertex transitive group of automorphisms of a connected graph Σ of valency 3. Let H be the stabilizer of a vertex v of Σ and let X be the set of three vertices adjacent to v.*

Let G_1 be a subgroup of G minimal among the subgroups of G which act transitively on the vertices of Σ and have the property that their intersection with H has the same orbits in X that H does.

Let G_2 be a subgroup of G_1 maximal among the normal subgroups of G_1 which contain no member except 1 fixing a vertex of Σ.

Then either Σ is one of the graphs listed under (I), (II), (III) or (IV) in Section 2 or $G_1|G_2$ is a simple group which acts as a vertex-transitive group of automorphisms of a graph of valency 3 in such way that the stabilizer of each vertex has the same orbit structure on the vertices adjacent to it that H has on X.

This statement is necessarily a little vague at this stage but its various components will be made precise later. In particular, the relationship between the graph Σ and the graph on which $G_1|G_2$ acts will be defined exactly.

1. PRELIMINARIES

The subject of this paper is a connected graph Σ of valency 3 on which a group G acts as a vertex-transitive group of automorphisms. Let v be a fixed, but arbitrarily

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chosen vertex of Σ and let H be its stabilizer in G. Let $X=\{x, y, z\}$ be the vertices of Σ adjacent to v and let a, b, c be members of G with the properties $a(v)=x$, $b(v)=y$, $c(v)=z$ respectively. Let D be the subset of G containing all members g of G with the property that $g(v)$ is adjacent to v.

The notations in the previous paragraph will be maintained throughout the paper.

The first proposition in this section contains the basic properties of the set D.

PROPOSITION 1.
(i) $D=D^{-1}$.
(ii) $D=aH\cup bH\cup cH$.
(iii) *D is a union of double cosets of H in G.*

PROOF: A proof of this can be found in Theorem 1 of [3].

In these circumstances the vertices of the graph can be identified with the left cosets of H in G and two cosets αH and βH are adjacent if and only if $\alpha^{-1}\beta\in D$: for details see [3, Theorems 1 and 2].

The subgroup H of G is the stabilizer of v. As it acts as a group of automorphisms of Σ it necessarily acts as a permutation group on the set X of vertices adjacent to v. The graphs which are the subject of this paper split into three types, depending on the action of H on X and these three types are considered separately in Sections 2, 3 and 4. In terms of what has been written so far, they can be described as follows.

PROPOSITION 2. *One of the following is true.*
(I) *$H=\{1\}$ and, equivalently, G acts regularly on Σ. $D=\{a, b, c\}$ and G is generated by a, b and c.*
(II) *X has two orbits under the action of H and, without loss of generality, these can be taken as $\{x, y\}$ and $\{z\}$; c lies in the normalizer of H and $HaH=HbH=aH\cup bH$; $D=HaH\cup cH$; G is generated by H, a and c; $|H|=2|H\cap aHa^{-1}|$.*
(III) *H acts transitively on X; $D=HaH=HbH=HcH$ and G is generated by H and a; $|H|=3|H\cap aHa^{-1}|$.*

PROOF. In the action of H on X, the members of X fall into three orbits of length 1, one of length one and one of length 2 or one of length 3.

(I) Suppose that the orbits of X under H all have length 1. Then every member of G which fixes v, also fixes every vertex adjacent to v. Because G is vertex transitive the same is true for each vertex of Σ in place of v. The connectedness of Σ then implies that every member of H fixes every vertex of Σ. Hence $H=\{1\}$.

As Σ is connected, it follows from [3, Theorem 6] that G is generated by D and hence by a, b and c.

(II) As $\{x, y\}$ is an orbit under H, there is a member h of H with $h(x)=y$. Then $ha(v)=b(v)$ and $b\in haH\subseteq HaH$; hence $HaH=HbH$. On the other hand, if $g\in HaH$, say $g=h_1ah_2$ with h_1, $h_2\in H$, then $g(v)=h_1ah_2(v)=h_1(x)$. As $\{x, y\}$ is an orbit under H, either $h_1(x)=x$, in which case $g\in aH$, or $h_1(x)=y$ in which case $g\in bH$. Hence $HaH=aH\cup bH$. As $\{x, y\}$ is an orbit under H, the stabilizer of x in H has just two cosets, i.e. $|H|=2|H\cap aHa^{-1}|$.

As $\{z\}$ is an orbit under H, this subgroup is also the stabilizer of z in G; i.e. $H=cHc^{-1}$ Hence $cH=Hc=HcH$.

As Σ is connected, it follows from [3, Theorem 6] that G is generated by D. As $D=HaH\cup cH$, it is generated by H, a and c.

(III) The proof follows the form of (II); in this case $\{x, y, z\}$ is the only orbit of X under H.

The theorems in this paper are established by examining the normal structure of G and what were called, in [3], the quotient graphs of Σ and their factor subgraphs. Here is the way that these things are related.

Let N be a normal subgroup of G. Put $K = HN$ and $E = KDK - K$; K is a subgroup of G. A graph Λ, called a quotient graph of Σ, can be defined as follows: the vertices are the left cosets of K in G and two cosets αK and βK are adjacent in Λ if and only if $\alpha^{-1}\beta \in E$. If, as suggested earlier, the vertices of Σ are identified with the left cosets of H in G, then each vertex of Λ is a union of vertices of Σ, and, in this way, the vertices of Λ partition the vertices of Σ. Let $g_1H, \ldots, g_rH$ be the cosets of H in one coset of K. If $g_i^{-1}g_j$ is a member of D, then $g_i^{-1}g_j \in D \cap K$. Hence, the induced subgraph of Σ with these vertices has two vertices g_iH, g_jH adjacent if and only if $g_i^{-1}g_j \in D \cap K$. This is called a factor subgraph defined by HN.

There are a number of vertex-transitive graphs of valency 3 which are easily described because of their simple structure. As these necessarily play a role in the statements of the theorems of the paper, they are now described. The numbers established here will be used throughout the paper to refer to these graphs.

(I) Σ could be a bipartite graph.

(II) The vertices of Σ could be partitioned into two equal sets each consisting of a disjoint union of cycles of the same size, with each vertex in one set joined to exactly one vertex of the other.

(III) The vertices of Σ could be partitioned into two equal sets each consisting of a disjoint union of paths of length 2, with each vertex of one set joined to exactly two vertices of the other.

(IV) The vertices of Σ could be partitioned into $r > 2$ equal sets, arranged in a cycle, with each set of the partition consisting of a disjoint union of paths of length 2, and with each vertex of Σ joined to one vertex before it and one after it in the cycle of the partition.

2. G Is Regular

In this section G is supposed to act regularly on Σ. Two theorems are proved: the first is a structure theorem and the second describes this case in terms of simple groups and the graphs described in Section 1.

Theorem 3. *Suppose that G acts regularly on Σ and N is a proper normal subgroup of G. Then either Σ is a graph of type I, II, III or IV in which N maps each set of the partition mentioned onto itself or the quotient graph Λ defined by N has valency 3, $G|N$ acts on it as a regular group of automorphisms and the factor subgraphs defined by HN have no edges.*

Proof. $D = \{a, b, c\}$. As $D = D^{-1}$ there are two possibilities for the orders of the members of D which can be described, without loss of generality, as $a^2 = 1$, $b^{-1} = c \neq b$ and $a^2 = b^2 = c^2 = 1$.

The different possibilities that can arise for $D \cap N$ need to be considered separately. Notice first that, as D generates G and $N \neq G$, D is not a subset of N.

Suppose that $D \cap N = \emptyset$. In this case the factor subgraphs defined by HN have no edges and the quotient graph has valency at most 3. (A proof of this latter fact can be found in [3 Theorem 5]). If the quotient graph Λ has valency smaller than 3 it is a path of length 2 or a cycle, because it must be connected. If it is a path of length 2 the graph is bipartite. Suppose that it is a cycle. As $D \cap N = \emptyset$, G/N contains an involution and, as $G|N$ acts regularly, has an even number of vertices: in this case also, Σ is bipartite.

Otherwise $D \cap N$ contains one or two members. For the rest of the proof the two possible types of sets that D can be must be considered separately.

Suppose that $D = \{a, b, c\}$ where $a^2 = 1, b^{-1} = c \neq b$ and that $D \cap N = \{a\}$. Because $D \cap N = \{a\}$, the factor subgraphs induced by N have valency one and are disjoint unions of paths of length 2. As D generates G, the quotient group $G|N$ is generated by b and $c = b^{-1}$ and is cyclic. The quotient graph Λ is a cycle and Λ is described by type (IV) of the first section.

Suppose that D is as in the last paragraph and that $D \cap N = \{b, c\}$. Because $D \cap N = \{b, c\}$, the factor subgraphs induced by N have valency two and are a disjoint union of cycles which must all be of the same length. As D generates G, $G = N \cup aN$ and the quotient graph defined by N is a path of length 2. Thus Σ is defined by type (II) of the last section.

Now suppose that $D = \{a, b, c\}$ where $a^2 = b^2 = c^2 = 1$ and that $D \cap N$ contains just one member. Without loss of generality it may be supposed that $D \cap N = \{a\}$. In this case the factor subgraphs defined by N have valency 1 and are a disjoint union of paths of length 2. The quotient graph is connected and as $NDN = DN = bN \cup cN$, the quotient graph is either a path of length 2, (if $bN = cN$) or a cycle (if $bN \neq cN$). Thus Σ is of type (III) or (IV).

Finally, suppose that D is as in the last paragraph and that $D \cap N$ contains two members. Without loss of generality it may be supposed that $D \cap N = \{a, b\}$. Then the factor subgraphs have valency 2 and are a disjoint union of cycles. $G|N = N \cup cN$ and the quotient graph defined by N is a path of length 2. Thus Σ is described by type (II) of the last section.

This completes the proof of Theorem 3.

The consequence of this theorem to be noted in building up the proof of the Theorem mentioned in the introduction is

THEOREM 4. *Suppose that G acts regularly on Σ and let N be a maximal proper normal subgroup of G (permitting the possibility that $N = 1$). Then either*
(1) *Σ is a graph of type I, II, III or IV and N is the subgroup of G containing those members of G which fix, as a whole, any one of the subgraphs into which Σ is partitioned in the descriptions of types I, II, III or IV. $G|N$ is cyclic of prime order, or*
(2) *the factor graph Λ has valency 3 and $G|N$ is a simple nonabelian group which acts regularly as a group of automorphisms of Λ.*

PROOF. Notice, first, that if $G|N$ is abelian then it is cyclic of prime order. The result then follows directly from Theorem 3.

3. X Has Two Orbits under H

The subgroup H which is the stabilizer of the vertex v acts as a permutation group on the set X of vertices adjacent to v. As such it is possible that it splits X into two orbits, one of length 1 and the other of length 2. That is the case dealt with in this section. The main structure theorem is:

THEOREM 5. *Suppose that, as a permutation group on X, H has two orbits. Let N be a normal subgroup of G. Then either*

(1) *Σ is a graph of type (I), or*
(2) *N acts regularly on Σ, or*

(3) *H has order 2, $G|N$ is a dihedral group and N is a subgroup of a normal subgroup of G which acts regularly on Σ, or*
(4) *N acts transitively on Σ and $H \cap N$ has the same orbits as a permutation group on X that H does, or*
(5) *HN is a normal subgroup of index 2 in G, Σ is a graph of type (II) or (III), HN is the stabilizer of each of the two sets into which Σ is partitioned and each member of $G - HN$ interchanges these two sets, or*
(6) *$H \cap N = 1$, the quotient graph Λ induced by HN has valency 3 and the factor subgraphs have no edges. The vertices v, x, y, z of Σ lie in four different vertices v^*, x^*, y^*, z^*, say, of Λ. The kernel of the representation of G on Λ is N, HN is the stabilizer of v^* and the orbits of HN on the set $\{x^*, y^*, z^*\}$ are those induced by the orbits of H on $\{x, y, z\}$.*

PROOF. Without loss of generality it may be supposed that the orbits of H on X are $\{x, y\}$ and $\{z\}$. By Proposition 2 (II) it follows that c lies in the normalizer of H, $HaH = HbH = aH \cup bH$, $D = HaH \cup cH$, G is generated by H, a and c, and $|H| = 2|H \cap aHa^{-1}|$.
As the following facts are needed throughout the proof they are separated as a Lemma

LEMMA.

(1) $$\frac{|HaH|}{|H|} = \frac{|H|}{|H \cap aHa^{-1}|};$$

(2) $$\frac{|HN|}{|HN \cap aHa^{-1}N|}\,\frac{|H \cap N|}{|H \cap aHa^{-1} \cap N|}\,\frac{|HN \cap aHa^{-1}N|}{|(H \cap aHa^{-1})N|} = \frac{|H|}{|H \cap aHa^{-1}|};$$

(3) one of the numbers

$$\frac{|HN|}{|HN \cap aHa^{-1}N|}, \qquad \frac{|H \cap N|}{|H \cap aHa^{-1} \cap N|} \quad \text{and} \quad \frac{|HN \cap aHa^{-1}N|}{|(H \cap aHa^{-1})N|}$$

is 2 and the other two are equal to 1.

PROOF. (1) It is a standard result of group theory that

$$|HaHa^{-1}| = \frac{|H|\,|aHa^{-1}|}{|H\,aHa^{-1}|}$$

See, for example, [2 page 8]. (1) follows because $|HaHa^{-1}| = |HaH|$ and $|aHa^{-1}| = |H|$.
(2) is proved using the identities

$$|HN| = \frac{|H|\,|N|}{|H \cap N|} \quad \text{and} \quad |(H \cap aHa^{-1})N| = \frac{|H \cap aHa^{-1}|\,|N|}{|H \cap aHa^{-1} \cap N|}.$$

(3) In each of the three quotients mentioned, the group in the denominator is a subgroup of the group in the numerator, and hence, by Lagrange's Theorem, each quotient is an integer. As their product is equal to $\frac{|H|}{|H \cap aHa^{-1}|}$ which has been shown to be equal to 2, this result is established.
This completes the proof of the Lemma and we return to the proof of the theorem.

Suppose, first, that $HN = G$. Then, also, $aHa^{-1}N = G$ and it follows from the Lemma that one of

$$\frac{|H \cap N|}{|H \cap aHa^{-1} \cap N|}, \qquad \frac{|HN \cap aHa^{-1}N|}{|(H \cap aHa^{-1})N|}$$

is equal to 2 and the other is equal to 1. If $|H \cap N| = |H \cap aHa^{-1} \cap N|$, then $a(H \cap N)a^{-1} = aHa^{-1} \cap N = H \cap N$. As G is generated by H, a and c, and c normalizes H, it follows that $H \cap N$ is a normal subgroup of G. As G is a permutation group in which H is the stabilizer of a point, $H \cap N = 1$ and in this case N is a regular subgroup of G, a conclusion which is in the Theorem. Alternatively,

$$\frac{|H \cap N|}{|H \cap aHa^{-1} \cap N|} = 2 \quad \text{and} \quad HN \cap aHa^{-1}N = (H \cap aHa^{-1})N.$$

Consider the action of N on Σ. As $HN = G$, N acts vertex-transitively. In N, the stabilizer of the vertex v is $H \cap N$ and, as $|H \cap N| = 2|H \cap aHa^{-1} \cap N| = 2|(H \cap N) \cap a(H \cap N)a^{-1}|$ it follows that the vertices of Σ adjacent to v fall into the orbits $\{x, y\}$ and $\{z\}$ under the action of $H \cap N$. This is conclusion (4) of the Theorem. This completes consideration of the case that $HN = G$.

Suppose next that HN is a normal subgroup of G, but $HN \neq G$.

As $a^{-1} \in HaH$, $Ha^{-1}H = HaH$ and $aHN = HNaHN = HNa^{-1}HN = a^{-1}HN$. Hence $(aHN)^2 = HN$ and either $a \in HN$ or aHN is an involution in $G|HN$. As $c^{-1} \in HcH$ and $c \in N(H)$, $(cH)^2$ and hence $(cHN)^2 = HN$.

As G is generated by H, a and c, but $G \neq HN$, either $a \notin HN$ or $c \notin HN$. If $a \notin HN$ and $c \notin HN$ then $D \cap HN = \varnothing$ and the factor subgraphs defined by HN have no edges. As $HNDHN = HNaHN \cup HNcHN = aHN \cup cHN$, the quotient graph Λ has valency 2. As G is generated by H, a and c it is connected. As $G|HN$ contains an element aHN of order 2, Λ has an even number of edges. Thus Σ is bipartite. Suppose that $a \in HN$ and $c \notin HN$. Then $D \cap HN = HaH$ and the factor subgraphs induced by HN have valency 2. Thus the factor subgraphs are all a disjoint union of cycles. As G is generated by H, a, c, G is also generated by HN, c. As $(cHN)^2 = HN$, $G = HN \cup cHN$ and the quotient graph induced by HN has just vertices, HN and cHN. As it is connected, it is a path of length 2. Finally, suppose that $c \in HN$ and $a \notin HN$. Then $D \cap HN = HcH = cH$ and the factor subgraphs defined by HN all have valency 1; they are thus a disjoint union of paths of length 2. As G is generated by H, a, c it is also generated by HN and a; then $G = HN \cup aHN$ and, as in the last case, the quotient graph induced by HN is a path of length 2.

All the outcomes in the last paragraph are among the conclusions of the theorem and the possibility that HN is a normal subgroup of G has been completely covered.

Suppose, for the rest of the proof, that HN is not a normal subgroup of G. As G is generated by H, a, and c, and c normalizes H, it must be that $aHNa^{-1} \neq HN$. Hence, by the Lemma at the beginning of this proof $|HN| = 2|HN \cap aHa^{-1}N|$, $H \cap N = H \cap aHa^{-1} \cap N$ and $HN \cap aHa^{-1}N = (H \cap aHa^{-1})N$.

As $H \cap N = H \cap aHa^{-1} \cap N$, $a(H \cap N)a^{-1} = H \cap N$. As H is normalized by H and c which, together with a, generate G, $H \cap N$ must be a normal subgroup of G. As G is a permutation group, transitive on the vertices of Σ and H is the stabilizer of one of these vertices it follows that $H \cap N = 1$.

Let Λ be the quotient graph of Σ induced by HN. Let K be the kernel of the representation of G as a group of automorphisms of Λ. As G is transitive and HN is the stabilizer of a point of Λ, K is the intersection of HN and all its conjugates. As N is a normal subgroup of G and $N \subseteq HN$, it follows that $N \subseteq K$. On the other hand, $K \subseteq HN$ so that $HK = HN$. As HN is not a normal subgroup of G, neither is HK. Now apply what has passed so far in this proof to K in place of N. As HK is not a normal subgroup of G, $|HK| = 2|HK \cap aHa^{-1}K|$, $H \cap K = H \cap aHa^{-1} \cap K$, and thus $H \cap K = 1$. Thus $N \subseteq K$, $HK = HN$, $H \cap N = H \cap K = 1$ and $N = K$. This show that $G|N$ acts as a group of automorphisms of Λ; because $H \cap N = 1$, the representation of H on Λ is faithful.

As c normalizes H, it also normalizes HN. As G is generated by H, a and c but HN is not a normal subgroup of G, a does not normalize HN. In particular, $a \notin HN$.

Suppose that $c \in HN$. Then $HNDHN - HN = HN(HaH \cup HcH)HN - HN = HNaHN = aHN \cup bHN$. Hence, the quotient graph Λ induced by HN has valency 2 if $aHN \neq bHN$ or 1 if $aHN = bHN$. As G is generated by HN and a, Λ is connected and hence is a cycle or a path of length 2. As $HN \cap D = HcH = cH$, each factor graph is a disjoint union of paths of length 2. Now $G|N$ acts faithfully as a group of automorphisms of Λ. As H also acts faithfully on Λ, $HN|N$ is not a normal subgroup of $G|N$. Hence Λ is a cycle of length greater than 2 and the automorphism group of Λ is dihedral. $G|N$ acts transitively on Λ and the stabilizer of a vertex is not trivial. Hence $G|N$ is the full automorphism group of Λ. Thus G contains a normal subgroup L such that $L|N$ is the cyclic subgroup of index 2 in $G|N$. Then L is a transitive normal subgroup of G and conclusion (3) of the Theorem is true.

Finally, suppose that $c \notin HN$. As c normalizes HN but a does not, $cHN \neq aHN$ and $cHN \neq bHN$. As a does not normalize HN, $aHN \neq bHN$. Hence $HNDHN - HN = aHN \cup bHN \cup cHN$ contains three cosets of HN. Thus the quotient graph Λ has valency 3. As $HNaHN = aHN \cup bHN$ and $HNcHN = cHN$, the orbits of $HN|N$ on the vertices adjacent to the one which it stabilizers have lengths 2 and 1. This situation is described by conclusion (6) of the Theorem. This completes the proof of Theorem 5.

This structure theorem has the following consequence:

THEOREM 6. *Suppose that Σ is not a group of type (I), (II) or (III).*

Let G_1 be a subgroup of G which is minimal among the subgroups of G which act transitively on the vertices of Σ and have the property that the stabilizer in them of each vertex has two orbits on the vertices adjacent to that vertex.

Let G_2 be a subgroup of G_1 which is maximal among those normal subgroups of G_1 which contain no member, except 1, *which fixes a vertex.*

Let H be the stabilizer of one of the vertices of Σ and let Λ be the quotient graph of Σ defined by $(H \cap G_2)G_2$.

Then either G_1 acts regularly on Σ or $G_1|G_2$ is a simple group which acts faithfully as a group of automorphisms of Λ and the stabilizer in $G_1|G_2$ of a vertex of Λ has orbits of length 1 *and* 2 *on the vertices adjacent to that vertex. Λ has valency 3.*

4. H ACTS TRANSITIVELY ON X

Another possibility that can occur is that the stabilizer in G of each vertex acts transitively on the vertices adjacent to that one. In these circumstances G is said to act symmetrically on Σ. Graphs like these having prime valency, rather than just three, are the subject of another paper [3] by the author and the following two theorems come from there.

THEOREM 7. *Suppose that G acts symmetrically on Σ. If N is a normal subgroup of G then either*

(1) *Σ is a bipartite graph, or*

(2) *N acts regularly on the vertices of Σ, or*

(3) *N acts symmetrically on Σ, or*

(4) *$H \cap N = 1$, the quotient graph Λ induced by HN has valency* 3, *the factor subgraphs have no edges, N is the kernel of the representation of G on Λ and $G|N$ acts symmetrically on Λ.*

THEOREM 8. *Suppose that Σ is not a bipartite graph and let G act symmetrically on Σ.*

Let G_1 be a subgroup of G which is minimal among the subgroups of G which act symmetrically on the vertices of Σ.

Let G_2 be a subgroup of G_1 which is maximal among the normal subgroups of G_1 having no element except the identity fixing a vertex of Σ.

Then either G_1 acts regularly on Σ or $G_1|G_2$ is a simple group which acts symmetrically as a group of automorphisms of the quotient graph Λ defined by $(G_1 \cap H)G_2$. Λ has valency 3.

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