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On dimension of inverse limits with upper semicontinuous set-valued bonding functions

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Abstract

We give results about the dimension of continua, obtained by combining inverse limits of inverse sequences of metric spaces and one-valued bonding maps with inverse limits of inverse sequences of metric spaces and upper semicontinuous set-valued bonding functions, by standard procedure introduced in [I. Banič, Continua with kernels, Houston J. Math. (2006), in press].
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1. Introduction

As very complicated continua may be presented as inverse limits of inverse sequences of very simple spaces and bonding maps, a representation of continua as inverse limits can be very useful when studying their properties. For example, the Knaster continuum [11, p. 205] is the inverse limit of an inverse sequence of unit intervals $[0, 1]$ and very simple bonding maps [13, p. 22] (for more examples see [2,6,10,13], etc.). One can construct new examples of continua by constructing the inverse sequences of well known continua by choosing appropriate bonding maps among them.

W.T. Ingram [9] and W.S. Mahavier [9,12] introduced the concept of inverse limits of inverse sequences of compact Hausdorff spaces with upper semicontinuous bonding functions. They gave several sets of sufficient conditions under which the inverse limit is a Hausdorff continuum, provided some interesting examples, and discussed their dimension.

The author [1] introduced the inverse limits of inverse sequences of unit intervals $[0, 1]$ and upper semicontinuous bonding functions $\tilde{f}_{n,t} : [0, 1] \to 2^{[0,1]}$, obtained from one-valued maps f_n on $[0, 1]$, such that the graph $\Gamma(\tilde{f}_{n,t})$ is the union of $\Gamma(f_n)$ and the segment $\{t\} \times [0, 1]$, $t \in [0, 1]$. For

$$K = \varprojlim\{[0, 1], f_n\}_{n=1}^{\infty},$$

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$$\tilde{K}_t = \varprojlim\{[0,1], \tilde{f}_{n,t}\}_{n=1}^{\infty}$$

he proved the following theorem.

Theorem 1.1. *Let f be a map from $[0,1]$ to $[0,1]$ and $K = \varprojlim\{[0,1], f_n\}_{n=1}^{\infty}$. Then $\tilde{K}_t$ has dimension either 1 or ∞ for all $t \in [0,1]$.*

In this article we continue the research of such inverse limits but instead of inverse sequences of unit intervals $[0,1]$, we consider the inverse sequences of arbitrary continua X_n and upper semicontinuous multi-valued bonding functions $\tilde{f}_n : X_{n+1} \to X_n$, obtained from one-valued maps f_n by the similar procedure as used in [1]: the graph $\Gamma(\tilde{f}_n)$ is the union of $\Gamma(f_n)$ and the product $A_{n+1} \times X_n$, for a closed subset A_{n+1} of X_{n+1}.

Our main goal is to prove the following generalization of Theorem 1.1:

Theorem 1.2. *Let X be a nondegenerate continuum, A a closed subset of X, $f : X \to X$ a map, and $K = \varprojlim\{X, f\}_{n=1}^{\infty}$. Then $\tilde{K}$ has dimension equal to either $\dim(X)$ or ∞.*

2. Definitions and notations

A continuum is a nonempty, compact and connected metric space. When referring to a space, the term degenerate is synonymous to being a one-point space, while the term nondegenerate means that the space consists of more than one point.

A map is a continuous function. Let f be a function from X onto X, then f^2 denotes the composition $f \circ f$, and inductively f^n denotes the composition $f^{n-1} \circ f$. We use $f^{-n}(Y)$ to denote $(f^n)^{-1}(Y) = \{x \in X \mid f^n(x) \in Y\}$.

Let $(X_1, d_1), (X_2, d_2), (X_3, d_3), \ldots$ be metric spaces, such that the metric d_n is bounded by 1 for all n. The metric we use on product $\prod_{n=1}^{\infty} X_n$ is given by

$$\mathrm{d}\big((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)\big) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}.$$

For details see [3, p. 259].

Given an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ of compact metric spaces X_n and maps $f_n : X_{n+1} \to X_n$, we define the inverse limit space $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ as the subspace of the product $\prod_{n=1}^{\infty} X_n$, which consists of all sequences $\{x_n\}_{n=1}^{\infty}$ such that $f_n(x_{n+1}) = x_n$ for every positive integer n.

Let X and Y be compact metric spaces, 2^Y be the set of all nonempty closed subsets of Y, and let $f : X \to 2^Y$ be a function. The function f is upper semicontinuous at a point $x \in X$ provided that for each open set V in Y containing $f(x)$, there is an open set U in X containing x such that if $y \in U$, then $f(y) \subseteq V$. The function f is upper semicontinuous, if it is upper semicontinuous at $x \in X$ for all $x \in X$. The graph $\Gamma(f)$ of f is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

For a given inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$, where all X_n are compact metric spaces and every f_n is an upper semicontinuous function $X_{n+1} \to 2^{X_n}$, the inverse limit $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is the subspace of the product $\prod_{n=1}^{\infty} X_n$, which consists of all sequences $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in f_n(x_{n+1})$ for every positive integer n.

We will use the following theorems:

Theorem 2.1. *(See [9, p. 120].) Suppose each of X and Y is a compact metric space and M is a subset of $X \times Y$ such that if $x \in X$ then there is a point $y \in Y$ such that $(x, y) \in M$. Then M is closed if and only if there is an upper semicontinuous function $f : X \to 2^Y$ such that $M = \Gamma(f)$.*

Theorem 2.2. *(See [9, p. 121].) Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of nonempty compact metric spaces and upper semicontinuous functions $f_n : X_{n+1} \to X_n$. Then $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is a nonempty compact metric space.*

Theorem 2.3. *(See [9, p. 124].) Assume that for each positive integer i, X_i is a continuum and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function such that for each $x \in X_{i+1}$, $f_i(x)$ is connected. Then $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is a continuum.*

For any compact metric space X we will use $\dim(X)$ for the topological (covering) dimension of X (for the definition see [3, p. 385], [14, p. 10] or [5]).

For the reader's convenience we list the following well-known results that will be used later:

Theorem 2.4. *(See [8, p. 19], [4, p. 261].) Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse system of compact metric spaces X_n and surjective bonding maps f_n. If for some nonnegative integer k, $\dim(X_n) \leqslant k$ for all n, then*

$$\dim\bigl(\varprojlim\{X_n, f_n\}_{n=1}^{\infty}\bigr) \leqslant k.$$

Theorem 2.5. *(See [14, p. 15].) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of compact metric spaces such that for some nonnegative integer k, $\dim(X_n) \leqslant k$ for all n. Then*

$$\dim\left(\bigcup_{n=1}^{\infty} X_n\right) \leqslant k.$$

Theorem 2.6. *(See [4, p. 33].) Let X be a metric space such that $\dim(X) = 0$. Then X is totally disconnected space.*

We shall also use the following theorem.

Theorem 2.7. *(See [7, p. 194].) Let X be a nonempty compact metric space with finite dimension and Y a separable metric space with dimension greater than 0. Then*

$$\dim(X \times Y) > \dim(X).$$

We shall need also the characterization of the Cantor set.

Theorem 2.8. *(See [15, p. 217].) A space X is homeomorphic to the Cantor set if and only if X is totally disconnected compact metric space without isolated points.*

The following results are obvious, but we state them for later use.

Theorem 2.9. *Let for each positive integer n, X_n be a totally disconnected metric space. Then $\prod_{n=1}^{\infty} X_n$ is also totally disconnected metric space.*

Theorem 2.10. *Let X and Y be compact metric spaces such that $\dim(X) = 0$. Then*

$$\dim(X \times Y) = \dim(X).$$

3. Preliminaries

Let X and Y be compact metric spaces and $f : X \to Y$ a map. For a closed subset $A \subseteq X$ we define the multi-valued function $\tilde{f} : X \to Y$ as

$$\tilde{f}(x) = \begin{cases} \{f(x)\}, & x \notin A, \\ Y, & x \in A. \end{cases}$$

Theorem 3.1. *Let X and Y be compact metric spaces, $f : X \to Y$ a map, and A be a closed subset of X. Then $\tilde{f}$ is an upper semicontinuous set-valued function from X to Y.*

Proof. The graph $\Gamma(f)$ and $A \times Y$ are closed subsets of $X \times Y$. Therefore $\Gamma(\tilde{f}) = (\Gamma(f)) \cup (A \times Y)$ is a closed subset of $X \times Y$, and so by Theorem 2.1, $\tilde{f}$ is an upper semicontinuous set-valued function from X to Y. □

Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n and maps $f_n : X_{n+1} \to X_n$ and let for each $n = 1, 2, 3, \dots, A_n \subseteq X_n$. If we use K to denote $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ then $\tilde{K}$ will denote the inverse limit $\varprojlim\{X_n, \tilde{f}_n\}_{n=1}^{\infty}$.

Theorem 3.2. *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of continua X_n and maps $f_n : X_{n+1} \to X_n$, and let for each $n = 1, 2, 3, \ldots, A_n$ be a closed subset of X_n. Then $\varprojlim\{X_n, \tilde{f}_n\}_{n=1}^{\infty}$ is a continuum.*

Proof. It follows from Theorem 3.1 that for each n, $\tilde{f}_n : X_{n+1} \to X_n$ is an upper semicontinuous set-valued bonding function. As $\tilde{f}_n(x)$ is connected for all n and $x \in X_{n+1}$, hence by Theorem 2.3, the inverse limit $\varprojlim\{X_n, \tilde{f}_n\}_{n=1}^{\infty}$ is a continuum. □

Definition 3.3. Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n and maps $f_n : X_{n+1} \to X_n$, and let for each $n = 1, 2, 3, \ldots, A_n$ be a closed subset of X_n. For each $m = -1, 0, 1, 2, \ldots$ we define D_m as the subspace of the product $\prod_{i=m+2}^{\infty} X_i$, consisting of all points $\underline{x}$, $\underline{x} = (x_1, x_2, x_3, \ldots)$, such that

(1) $x_1 \in A_{m+2}$;
(2) for each $i = 1, 2, 3, \ldots, x_i \in \tilde{f}_{i+m+1}(x_{i+1})$.

We call D_m the m-tree of the sequences $\{X_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$. If for certain $\{X_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ all the m-trees are equal, we call the m-tree simply the tree of the corresponding sequences and write D instead of D_m.

Remark 3.4. Obviously, $D_m = \pi(\pi_{m+2}^{-1}(\{A_{m+2}\}) \cap (\varprojlim\{X_n, \tilde{f}_n\}_{n=1}^{\infty}))$ for each m, where $\pi_m : \prod_{n=1}^{\infty} X_n \to X_m$ is the projection on mth factor, and $\pi : \prod_{n=1}^{\infty} X_n \to \prod_{n=m+2}^{\infty} X_n$ is defined by $\pi(x_1, x_2, x_3, \ldots) = (x_{m+2}, x_{m+3}, x_{m+4}, \ldots)$.

Example 3.5. Let for each n, $X_n = [0, 1]$, $f_n : [0, 1] \to [0, 1]$ be defined by $f_n(x) = 1 - x$, and $A_n = \{1\}$. In this case all the m-trees D_m of the sequences $\{X_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ are equal to the tree, shown on Fig. 1. D_m is the set of all $\underline{x} = (x_1, x_2, x_3, \ldots)$, where for each n, x_n lies on the nth level on Fig. 1, and there is an arrow pointing from x_{n+1} to x_n. For details see [1].

Example 3.6. Let $X_n = [0, 1]$, $f_n : [0, 1] \to [0, 1]$ be defined with $f_n(x) = 1 - x$, and

$$A_n = \begin{cases} \{1\}, & n = 2k - 1, \\ \{0\}, & n = 2k \end{cases}$$

for each n. In this case the m-trees D_m of the sequences $\{X_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ are shown on Fig. 2. The left-hand side m-tree of Fig. 2 belongs to even m and the right-hand side m-tree on Fig. 2 belongs to odd m. We will see later that all the m-trees of the sequences $\{X_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$, given in Example 3.6, are 0-dimensional, and this will be essential in proving that in this case $\dim(\varprojlim\{[0, 1], \tilde{f}_n\}_{n=1}^{\infty}) = 1$.

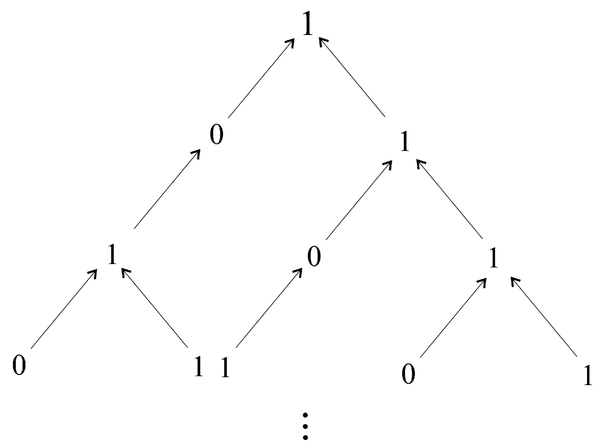

Fig. 1. The m-tree D_m from Example 3.5.

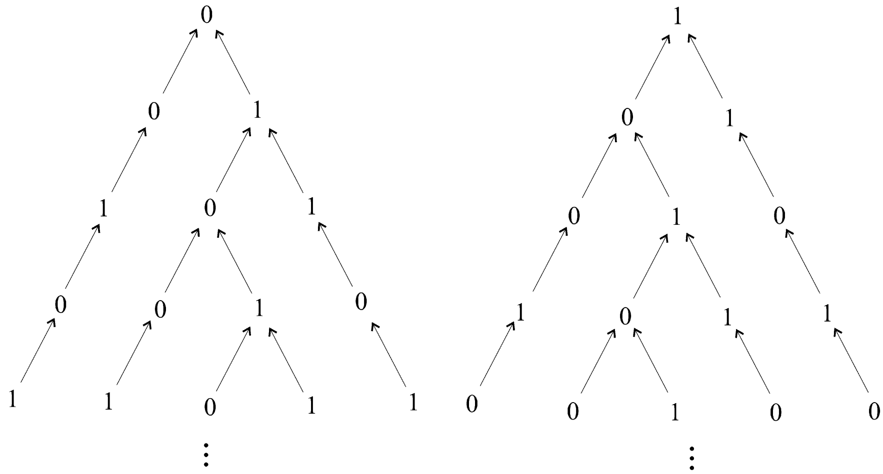

Fig. 2. The m-trees D_m from Example 3.6.

Theorem 3.7. *Let for each $n = 1, 2, 3, \ldots, X_n$ be a compact metric space, f_n a map from X_{n+1} to X_n, and A_n a closed subset of X_n. Then for each m, D_m is compact.*

Proof. It follows from Theorem 2.2, that $\varprojlim\{X_n, \tilde{f}_n\}_{n=1}^{\infty}$ is compact. As the projection π_{m+2} is continuous and A_{m+2} a closed subset of X_{m+2}, also $\pi_{m+2}^{-1}(\{A_{m+2}\})$ is compact. Therefore, as $\pi : \prod_{n=1}^{\infty} X_n \to \prod_{n=m+2}^{\infty} X_n$ is continuous, it follows from Remark 3.4, that D_m is compact for each m. $\square$

Definition 3.8. Let for each $n = 1, 2, 3, \ldots, X_n$ be a compact metric space, f_n a map from X_{n+1} to X_n, and A_n a closed subset of X_n. For all $m = 0, 1, 2, \ldots$ and each $\underline{a} \in D_m$ we define $L(\underline{a}, m)$ as follows.

$$L(\underline{a}, m) = \big\{\big(f_1\big(f_2\big(\ldots f_m(x)\ldots\big)\big), \ldots, f_{m-1}\big(f_m(x)\big), f_m(x), x, \underline{a}\big) \mid x \in X_{m+1}\big\}$$

for $m \geqslant 1$, and

$$L(\underline{a}, 0) = \big\{(x, \underline{a}) \mid x \in X_1\big\}.$$

Lemma 3.9. *For all $m \geqslant 0$ and all $\underline{a} \in D_m$, $L(\underline{a}, m)$ is homeomorphic to X_{m+1}.*

Proof. The map $f(x) = (f_1(f_2(\ldots f_m(x)\ldots)), \ldots, f_{m-1}(f_m(x)), f_m(x), x, \underline{a})$, $f : X_{m+1} \to L(\underline{a}, m)$, is a continuous bijection from a compact space onto a Hausdorff space, and is therefore a homeomorphism. $\square$

Lemma 3.10. *Let K be the inverse limit of the inverse sequence of compact metric spaces X_n and maps f_n from X_{n+1} to X_n, and for each n, let A_n be a closed subset of X_n. Then*

$$\tilde{K} = K \cup \left(\bigcup_{m=0}^{\infty}\left(\bigcup_{\underline{a} \in D_m} L(\underline{a}, m)\right)\right).$$

Proof. If $\underline{x} \in \tilde{K} \backslash K$, let $\ell = \min\{k \in \{1, 2, 3, \ldots\} \mid x_{k+1} \in A_{k+1}, x_k \neq f_k(x_{k+1})\}$ and $m = \ell - 1$. Then

$$\underline{x} = \big(f_1\big(f_2\big(\ldots f_{m-1}(f_m(x_\ell))\ldots\big)\big), f_2\big(\ldots f_{m-1}(f_m(x_\ell))\ldots\big), \ldots, f_m(x_\ell), x_\ell, \underline{a}\big),$$

where $\underline{a} = (x_{\ell+1}, x_{\ell+2}, x_{\ell+3}, \ldots)$. For all $i \geqslant 1$, $x_{\ell+i} \in \tilde{f}_{\ell+i}(x_{\ell+i+1})$, hence $\underline{a} \in D_m$ and $\underline{x} \in L(\underline{a}, m)$.

Also $K \subseteq \tilde{K}$, and it follows from the definition of $L(\underline{a}, m)$, that $L(\underline{a}, m) \subseteq \tilde{K}$ for all m and all $\underline{a} \in D_m$. $\square$

Theorem 3.11. *For every inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ of compact metric spaces and maps $f_n : X_{n+1} \to X_n$, $\bigcup_{\underline{a} \in D_m} L(\underline{a}, m)$ is homeomorphic to $D_m \times X_{m+1}$ for all integers $m = 0, 1, 2, \ldots$.*

Proof. It follows from Theorem 3.7 that D_m is compact, hence $D_m \times X_{m+1}$ is compact. Define $F : D_m \times X_{m+1} \to \bigcup_{\underline{a} \in D_m} L(\underline{a}, m)$ as

$$F(\underline{a}, x) = \big(f_1\big(f_2\big(\dots f_{m-1}(f_m(x))\dots\big)\big), f_2\big(\dots f_{m-1}(f_m(x))\dots\big), \dots, f_m(x), x, \underline{a}\big).$$

Clearly F is continuous bijection from the compact space $D_m \times X_{m+1}$ onto the Hausdorff space $\bigcup_{\underline{a} \in D_m} L(\underline{a}, m)$. Therefore F is a homeomorphism. □

Theorem 3.12. *Let K be inverse limit of the inverse sequence of compact metric spaces X and maps f_n from X to X and A_n a closed subset of X for each n. Then* $\dim(\tilde{K}) = \dim(D_m \times X)$ *for some m or* $\dim(\tilde{K}) = \infty$.

Proof. Let $M = \text{l.u.b.}\{\dim(D_n) \mid n = -1, 0, 1, 2, 3, \dots\}$. If $M = \infty$, then as $\tilde{K} = K \cup (\bigcup_{n=0}^{\infty}(\bigcup_{\underline{a} \in D_n}(L(\underline{a}, n)))$ and each $\bigcup_{\underline{a} \in D_n} L(\underline{a}, n)$ is homeomorphic to $D_n \times X$, we have $\dim(\tilde{K}) = \infty$. If $M < \infty$, there is a positive integer m such that $M = \dim(D_m)$. It follows from Theorem 2.4 that $\dim(K) \leqslant \dim(X)$. As $D_m \times X$ is homeomorphic to a subspace of $\tilde{K}$, therefore $\dim(D_m \times X) \leqslant \dim(\tilde{K})$. $\tilde{K}$ is the union of K and the space, which is the union of countable many compact metric spaces $K_n = \bigcup_{\underline{a} \in D_n} L(\underline{a}, n)$, which are homeomorphic to spaces $D_n \times X$. As $\dim(K), \dim(D_n \times X) \leqslant \dim(D_m \times X)$ for each n, hence $\dim(\tilde{K}) \leqslant \dim(D_m \times X)$. Therefore $\dim(\tilde{K}) = \dim(D_m \times X)$. □

Corollary 3.13. *Let K be the inverse limit of the inverse sequence of compact metric space X and maps f_n from X to X, and A_n a closed subset of X for each n. If* $\dim(D_n) = 0$ *for each n, then* $\dim(\tilde{K}) = \dim(X)$.

Proof. As $\dim(\tilde{K}) = \dim(D_m \times X)$ for all m, it follows from Theorem 2.10, that $\dim(\tilde{K}) = \dim(X)$.

Let us look back to Example 3.6. Recall the standard homeomorphism, defined by

$$(x_1, x_2, x_3, \dots) \mapsto \sum_{n=1}^{\infty} \frac{2x_n}{3^n},$$

mapping $\{0, 1\}^{\aleph_0}$ onto C, where $C \subset [0, 1]$ is the ternary Cantor set. Using its restriction on D_m, we see that both the m-trees from the Example 3.6 are homeomorphic to a countable closed subset of $[0, 1]$ with exactly one cluster point, therefore they both have dimension 0, and hence $\tilde{K}$ has dimension 1 (by Corollary 3.13, $\dim(\tilde{K}) = \dim([0, 1]) = 1$).

Note also that, if $A_n = \{1\}$ for each positive integer n, all the 0-dimensional m-trees are either degenerate or are homeomorphic to the Cantor set C (see [1]). But in Example 3.6, where for each n, $A_{2n} \neq A_{2n-1}$, all the m-trees are nondegenerate and none of them contains a Cantor set.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Before proving it, let us prove the following lemmas, which are essential for the proof of Theorem 1.2.

Lemma 4.1. *Let for each positive integer n, X_n be a compact metric space such that* $\dim(X_n) > 0$. *Then*

$$\dim\left(\prod_{n=1}^{\infty} X_n\right) = \infty.$$

Proof. First we show that for each n, $\dim(\prod_{i=1}^{n} X_i) \geqslant n$. As X_2 is compact and therefore separable metric space, it follows from Theorem 2.7 that $\dim(X_1 \times X_2) > \dim(X_1) > 0$. Therefore $\dim(X_1 \times X_2) \geqslant 2$. By induction on n, applying Theorem 2.7, we get $\dim(\prod_{i=1}^{n+1} X_i) > \dim(\prod_{i=1}^{n} X_i) \geqslant n$ and therefore $\dim(\prod_{i=1}^{n+1} X_i) \geqslant n + 1$. For each positive integer n, the finite product $\prod_{i=1}^{n} X_i$ may be embedded into the infinite product $\prod_{i=1}^{\infty} X_i$, hence $\dim(\prod_{i=1}^{\infty} X_i) \geqslant n$ for all positive integers n, and therefore $\dim(\prod_{i=1}^{\infty} X_i) = \infty$. □

Lemma 4.2. *Let X be a nondegenerate continuum, A a nonempty closed subset of X, and $f : X \to X$ a map. Then the tree of the sequences $\{X\}_{n=1}^{\infty}$, $\{f\}_{n=1}^{\infty}$ and $\{A\}_{n=1}^{\infty}$ is either a 0-dimensional or an ∞-dimensional compactum.*

Proof. Let D be the tree of the sequences $\{X\}_{n=1}^{\infty}$, $\{f\}_{n=1}^{\infty}$ and $\{A\}_{n=1}^{\infty}$. As for each $a, b \in A$ $a \in \tilde{f}(b)$, therefore $\prod_{n=1}^{\infty} A \subseteq D$.

If $\dim(A) > 0$, it follows from Lemma 4.1 that $\dim(\prod_{n=1}^{\infty} A) = \infty$. As $\prod_{n=1}^{\infty} A \subseteq D$, $\dim(D) = \infty$.

If $\dim(A) = 0$, only the next two cases are possible.

Case 1. For all positive integers n, $\dim(\tilde{f}^{-n}(A)) = 0$. By Theorem 2.6, $\tilde{f}^{-n}(A)$ is totally disconnected space for all n. In case 1 we will consider the next two subcases.

Subcase 1: A is a degenerate space and $f^{-1}(A) \subseteq A$. Since $\tilde{f}^{-1}(A) = A$, it follows that D is degenerate and so $\dim(D) = 0$.

Subcase 2: A is a nondegenerate space or $f^{-1}(A) \nsubseteq A$. We will show that in this subcase, D contains no isolated points, and is totally disconnected compact metric space, and is by Theorem 2.8 homeomorphic to the Cantor set.

(1) By Theorem 3.7, D is a compact metric space.
(2) It follows from the definition of D, that $D \subseteq \prod_{n=0}^{\infty} \tilde{f}^{-n}(A)$. As $\tilde{f}^{-n}(A)$ is totally disconnected for all n, it follows from Theorem 2.9 that $\prod_{n=0}^{\infty} \tilde{f}^{-n}(A)$ is also totally disconnected. As $D \subseteq \prod_{n=0}^{\infty} \tilde{f}^{-n}(A)$, therefore D is totally disconnected.
(3) Next we show that D has no isolated points. Let $\underline{a} = (a_1, a_2, a_3, \ldots) \in D$ and $\varepsilon > 0$. Take positive integer n such that $\sum_{i=n+1}^{\infty} \frac{\operatorname{diam}(X)}{2^i} < \varepsilon$. Take for $(b_1, b_2, b_3, \ldots, b_n) = (a_1, a_2, a_3, \ldots, a_n)$.

If A is nondegenerate, we may take $b_{n+1} \in A \setminus \{a_{n+1}\}$. For any $k \geqslant n+2$, we take any element from $\tilde{f}^{-1}(\{b_{k-1}\})$ for b_k. For $\underline{b} = (b_1, b_2, b_3, \ldots)$, $\underline{b} \in D$ and $\mathrm{d}(\underline{a}, \underline{b}) < \sum_{i=n+1}^{\infty} \frac{\operatorname{diam}(X)}{2^i} < \varepsilon$.

If A is degenerate and $f^{-1}(A) \nsubseteq A$, we consider the next two cases.

- If $a_n \in A$, then we take for b_{n+1} an element from $\tilde{f}^{-1}(A) = f^{-1}(A) \cup A$, which has at least two elements, different from a_{n+1}. For any $k \geqslant n+2$, we take any element from $\tilde{f}^{-1}(\{b_{k-1}\})$ for b_k. Clearly, if $\underline{b} = (b_1, b_2, b_3, \ldots)$, $\underline{b} \in D$ and $\mathrm{d}(\underline{a}, \underline{b}) < \sum_{i=n+1}^{\infty} \frac{\operatorname{diam}(X)}{2^i} < \varepsilon$.
- If $a_n \notin A$, then we take $b_{n+1} \in A$ and we choose b_{n+2} from $\tilde{f}^{-1}(A)$, different from a_{n+2}, as in previous case. For b_k, $k \geqslant n+3$, we can take any element from $\tilde{f}^{-1}(\{b_{k-1}\})$. Again for $\underline{b} = (b_1, b_2, b_3, \ldots)$, $\underline{b} \in D$ and $\mathrm{d}(\underline{a}, \underline{b}) < \sum_{i=n+1}^{\infty} \frac{\operatorname{diam}(X)}{2^i} < \varepsilon$.

Case 2. There is a positive integer n such that $\dim(\tilde{f}^{-n}(A)) \geqslant 1$. Now let H be defined as $H = \{(f^n(x_1), f^{n-1}(x_1), \ldots, f(x_1), x_1, f^n(x_2), f^{n-1}(x_2), \ldots, f(x_2), x_2, f^n(x_3), f^{n-1}(x_3), \ldots, f(x_3), x_3, \ldots) \mid x_1, x_2, x_3, \ldots \in \tilde{f}^{-n}(A)\}$. Clearly for all positive integers k, $f^n(x_k) \in \tilde{f}^n(A) \subseteq A$ and hence $\tilde{f}(f^n(x_{k+1})) = X$. Therefore $x_k \in \tilde{f}(f^n(x_{k+1}))$, and hence $H \subseteq D$. Furthermore H is homeomorphic to the product space $\prod_{n=1}^{\infty} \tilde{f}^{-n}(A)$ (note that the function $(x_1, x_2, x_3, \ldots) \mapsto (f^n(x_1), f^{n-1}(x_1), \ldots, f(x_1), x_1, f^n(x_2), f^{n-1}(x_2), \ldots, f(x_2), x_2, f^n(x_3), f^{n-1}(x_3), \ldots, f(x_3), x_3, \ldots)$ is a homeomorphism from $\prod_{n=1}^{\infty} \tilde{f}^{-n}(A)$ to H). As $\dim(f^{-n}(A)) \geqslant 1$, therefore by Theorem 4.1,

$$\dim(H) = \dim\left(\prod_{n=1}^{\infty} \tilde{f}^{-n}(A)\right) = \infty$$

and so $\dim(D) = \infty$. □

Finally we prove Theorem 1.2.

Proof. Let D be the tree of the sequences $\{X\}_{n=1}^{\infty}$, $\{f\}_{n=1}^{\infty}$ and $\{A\}_{n=1}^{\infty}$. By Lemma 3.10,

$$\tilde{K} = K \cup \left(\bigcup_{n=0}^{\infty}\left(\bigcup_{\underline{a} \in D} L(\underline{a}, n)\right)\right).$$

By Theorem 2.4, $\dim(K) \leqslant \dim(X)$. By Theorem 3.11, $\bigcup_{\underline{a} \in D} L(\underline{a}, n)$ is homeomorphic to $D \times X$ for all integers $n = 0, 1, 2, \ldots$. By Lemma 4.2, only one of the following is true:

(1) $\dim(D) = 0$.
(2) $\dim(D) = \infty$.

In the first case, when D is 0-dimensional, as $\bigcup_{\underline{a}\in D} L(\underline{a}, n)$ is homeomorphic to $D \times X$ (by Theorem 3.11), it follows from Theorem 2.10 that

$$\dim\left(\bigcup_{\underline{a}\in D} L(\underline{a}, n)\right) = \dim(X).$$

By Theorems 3.7 and 3.11, each $\bigcup_{\underline{a}\in D} L(\underline{a}, n)$ is compact, hence it follows from Theorem 2.5 that

$$\dim(\tilde{K}) = \dim\left(K \cup \left(\bigcup_{n=0}^{\infty}\left(\bigcup_{\underline{a}\in D} L(\underline{a}, n)\right)\right)\right) = \dim(X).$$

In the second case, when $\dim(D) = \infty$, as $D \subseteq \tilde{K}$, it follows that $\dim(\tilde{K}) = \infty$. □

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