Periodic solutions for a semi-ratio-dependent predator–prey system with functional responses

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Abstract

A class of nonautonomous semi-ratio-dependent predator–prey systems with functional responses is investigated. New sufficient conditions for the existence of a positive periodic solution are obtained. Some known results are generalized and improved.

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1. Introduction

Recently, Wang et al. [1] considered the following semi-ratio-dependent predator–prey systems with functional responses:

\[
\begin{align*}
\frac{dH(t)}{dt} &= r_1(t) - b_1(t)H(t)H - f(t, H)P, \\
\frac{dP(t)}{dt} &= r_2(t) - a_2(t)\frac{P}{H} P,
\end{align*}
\]

(1.1)

and they obtained the following theorem.

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Theorem A ([1, Theorem 3.3]). Assume that

(A1) \( r_i(t), \ a_i(t), \ b_1(t) \) are continuous functions on \( \mathbb{R} \) and are bounded below and above by positive constants;

(A2) \( f(t, H) \) is continuous with respect to the first variable and is differentiable with respect to the second variable, and \( f(t, 0) = 0, \frac{\partial f(t, H)}{\partial H} > 0 \) for any \( t \in \mathbb{R}, \ H > 0 \), and also \( \frac{\partial f(t, H)}{\partial H} \) is bounded with respect to \( t \);

(A3) there exists a constant \( C_0 > 0 \) such that \( f(t, H) \leq C_0H \) for any \( t \in \mathbb{R}, \ H > 0 \); and

(A4) the parameters in system (1.1) are \( \omega \)-periodic with respect to \( t \).

hold. Then, if

\[ \frac{C_0}{b_1} \exp[2(\tau_1 + \tau_2)\omega] < 1, \]  

(1.2)

the system (1.1) has at least one positive \( \omega \)-periodic solution.

In particular, if \( f(t, H) = a_1(t)H(t) \), then system (1.1) can be formulated as follows:

\[
\begin{align*}
\frac{dH(t)}{dr} &= (r_1(t) - a_1(t)P(t) - b_1(t)H(t))H(t), \\
\frac{dP(t)}{dr} &= \left( r_2(t) - a_2(t) \frac{P(t)}{H(t)} \right) P(t),
\end{align*}
\]

(1.3)

which is known as the Leslie–Gower predator–prey model [2–4].

In [5], Huo and Li obtained the following theorem.

Theorem B. Assume that

(B1) \( r_i(t), \ a_i(t), \ b_1(t) \) are continuous functions on \( \mathbb{R} \) and are bounded below and above by positive constants; and

(B2) the parameters in system (1.3) are \( \omega \)-periodic with respect to \( t \)

hold. Then, if

\[ r_2(t) > a_2(t), \]  

(1.4)

the system (1.3) has at least one positive \( \omega \)-periodic solution.

In this paper, we also explore the system (1.1) and obtain a new existence theorem by using a new estimation method, which improved on Theorem A by dropping assumption (A3) and condition (1.2). When our results were applied to the system (1.3), we also improved on Theorem B by dropping condition (1.4).

First, we need some preparations as follows.

Let \( X \) and \( Z \) be two Banach spaces. Consider an operator equation

\[ Lx = \lambda Nx, \ \lambda \in (0, 1), \]

where \( L : \text{Dom} \ L \cap X \to Z \) is a linear operator and \( \lambda \) is a parameter. Let \( P \) and \( Q \) denote two projectors such that

\[ P : X \cap \text{Dom} \ L \to \text{Ker} \ L \quad \text{and} \quad Q : Z \to Z/\text{Im} \ L. \]

In the following, we will use the following result of Mawhin [6, p. 40].
Lemma 1.1. Let $X$ and $Z$ be two Banach spaces and $L$ be a Fredholm mapping of index zero. Assume that $N: \Omega \to Z$ is $L$-compact on $\Omega$ with $\Omega$ open bounded in $X$. Furthermore, assume:

(a) for each $\lambda \in (0, 1)$, $x \in \partial \Omega \cap \text{Dom } L$,
   \[ Lx \neq \lambda Nx; \]
(b) for each $x \in \partial \Omega \cap \text{Ker } L$,
   \[QNx \neq 0, \]
   and
   \[\text{deg}\{QNx, \Omega \cap \text{Ker } L, 0\} \neq 0.\]

Then the equation $Lx = Nx$ has at least one solution in $\Omega \cap \text{Dom } L$.

For convenience, we shall introduce the notation

$$ u = \frac{1}{\omega} \int_0^\omega u(t) \, dt, $$

where $u$ is a periodic continuous function with period $\omega$.

Theorem 1.2. If $(A_1)$, $(A_2)$, and $(A_4)$ hold, then system (1.1) has at least one positive $\omega$-periodic solution.

Proof. Since

$$ H(t) = H(0) \exp \left\{ \int_0^t \left[ r_1(s) - \frac{f(s, H(s))}{H(s)} P(s) - b_1(s) H(s) \right] \, ds \right\}, $$
$$ P(t) = P(0) \exp \left\{ \int_0^t \left[ r_2(s) - a_2(s) \frac{P(s)}{H(s)} \right] \, ds \right\}, $$

the solution of system (1.1) remains positive for $t \geq 0$; we can let

$$ H(t) = \exp[x_1(t)], \quad P(t) = \exp[x_2(t)] \quad (1.5) $$

and derive that

$$ \frac{dx_1(t)}{dr} = r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)], $$
$$ \frac{dx_2(t)}{dr} = r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]}. \quad (1.6) $$

In order to use Lemma 1.1 on system (1.1), we take

$$ X = Z = \{x(t) = (x_1(t), x_2(t))^T \in C(R, R^2) : x(t + \omega) = x(t)\} $$

and use the notation

$$ \|x\| = \|(x_1(t), x_2(t))^T\| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)|. $$

Then $X$ and $Z$ are Banach spaces when they are endowed with the norms $\| \cdot \|$.
Set
\[ N_x = \begin{bmatrix} r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)] \\ r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \end{bmatrix}, \]

and
\[ Lx = x', \quad Px = \frac{1}{\omega} \int_0^\omega x(t) dt, \quad x \in X, \quad Qz = \frac{1}{\omega} \int_0^\omega z(t) dt, \quad z \in Z. \]

Evidently, \( \ker L = \{ x \mid x \in X, x = R^2 \} \), \( \text{Im} L = \{ z \mid z \in Z, \int_0^\omega z(t) dt = 0 \} \) is closed in \( Z \), and \( \dim \ker L = \omega \dim \text{Im} L = 2 \). Hence, \( L \) is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to \( L \)) \( K_p : \text{Im} L \to \ker P \cap \text{dom} L \) has the form
\[ K_p(z) = \int_t^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_t^t z(s) ds dt. \]

Thus
\[ QN_x = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \left[ r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)] \right] dt \\ \frac{1}{\omega} \int_0^\omega \left[ r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right] dt \end{bmatrix}, \]

and
\[ K_p(I - Q)N = \begin{bmatrix} \int_t^t \left[ r_1(s) - \frac{f(s, \exp[x_1(s)])}{\exp[x_1(s)]} \exp[x_2(s)] - b_1(s) \exp[x_1(s)] \right] ds \\ \int_t^t \left[ r_2(s) - a_2(s) \frac{\exp[x_2(s)]}{\exp[x_1(s)]} \right] ds \end{bmatrix} \]
\[ - \frac{1}{\omega} \int_0^\omega \int_t^t \left[ r_1(s) - \frac{f(s, \exp[x_1(s)])}{\exp[x_1(s)]} \exp[x_2(s)] - b_1(s) \exp[x_1(s)] \right] ds dt \\
- \frac{1}{\omega} \int_0^\omega \int_t^t \left[ r_2(s) - a_2(s) \frac{\exp[x_2(s)]}{\exp[x_1(s)]} \right] ds dt \\
- \left[ \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)] \right] dt \right] \\
- \left[ \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right] dt \right]. \]

Clearly, \( QN \) and \( K_p(I - Q)N \) are continuous and, moreover, \( QN(\Omega), K_p(I - Q)N(\Omega) \) are relatively compact for any open bounded set \( \Omega \subset X \). Hence, \( N \) is \( L \)-compact on \( \Omega \); here \( \Omega \) is any open bounded set in \( X \).
By (1.8) and (1.12), we obtain
\[ (1.7) \] over the interval
Lemma 1.1. Corresponding to equation \( Lx = \lambda Nx, \lambda \in (0, 1) \), we have
\[
\begin{align*}
\begin{cases} 
  x_1'(t) = \lambda \left[ r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)] \right], \\
  x_2'(t) = \lambda \left[ r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right].
\end{cases}
\end{align*}
\]
Suppose that \( x(t) = (x_1, x_2) \in X \) is a solution of system (1.7) for a certain \( \lambda \in (0, 1) \). By integrating (1.7) over the interval \([0, \omega]\), we obtain
\[
\begin{align*}
\int_0^\omega \left[ r_1(t) - \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] - b_1(t) \exp[x_1(t)] \right] dt = 0, \\
\int_0^\omega \left[ r_2(t) - a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right] dt = 0.
\end{align*}
\]
Hence,
\[
\int_0^\omega \left[ \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] + b_1(t) \exp[x_1(t)] \right] dt = \overline{\tau}_1 \omega, \tag{1.8}
\]
and
\[
\int_0^\omega \left[ a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right] dt = \overline{\tau}_2 \omega. \tag{1.9}
\]
From (1.7)–(1.9), we obtain
\[
\int_0^\omega |x_1'(t)| dt < \int_0^\omega \left[ \frac{f(t, \exp[x_1(t)])}{\exp[x_1(t)]} \exp[x_2(t)] + b_1(t) \exp[x_1(t)] \right] dt + \int_0^\omega |r_1(t)| dt = 2\overline{\tau}_1 \omega, \tag{1.10}
\]
and
\[
\int_0^\omega |x_2'(t)| dt < \int_0^\omega \left[ a_2(t) \frac{\exp[x_2(t)]}{\exp[x_1(t)]} \right] dt + \overline{\tau}_2 \omega = 2\overline{\tau}_2 \omega. \tag{1.11}
\]
Note that \((x_1(t), x_2(t))^T \in X\); then there exists \( \xi_i, \eta_i \in [0, \omega], i = 1, 2 \), such that
\[
\begin{align*}
x_i(\xi_i) &= \min_{t \in [0, \omega]} x_i(t), \quad x_i(\eta_i) = \max_{t \in [0, \omega]} x_i(t), \quad i = 1, 2.
\end{align*}
\]
By (1.8) and (1.12), we obtain
\[
\overline{\tau}_1 \omega \geq \overline{\tau}_1 \omega \exp[x_1(\xi_1)], \quad x_1(\xi_1) \leq \ln \left\{ \frac{\overline{\tau}_1}{b_1} \right\},
\]
and
\[
x_1(t) \leq x_1(\xi_1) + \int_0^\omega |x_1'(t)| dt < \ln \left\{ \frac{\overline{\tau}_1}{b_1} \right\} + 2\overline{\tau}_1 \omega. \tag{1.13}
\]
In addition, from (1.12) and system (1.7), we obtain
\[
r_2(\xi_2) - a_2(\xi_2) \frac{\exp[x_2(\xi_2)]}{\exp[x_1(\xi_2)]} = 0,
\]
together with \(x_2(t) > 0\) and (1.12), we have
\[
r_2(\xi_2) - a_2(\xi_2) \frac{1}{\exp\{x_1(\eta_1)\}} \geq 0,
\]
and
\[
x_1(\eta_1) \geq \ln \left( \frac{a_2(\xi_2)}{r_2(\xi_2)} \right) \geq \min_{t \in [0, \infty]} \left\{ \ln \left( \frac{a_2(t)}{r_2(t)} \right) \right\}. \tag{1.14}
\]
Then
\[
x_1(t) \geq x_1(\eta_1) - \int_0^\omega |x_1'(t)| dt \geq \min_{t \in [0, \infty]} \left\{ \ln \left( \frac{a_2(t)}{r_2(t)} \right) \right\} - 2r_2\omega. \tag{1.15}
\]
It follows from (1.13) and (1.15) that
\[
\max_{t \in [0, \omega]} |x_1(t)| \leq \max \left\{ \ln \left( \frac{\mathcal{P}_1}{a_2} \right) + 2\mathcal{P}\omega, \ln \left( \frac{a_2(t)}{r_2(t)} \right) - 2r_2\omega \right\} \coloneqq M_1. \tag{1.16}
\]
In view of (1.9) and (1.12), we have
\[
\mathcal{P}_2\omega \leq \mathcal{A}_2\omega \exp\{x_2(\eta_2)\},
\]
that is
\[
x_2(\eta_2) \geq \ln \left( \frac{\mathcal{P}_2}{\mathcal{A}_2} \right).
\]
Then
\[
x_2(t) \geq x_2(\eta_2) - \int_0^\omega |x_2'(t)| dt < \ln \left( \frac{\mathcal{P}_2}{\mathcal{A}_2} \right) - 2\mathcal{P}_2\omega. \tag{1.17}
\]
By virtue of (1.8), (1.13), and (1.12), we obtain that
\[
\mathcal{P}_1\omega \geq \frac{f(t, 1)}{\exp\{M_1\}} \omega \exp\{x_2(\xi_2)\},
\]
and so
\[
x_2(\xi_2) \leq \ln \left( \frac{\mathcal{P}_1 \exp\{M_1\}}{f(t, 1)} \right).
\]
Then
\[
x_2(t) \leq x_2(\xi_2) + \int_0^\omega |x_2'(t)| dt < \ln \left( \frac{\mathcal{P}_1 \exp\{M_1\}}{f(t, 1)} \right) + 2\mathcal{P}_2\omega. \tag{1.18}
\]
It follows from (1.17) and (1.18) that
\[
\max_{t \in [0, \omega]} |x_2(t)| \leq \max \left\{ \ln \left( \frac{\mathcal{P}_1}{\mathcal{A}_2} \right) - 2\mathcal{P}_2\omega, \ln \left( \frac{\mathcal{P}_1 \exp\{M_1\}}{f(t, 1)} \right) + 2\mathcal{P}_2\omega \right\} \coloneqq M_2.
Clearly, $M_i$, $i = 1, 2$, are independent of $\lambda$. By assumption $(A_2)$, it is easy to show that the system of algebraic equations
\[
\begin{cases}
  r_1 - \frac{f(v_1)}{v_1} v_2 - b_1 v_1 = 0, \\
  r_2 - \frac{a_2}{v_1} v_2 = 0,
\end{cases}
\]
has a unique solution $(v^*_1, v^*_2)^T \in \text{int} \mathbb{R}^2$ with $v^*_i > 0$, $i = 1, 2$. Use the notation $M = M_1 + M_2 + M_3$, where $M_3 > 0$ is taken sufficiently large that
\[\|\ln\{v^*_1\}, \ln\{v^*_2\}\| = |\ln\{v^*_1\}| + |\ln\{v^*_2\}| < M_3,\]
and define
\[\Omega = \{x(t) \in X : \|x\| < M\}.
\]
It is clear that $\Omega$ satisfies condition (a) of the Lemma 1.1. When
\[x = (x_1, x_2)^T \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap \mathbb{R}^2,
\]
x is a constant vector in $\mathbb{R}^2$ with $\|x\| = M$. Then
\[QN_x = \begin{bmatrix}
  r_1 - \frac{f(\exp\{x_1\})}{\exp\{x_1\}} \exp\{x_2\} - b_1 \exp\{x_1\} \\
  r_2 - \frac{a_2}{\exp\{x_1\}} \exp\{x_2\}
\end{bmatrix} \neq 0.
\]
Furthermore, in view of the assumption in Theorem 1.2, it can easily be seen that
\[\text{deg}\{QN_x, \Omega \cap \text{Ker } L, 0\} \neq 0.
\]
Now we know that $\Omega$ verifies all the requirements of Lemma 1.1 and thus system (1.6) has at least one $\omega$-periodic solution. By means of Eq. (1.5), we derive that system (1.1) has at least one positive $\omega$-periodic solution. The proof is complete. □

Remark 1.3. Theorem 1.2 remains valid if some terms are replaced by terms with discrete time delays, distributed delays (finite or infinite), or state-dependent delays.

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