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A note on the topological transversality theorem for acyclic maps

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Abstract

A new topological transversality theorem is presented for acyclic maps. The analysis relies on Urysohn's Lemma and the fact that the unit sphere is contractible in infinite dimensional normed linear spaces.

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1. Introduction

This work establishes a topological transversality theorem of Granas type [5] for multivalued acyclic maps. The proof differs from that given for Kututani maps [5,6] and relies on the fact that in a infinite dimensional normed linear space there exists a retraction from the unit ball to the unit sphere [1].

For the remainder of this section we look at the results in [1,3]. Let $E = (E, \|.\|)$ be an infinite dimensional normed linear space with $B = \{x \in E : \|x\| < 1\}$ and $S = \{x \in E : \|x\| = 1\}$. From [1,3] we know that there exists a Lipschitz (Lipschitz constant k_0 say) retraction r from $\overline{B}$ onto S. Next fix $R > 0$ and let $B_R = \{x \in E : \|x\| < R\}$ and $S_R = \{x \in E : \|x\| = R\}$. Also let

$$r_1(x) = \frac{x}{R} \qquad \text{and} \qquad r_2(x) = Rx,$$

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so $r_1 : \overline{B_R} \to \overline{B}$ and $r_2 : S \to S_R$. It is easy to check that $r_R = r_2 r r_1 : \overline{B_R} \to S_R$ is a Lipschitz retraction (Lipschitz constant k_0) from $\overline{B_R}$ onto S_R. Now let U be an open convex subset of E with $0 \in U$. Then there exists $R > 0$ with $B_R \subseteq U$. Let

$$r_3(x) = \frac{x}{\max\{1, \mu_1(x)\}}, x \in E \qquad \text{and} \qquad r_4(x) = \frac{x}{\mu_2(x)}, x \in E\backslash\{0\}$$

where μ_1 is the Minkowski functional on $\overline{B_R}$ and μ_2 is the Minkowski functional on $\overline{U}$. Notice $r_3 : \overline{U} \to \overline{B_R}$ and $r_4 : S_R \to \partial U$. Then $r_4 r_R r_3 : \overline{U} \to \partial U$ is a continuous retraction from $\overline{U}$ onto ∂U.

2. Topological transversality

Let E be an infinite dimensional normed linear space and U an open convex subset of E with $0 \in U$.

Definition 2.1. We let $F \in M(\overline{U}, E)$ denote the set of all upper semicontinuous compact maps $F : \overline{U} \to AC(E)$; here $AC(E)$ denotes the family of nonempty, compact, acyclic [4] subsets of E.

Definition 2.2. We let $F \in M_{\partial U}(\overline{U}, E)$ if $F \in M(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$.

Definition 2.3. A map $F \in M_{\partial U}(\overline{U}, E)$ is essential in $M_{\partial U}(\overline{U}, E)$ if for every $G \in M_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$. Otherwise F is inessential in $M_{\partial U}(\overline{U}, E)$.

Definition 2.4. $F, G \in M_{\partial U}(\overline{U}, E)$ are homotopic in $M_{\partial U}(\overline{U}, E)$, written $F \cong G$ in $M_{\partial U}(\overline{U}, E)$, if there exists an upper semicontinuous compact map $N : \overline{U} \times [0, 1] \to AC(E)$ such that $N_t(u) = N(u, t) : \overline{U} \to AC(E)$ belongs to $M_{\partial U}(\overline{U}, E)$ for each $t \in [0, 1]$ and $N_0 = F$ with $N_1 = G$.

Remark 2.1. Notice that $\cong$ is an equivalence relation in $M_{\partial U}(\overline{U}, E)$.

Theorem 2.1. *Let E be an infinite dimensional normed linear space and U an open convex subset of E with $0 \in U$. Suppose that $F \in M_{\partial U}(\overline{U}, E)$. Then the following conditions are equivalent:*

(i) *F is inessential in $M_{\partial U}(\overline{U}, E)$;*
(ii) *there exists a map $G \in M_{\partial U}(\overline{U}, E)$ with $x \notin G(x)$ for $x \in \overline{U}$ and $F \cong G$ in $M_{\partial U}(\overline{U}, E)$.*

Proof. To show that (i) implies (ii) let $G \in M_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ and $x \notin Gx$ for $x \in \overline{U}$. From Section 1 we know there exists a continuous retraction $r : \overline{U} \to \partial U$. Let the map $F^\star$ be given by $F^\star(x) = F(r(x))$ for $x \in \overline{U}$. Of course $F^\star(x) = G(r(x))$ for $x \in \overline{U}$ since $G|_{\partial U} = F|_{\partial U}$. With

$$H(x, \lambda) = G(2\lambda r(x) + (1 - 2\lambda)x) = G \circ j(x, \lambda) \qquad \text{for } (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

(here $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ is given by $j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x$) it is easy to see that

$$G \cong F^\star \text{ in } M_{\partial U}(\overline{U}, E); \tag{2.1}$$

notice that if there exists $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ with $x \in H_\lambda(x)$ then since $r(x) = x$ we have $x \in G(2\lambda x + (1 - 2\lambda)x) = G(x)$, a contradiction. Similarly with

$$Q(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x) \qquad \text{for } (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$F^\star \cong F \text{ in } M_{\partial U}(\overline{U}, E). \tag{2.2}$$

Combining (2.1) and (2.2) gives $G \cong F$ in $M_{\partial U}(\overline{U}, E)$.

We next show that (ii) implies (i). Let $N : \overline{U} \times [0,1] \to AC(E)$ be an upper semicontinuous, compact map with $N_t \in M_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$ and $N_1 = F$ with $N_0 = G$. Let

$$B = \{x \in \overline{U} : x \in N(x,t) \quad \text{for some } t \in [0,1]\}.$$

If $B = \emptyset$ then in particular $x \notin N(x,1)$ for $x \in \overline{U}$ so F is inessential in $M_{\partial U}(\overline{U}, E)$. So it remains to consider the case when $B \neq \emptyset$. Clearly B is closed (and in fact compact). Also since $B \cap \partial U = \emptyset$ there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Define a map $J : \overline{U} \to AC(E)$ by $J(x) = N(x, \mu(x))$. It is clear that J is an upper semicontinuous, compact map. Also $J|_{\partial U} = F|_{\partial U}$ since if $x \in \partial U$ then $J(x) = N(x,1) = F(x)$. In addition note that $x \notin J(x)$ for $x \in \overline{U}$ since if $x \in J(x)$ for some $x \in \overline{U}$ then $x \in B$ and so $\mu(x) = 0$, i.e. $x \in N(x,0) = G(x)$, a contradiction. Thus $J \in M_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $x \notin J(x)$ for $x \in \overline{U}$. As a result F is inessential in $M_{\partial U}(\overline{U}, E)$. □

Theorem 2.1 immediately yields the following continuation theorem.

Theorem 2.2. *Let E be an infinite dimensional normed linear space and U an open convex subset of E with $0 \in U$. Suppose that F and G are two maps in $M_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $M_{\partial U}(\overline{U}, E)$. Then F is essential in $M_{\partial U}(\overline{U}, E)$ if and only if G is essential in $M_{\partial U}(\overline{U}, E)$.*

To complete our discussion we now supply an example of an essential map (this is called a normalization property).

Theorem 2.3. *Let E be an infinite dimensional normed linear space and U an open convex subset of E with $0 \in U$. Then the zero map is essential in $M_{\partial U}(\overline{U}, E)$.*

Proof. Let $G : \overline{U} \to AC(E)$ be a map in $M_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in G(x)$. Let

$$J(x) = \begin{cases} G(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Clearly $J : E \to AC(E)$ is an upper semicontinuous, compact map. Now [4, p. 161] guarantees that J has a fixed point $x \in E$. In fact $x \in U$ since $0 \in U$. Hence $x \in G(x)$ and we are finished. □

Remark 2.2. It is also possible to combine the homotopy and normalization properties to obtain a Leray–Schauder alternative [5–8].

Next we discuss maps with values in a cone. Let $E = (E, \|.\|)$ be a normed linear space (not necessarily infinite dimensional) and let $C \subseteq E$ be a cone (i.e. C is a closed, convex, invariant under multiplication by nonnegative real numbers and $C \cap (-C) = \{0\}$). Fix $R > 0$ and let

$$B_R = \{x \in C : \|x\| < R\} \qquad \text{and} \qquad S_R = \{x \in C : \|x\| = R\}.$$

Definition 2.5. We let $F \in CM(\overline{B_R}, C)$ denote the set of all upper semicontinuous compact maps $F : \overline{B_R} \to AC(C)$.

Definition 2.6. We let $F \in CM_{S_R}(\overline{B_R}, C)$ if $F \in CM(\overline{B_R}, C)$ with $x \notin F(x)$ for $x \in S_R$.

Definition 2.7. A map $F \in CM_{S_R}(\overline{B_R}, C)$ is essential in $CM_{S_R}(\overline{B_R}, C)$ if for every $G \in CM_{S_R}(\overline{B_R}, C)$ with $G|_{S_R} = F|_{S_R}$ there exists $x \in B_R$ with $x \in G(x)$. Otherwise F is inessential in $CM_{S_R}(\overline{B_R}, C)$.

Definition 2.8. $F, G \in CM_{S_R}(\overline{B_R}, C)$ are homotopic in $CM_{S_R}(\overline{B_R}, C)$, written $F \cong G$ in $CM_{S_R}(\overline{B_R}, C)$, if there exists an upper semicontinuous compact map $N : \overline{B_R} \times [0, 1] \to AC(C)$ such that $N_t(u) = N(u, t) : \overline{B_R} \to AC(C)$ belongs to $CM_{S_R}(\overline{B_R}, C)$ for each $t \in [0, 1]$ and $N_0 = F$ with $N_1 = G$.

Essentially the same reasoning as in Theorem 2.1 (once one realizes that there exists a continuous retraction $r : \overline{B_R} \to S_R$ (see [2])) establishes the next result.

Theorem 2.4. *Let E be a normed linear space, $C \subseteq E$ a cone and $R > 0$. Suppose F and G are two maps in $CM_{S_R}(\overline{B_R}, C)$ with $F \cong G$ in $CM_{S_R}(\overline{B_R}, C)$. Then F is essential in $CM_{S_R}(\overline{B_R}, C)$ if and only if G is essential in $CM_{S_R}(\overline{B_R}, C)$.*

Remark 2.3. The analogue of Theorem 2.3 is also immediate in this case.

3. Generalizations

In this section we generalize the topological transversality theorem of Section 2. We discuss in particular a subclass of the $\mathcal{U}_c^k$ maps of Park [7]. Let X and Y be Hausdorff topological vector spaces. Recall that a polytope P in X is any convex hull of a nonempty finite subset of X. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (the nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single-valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii) for any polytope P, $F \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 3.1. $F \in \mathcal{U}_c^k(X, Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall that $\mathcal{U}_c^k$ is closed under compositions. In this section we will consider a subclass $\mathcal{A}$ of the $\mathcal{U}_c^k$ maps. The following condition will be assumed throughout this section:

$$\begin{cases} \text{for Hausdorff topological spaces } X_1, X_2 \text{ and } X_3, \\ \text{if } F \in \mathcal{A}(X_1, X_3) \text{ and } f \in \mathcal{C}(X_2, X_1), \\ \text{then } F \circ f \in \mathcal{A}(X_2, X_3). \end{cases} \tag{3.1}$$

In this section X is an infinite dimensional normed linear space, Y a topological vector space and U an open convex subset of X with $0 \in U$. Also $L : dom\ L \subseteq X \to Y$ will be a linear (not necessarily continuous) single-valued map; here $dom\ L$ is a vector subspace of X. Finally $T : X \to Y$ will be a linear, continuous single-valued map with $L + T : dom\ L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(X, Y)$.

A multivalued map $F : \overline{U} \to 2^Y$ is said to be (L, T) upper semicontinuous if $(L + T)^{-1}F : \overline{U} \to K(X)$ is an upper semicontinuous map; here $K(X)$ denotes the family of nonempty, compact subsets of X. $F : \overline{U} \to 2^Y$ is said to be (L, T) compact if $(L + T)^{-1}F : \overline{U} \to K(X)$ is a compact map.

Definition 3.2. We let $F \in D(\overline{U}, Y; L, T)$ if $(L+T)^{-1}F \in \mathcal{A}(\overline{U}, X)$ and $F : \overline{U} \to 2^Y$ is an (L, T) upper semicontinuous, (L, T) compact map.

Definition 3.3. $D_{\partial U}(\overline{U}, Y; L, T)$ denotes the maps $F \in D(\overline{U}, Y; L, T)$ with $Lx \notin F(x)$ for $x \in \partial U \cap dom\ L$.

Definition 3.4. A map $F \in D_{\partial U}(\overline{U}, Y; L, T)$ is essential in $D_{\partial U}(\overline{U}, Y; L, T)$ if for every map $G \in D_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ we have that there exists $x \in \overline{U} \cap dom\ L$ with $Lx \in G(x)$. Otherwise F is inessential in $D_{\partial U}(\overline{U}, Y; L, T)$, i.e. there exists $G \in D_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and $Lx \notin G(x)$ for $x \in \overline{U} \cap dom\ L$.

Definition 3.5. Two maps $F, G \in D_{\partial U}(\overline{U}, Y; L, T)$ are homotopic in $D_{\partial U}(\overline{U}, Y; L, T)$, written $F \cong G$ in $D_{\partial U}(\overline{U}, Y; T, T)$, if there exists an (L, T) upper semicontinuous, (L, T) compact mapping $N : \overline{U} \times [0, 1] \to 2^Y$ such that $N_t(u) = N(u, t) : \overline{U} \to 2^Y$ belongs to $D_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0, 1]$ and $N_0 = F$ with $N_1 = G$.

The following condition will be assumed throughout this section:

$$\cong \text{ is an equivalence relation in } D_{\partial U}(\overline{U}, Y; L, T). \tag{3.2}$$

Theorem 3.1. *Let X, Y, U, L and T be as above and assume that (3.1) and (3.2) hold. Suppose $F \in D_{\partial U}(\overline{U}, Y; L, T)$. Then the following conditions are equivalent:*

(i) *F is inessential in $D_{\partial U}(\overline{U}, Y; L, T)$;*
(ii) *there exists a map $G \in D_{\partial U}(\overline{U}, Y; L, T)$ with $Lx \notin G(x)$ for $x \in \overline{U} \cap dom\ L$ and $F \cong G$ in $D_{\partial U}(\overline{U}, Y; L, T)$.*

Proof. To show that (i) implies (ii) let $G \in D_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and $Lx \notin G(x)$ for $x \in \overline{U} \cap dom\ L$. Also let r be as in Theorem 2.1 and $F^\star(x) = F(r(x)) = G(r(x))$ for $x \in \overline{U}$. Let

$$H(x, \lambda) = G(2\lambda r(x) + (1 - 2\lambda)x) = G \circ j(x, \lambda) \qquad \text{for } (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

where $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ is given by $j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x$. Clearly H is an (L, T) upper semicontinuous, (L, T) compact map. In addition, assumption (3.1) guarantees that $(L+T)^{-1}H \in \mathcal{A}\left(\overline{U} \times \left[0, \frac{1}{2}\right], X\right)$. Now if there exists $x \in \partial U \cap dom\ L$ and $\lambda \in \left[0, \frac{1}{2}\right]$ with $Lx \in H_\lambda(x)$ then since $r(x) = x$ we have $Lx \in Gx$, a contradiction. Consequently

$$G \cong F^\star \text{ in } D_{\partial U}(\overline{U}, Y; L, T). \tag{3.3}$$

Similarly with

$$Q(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x) \qquad \text{for } (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

we have

$$F^\star \cong F \text{ in } D_{\partial U}(\overline{U}, Y; L, T). \tag{3.4}$$

Now (3.2)–(3.4) imply $G \cong F$ in $D_{\partial U}(\overline{U}, Y; L, T)$.

We next show that (ii) implies (i). Let $N : \overline{U} \times [0, 1] \to 2^Y$ denote the (L, T) upper semicontinuous, (L, T) compact map with $N_t \in D_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0, 1]$ and with $N_1 = F$ and $N_0 = G$

[in particular $Lx \notin N_t(x)$ for $x \in \partial U \cap dom\ L$ and for $t \in [0, 1]$]. Let

$$B = \left\{x \in \overline{U} \cap dom\ L : Lx \in N(x, t) \text{ for some } t \in [0, 1]\right\}.$$

Of course, it is immediate that

$$B = \left\{x \in \overline{U} : x \in (L + T)^{-1}(N_t + T)(x) \text{ for some } t \in [0, 1]\right\}.$$

If $B = \emptyset$ then F is inessential in $D_{\partial U}(\overline{U}, Y; L, T)$. So it remains to consider the case when $B \neq \emptyset$. Now B is closed and $\partial U \cap B = \emptyset$ so there exists a continuous function $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Define a map J by $J(x) = N(x, \mu(x)) = N \circ j(x)$ where $j : \overline{U} \to \overline{U} \times [0, 1]$ is given by $j(x) = (x, \mu(x))$. Clearly J is an (L, T) upper semicontinuous, (L, T) compact map. Also from (3.1) we know that $(L + T)^{-1} J \in \mathcal{A}(\overline{U}, X)$. Note also that $J|_{\partial U} = F|_{\partial U}$. Finally $Lx \notin J(x)$ for $x \in \overline{U} \cap dom\ L$ since if $Lx \in J(x)$ for $x \in \overline{U} \cap dom\ L$ then $x \in B$ and so $\mu(x) = 0$ (i.e. $Lx \in G(x)$), a contradiction. Thus $J \in D_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $Lx \notin J(x)$ for $x \in \overline{U} \cap dom\ L$. As a result F is inessential in $D_{\partial U}(\overline{U}, Y; L, T)$ and we are finished. □

Now (3.2) together with Theorem 3.1 yields the following continuation theorem.

Theorem 3.2. *Let X, Y, U, L and T be as above and assume that (3.1) and (3.2) hold. Suppose F and G are two maps in $D_{\partial U}(\overline{U}, Y; L, T)$ with $F \cong G$ in $D_{\partial U}(\overline{U}, Y; L, T)$. Then F is essential in $D_{\partial U}(\overline{U}, Y; L, T)$ if and only if G is essential in $D_{\partial U}(\overline{U}, Y; L, T)$.*

Remark 3.1. If $L = I$ and $T = 0$ the results in this section improve on those in Section 2. In this case also we could discuss maps in $\mathcal{A}(\overline{U}, C)$ where $\overline{B_R}$ and C are as defined in Section 2.

References

[1] Y. Benyamini, Y. Sternfelf, Spheres in infinite dimensional normed spaces and Lipschitz contractibility, Proc. Amer. Math. Soc. 88 (1983) 439–445.

[2] G. Fournier, H.O. Peitgen, On some fixed point principles for cones in linear normed spaces, Math. Ann. 225 (1977) 205–218.

[3] K. Goebel, W. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[4] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Acad. Publishers, Dordrecht, 1999.

[5] A. Granas, Sur la méthode de continuité de Poincaré, C. R. Acad. Sci. Paris Sér. I Math. 282 (1976) 983–985.

[6] D. O'Regan, Continuation methods based on essential and 0–epi maps, Acta Appl. Math. 54 (1998) 319–330.

[7] S. Park, Generalized Leray–Schauder principles for condensing admissible multifunctions, Ann. Mat. Pura Appl. 172 (1997) 65–85.

[8] S. Park, Generalized Leray–Schauder principles for compact admissible multifunctions, Topol. Methods Nonlinear Anal. 5 (1995) 271–277.