A note on the topological transversality theorem for acyclic maps

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Abstract

A new topological transversality theorem is presented for acyclic maps. The analysis relies on Urysohn’s Lemma and the fact that the unit sphere is contractible in infinite dimensional normed linear spaces.

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1. Introduction

This work establishes a topological transversality theorem of Granas type \cite{5} for multivalued acyclic maps. The proof differs from that given for Kuttani maps \cite{5,6} and relies on the fact that in a infinite dimensional normed linear space there exists a retraction from the unit ball to the unit sphere \cite{1}.

For the remainder of this section we look at the results in \cite{1,3}. Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space with $B = \{x \in E : \|x\| < 1\}$ and $S = \{x \in E : \|x\| = 1\}$. From \cite{1,3} we know that there exists a Lipschitz (Lipschitz constant $k_0$ say) retraction $r$ from $\overline{B}$ onto $S$. Next fix $R > 0$ and let $B_R = \{x \in E : \|x\| < R\}$ and $S_R = \{x \in E : \|x\| = R\}$. Also let

$$r_1(x) = \frac{x}{R} \quad \text{and} \quad r_2(x) = Rx,$$

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so \( r_1 : \overline{B_R} \to B \) and \( r_2 : S \to S_R \). It is easy to check that \( r_R = r_2 r_1 : \overline{B_R} \to S_R \) is a Lipschitz retraction (Lipschitz constant \( k_0 \)) from \( \overline{B_R} \) onto \( S_R \). Now let \( U \) be an open convex subset of \( E \) with \( 0 \in U \). Then there exists \( R > 0 \) with \( B_R \subseteq U \). Let

\[
  r_3(x) = \frac{x}{\max\{1, \mu_1(x)\}}, \quad x \in E \quad \text{and} \quad r_4(x) = \frac{x}{\mu_2(x)}, \quad x \in E \setminus \{0\}
\]

where \( \mu_1 \) is the Minkowski functional on \( \overline{B_R} \) and \( \mu_2 \) is the Minkowski functional on \( \overline{U} \). Notice \( r_3 : \overline{U} \to \overline{B_R} \) and \( r_4 : S_R \to \partial U \). Then \( r_4 r_3 : \overline{U} \to \partial U \) is a continuous retraction from \( \overline{U} \) onto \( \partial U \).

2. Topological transversality

Let \( E \) be an infinite dimensional normed linear space and \( U \) an open convex subset of \( E \) with \( 0 \in U \).

**Definition 2.1.** We let \( F \in M(U, E) \) denote the set of all upper semicontinuous compact maps \( F : \overline{U} \to AC(E) \); here \( AC(E) \) denotes the family of nonempty, compact, acyclic \([4]\) subsets of \( E \).

**Definition 2.2.** We let \( F \in M_{acU}(U, E) \) if \( F \in M(U, E) \) with \( x \notin F(x) \) for \( x \in \partial U \).

**Definition 2.3.** A map \( F \in M_{acU}(U, E) \) is essential in \( M_{acU}(U, E) \) if for every \( G \in M_{acU}(U, E) \) with \( G|_{\partial U} = F|_{\partial U} \) there exists \( x \in U \) with \( x \in G(x) \). Otherwise \( F \) is inessential in \( M_{acU}(U, E) \).

**Definition 2.4.** \( F, G \in M_{acU}(U, E) \) are homotopic in \( M_{acU}(U, E) \), written \( F \cong G \) in \( M_{acU}(U, E) \), if there exists an upper semicontinuous compact map \( N : \overline{U} \times [0, 1] \to AC(E) \) such that \( N_t(u) = N(u, t) = \partial U \to AC(E) \) belongs to \( M_{acU}(U, E) \) for each \( t \in [0, 1] \) and \( N_0 = F \) with \( N_1 = G \).

**Remark 2.1.** Notice that \( \cong \) is an equivalence relation in \( M_{acU}(U, E) \).

**Theorem 2.1.** Let \( E \) be an infinite dimensional normed linear space and \( U \) an open convex subset of \( E \) with \( 0 \in U \). Suppose that \( F \in M_{acU}(U, E) \). Then the following conditions are equivalent:

(i) \( F \) is inessential in \( M_{acU}(U, E) \);

(ii) there exists a map \( G \in M_{acU}(U, E) \) with \( x \notin G(x) \) for \( x \in U \) and \( F \cong G \) in \( M_{acU}(U, E) \).

**Proof.** To show that (i) implies (ii) let \( G \in M_{acU}(U, E) \) with \( G|_{\partial U} = F|_{\partial U} \) and \( x \notin Gx \) for \( x \in \overline{U} \). From Section 1 we know there exists a continuous retraction \( r : \overline{U} \to \partial U \). Let the map \( F^* \) be given by \( F^*(x) = F(r(x)) \) for \( x \in \overline{U} \). Of course \( F^*(x) = G(r(x)) \) for \( x \in \overline{U} \) since \( G|_{\partial U} = F|_{\partial U} \).

\[
  H(x, \lambda) = G(2\lambda r(x) + (1 - 2\lambda)x) = G \circ j(x, \lambda)
\]

for \( (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right] \).

(here \( j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U} \) is given by \( j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x \) it is easy to see that

\[
  G \cong F^* \quad \text{in} \quad M_{acU}(U, E).
\]

(2.1)

notice that if there exists \( x \in \partial U \) and \( \lambda \in \left[0, \frac{1}{4}\right] \) with \( x \in H_\lambda(x) \) then since \( r(x) = x \) we have \( x \in G(2\lambda x + (1 - 2\lambda)x) = G(x) \), a contradiction. Similarly with

\[
  Q(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x)
\]

for \( (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right] \).

(2.2)

it is easy to see that

\[
  F^* \cong F \quad \text{in} \quad M_{acU}(U, E).
\]
Combining (2.1) and (2.2) gives $G \cong F$ in $M_{\partial U}(\overline{U}, E)$.

We next show that (ii) implies (i). Let $N : \overline{U} \times [0, 1] \to AC(E)$ be an upper semicontinuous, compact map with $N_t \in M_{\partial U}(\overline{U}, E)$ for each $t \in [0, 1]$ and $N_1 = F$ with $N_0 = G$. Let

$$B = \{ x \in \overline{U} : x \in N(x, t) \text{ for some } t \in [0, 1] \}.$$ 

If $B = \emptyset$ then in particular $x \notin N(x, 1)$ for $x \in \overline{U}$ so $F$ is inessential in $M_{\partial U}(\overline{U}, E)$. So it remains to consider the case when $B \neq \emptyset$. Clearly $B$ is closed (and in fact compact). Also since $B \cap \partial U = \emptyset$ there exists a continuous $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Define a map $J : \overline{U} \to AC(E)$ by $J(x) = N(x, \mu(x))$. It is clear that $J$ is an upper semicontinuous, compact map. Also $J|_{\partial U} = F|_{\partial U}$ since if $x \in \partial U$ then $J(x) = N(x, 1) = F(x)$. In addition note that $x \notin J(x)$ for $x \in \overline{U}$ since if $x \in J(x)$ for some $x \in \overline{U}$ then $x \in B$ and so $\mu(x) = 0$, i.e. $x \in N(x, 0) = G(x)$, a contradiction. Thus $J \in M_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $x \notin J(x)$ for $x \in \overline{U}$. As a result $F$ is inessential in $M_{\partial U}(\overline{U}, E)$. □

Theorem 2.1 immediately yields the following continuation theorem.

**Theorem 2.2.** Let $E$ be an infinite dimensional normed linear space and $U$ an open convex subset of $E$ with $0 \in U$. Suppose that $F$ and $G$ are two maps in $M_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $M_{\partial U}(\overline{U}, E)$. Then $F$ is essential in $M_{\partial U}(\overline{U}, E)$ if and only if $G$ is essential in $M_{\partial U}(\overline{U}, E)$.

To complete our discussion we now supply an example of an essential map (this is called a normalization property).

**Theorem 2.3.** Let $E$ be an infinite dimensional normed linear space and $U$ an open convex subset of $E$ with $0 \in U$. Then the zero map is essential in $M_{\partial U}(\overline{U}, E)$.

**Proof.** Let $G : \overline{U} \to AC(E)$ be a map in $M_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in G(x)$. Let

$$J(x) = \begin{cases} G(x), & x \in \overline{U} \\ \{0\}, & x \in E \setminus \overline{U}. \end{cases}$$

Clearly $J : E \to AC(E)$ is an upper semicontinuous, compact map. Now [4, p. 161] guarantees that $J$ has a fixed point $x \in E$. In fact $x \in U$ since $0 \in U$. Hence $x \in G(x)$ and we are finished. □

**Remark 2.2.** It is also possible to combine the homotopy and normalization properties to obtain a Leray–Schauder alternative [5–8].

Next we discuss maps with values in a cone. Let $E = (E, \|\cdot\|)$ be a normed linear space (not necessarily infinite dimensional) and let $C \subseteq E$ be a cone (i.e. $C$ is a closed, convex, invariant under multiplication by nonnegative real numbers and $C \cap (-C) = \{0\}$). Fix $R > 0$ and let

$$B_R = \{ x \in C : \|x\| < R \} \quad \text{and} \quad S_R = \{ x \in C : \|x\| = R \}.$$ 

**Definition 2.5.** We let $F \in CM(B_R, C)$ denote the set of all upper semicontinuous compact maps $F : B_R \to AC(C)$.

**Definition 2.6.** We let $F \in CM_{S_R}(B_R, C)$ if $F \in CM(B_R, C)$ with $x \notin F(x)$ for $x \in S_R$. 

Definition 2.7. A map \( F \in \text{CM}_{SR}(\overline{B_R}, C) \) is essential in \( \text{CM}_{SR}(\overline{B_R}, C) \) if for every \( G \in \text{CM}_{SR}(\overline{B_R}, C) \) with \( G|_{SR} = F|_{SR} \) there exists \( x \in B_R \) with \( x \in G(x) \). Otherwise \( F \) is inessential in \( \text{CM}_{SR}(\overline{B_R}, C) \).

**Definition 2.8.** \( F, G \in \text{CM}_{SR}(\overline{B_R}, C) \) are homotopic in \( \text{CM}_{SR}(\overline{B_R}, C) \), written \( F \simeq G \) in \( \text{CM}_{SR}(\overline{B_R}, C) \), if there exists an upper semicontinuous compact map \( N : \overline{B_R} \times [0, 1] \rightarrow AC(C) \) such that \( N_t(u) = N(u, t) : B_R \rightarrow AC(C) \) belongs to \( \text{CM}_{SR}(\overline{B_R}, C) \) for each \( t \in [0, 1] \) and \( N_0 = F \) with \( N_1 = G \).

Essentially the same reasoning as in Theorem 2.1 (once one realizes that there exists a continuous retraction \( r : \overline{B_R} \rightarrow S_R \) (see [2])) establishes the next result.

**Theorem 2.4.** Let \( E \) be a normed linear space, \( C \subseteq E \) a cone and \( R > 0 \). Suppose \( F \) and \( G \) are two maps in \( \text{CM}_{SR}(\overline{B_R}, C) \) with \( F \simeq G \) in \( \text{CM}_{SR}(\overline{B_R}, C) \). Then \( F \) is essential in \( \text{CM}_{SR}(\overline{B_R}, C) \) if and only if \( G \) is essential in \( \text{CM}_{SR}(\overline{B_R}, C) \).

**Remark 2.3.** The analogue of Theorem 2.3 is also immediate in this case.

### 3. Generalizations

In this section we generalize the topological transversality theorem of Section 2. We discuss in particular a subclass of the \( \mathcal{U}_k^k \) maps of Park [7]. Let \( X \) and \( Y \) be Hausdorff topological vector spaces. Recall that a polytope \( P \) in \( X \) is any convex hull of a nonempty finite subset of \( X \). Given a class \( \mathcal{X} \) of maps, \( \mathcal{X}(X, Y) \) denotes the set of maps \( F : X \rightarrow 2^Y \) (the nonempty subsets of \( Y \)) belonging to \( \mathcal{X} \), and \( \mathcal{X}_c \) the set of finite compositions of maps in \( \mathcal{X} \). A class \( \mathcal{U} \) of maps is defined by the following properties:

(i) \( \mathcal{U} \) contains the class \( C \) of single-valued continuous functions;
(ii) each \( F \in \mathcal{U} \) is upper semicontinuous and compact valued; and
(iii) for any polytope \( P, F \in \mathcal{U}(P, P) \) has a fixed point, where the intermediate spaces of composites are suitably chosen for each \( \mathcal{U} \).

**Definition 3.1.** \( F \in \mathcal{U}_k^k(X, Y) \) if for any compact subset \( K \) of \( X \), there is a \( G \in \mathcal{U}_c(K, Y) \) with \( G(x) \subseteq F(x) \) for each \( x \in K \).

Recall that \( \mathcal{U}_k^k \) is closed under compositions. In this section we will consider a subclass \( \mathcal{A} \) of the \( \mathcal{U}_k^k \) maps. The following condition will be assumed throughout this section:

\[
\begin{align*}
\text{for Hausdorff topological spaces } X_1, X_2 \text{ and } X_3, \\
\text{if } F \in \mathcal{A}(X_1, X_3) \text{ and } f \in \mathcal{C}(X_2, X_1), \\
\text{then } F \circ f \in \mathcal{A}(X_2, X_3).
\end{align*}
\]

(3.1)

In this section \( X \) is an infinite dimensional normed linear space, \( Y \) a topological vector space and \( U \) an open convex subset of \( X \) with \( 0 \in U \). Also \( L : \text{dom } L \subseteq X \rightarrow Y \) will be a linear (not necessarily continuous) single-valued map; here \( \text{dom } L \) is a vector subspace of \( X \). Finally \( T : X \rightarrow Y \) will be a linear, continuous single-valued map with \( L + T : \text{dom } L \rightarrow Y \) an isomorphism (i.e. a linear homeomorphism); for convenience we say \( T \in \mathcal{H}_L(X, Y) \).

A multivalued map \( F : \overline{U} \rightarrow 2^Y \) is said to be \( (L, T) \) upper semicontinuous if \( (L + T)^{-1} F : \overline{U} \rightarrow K(X) \) is an upper semicontinuous map; here \( K(X) \) denotes the family of nonempty, compact subsets of \( X \). \( F : \overline{U} \rightarrow 2^Y \) is said to be \( (L, T) \) compact if \( (L + T)^{-1} F : \overline{U} \rightarrow K(X) \) is a compact map.
Definition 3.2. We let \( F \in D(U, Y; L, T) \) if \((L + T)^{-1}F \in \mathcal{A}(U, X)\) and \( F : \overline{U} \to 2^Y \) is an \((L, T)\) upper semicontinuous, \((L, T)\) compact map.

Definition 3.3. \( D_{\partial U}(U, Y; L, T) \) denotes the maps \( F \in D(U, Y; L, T) \) with \( Lx \notin F(x) \) for \( x \in \partial U \cap \text{dom} \ L \).

Definition 3.4. A map \( F \in D_{\partial U}(U, Y; L, T) \) is essential in \( D_{\partial U}(U, Y; L, T) \) if for every map \( G \in D_{\partial U}(U, Y; L, T) \) with \( G|_{\partial U} = F|_{\partial U} \) we have that there exists \( x \in \overline{U} \cap \text{dom} \ L \) with \( Lx \in G(x) \). Otherwise \( F \) is inessential in \( D_{\partial U}(U, Y; L, T) \), i.e. there exists \( G \in D_{\partial U}(U, Y; L, T) \) with \( G|_{\partial U} = F|_{\partial U} \) and \( Lx \notin G(x) \) for \( x \in \overline{U} \cap \text{dom} \ L \).

Definition 3.5. Two maps \( F, G \in D_{\partial U}(U, Y; L, T) \) are homotopic in \( D_{\partial U}(U, Y; L, T) \), written \( F \cong G \) in \( D_{\partial U}(U, Y; T, T) \), if there exists an \((L, T)\) upper semicontinuous, \((L, T)\) compact mapping \( N : \overline{U} \times [0, 1] \to 2^Y \) such that \( N_t(u) = N(u, t) : \overline{U} \to 2^Y \) belongs to \( D_{\partial U}(U, Y; L, T) \) for each \( t \in [0, 1] \) and \( N_0 = F \) with \( N_1 = G \).

The following condition will be assumed throughout this section:

\[ \cong \text{ is an equivalence relation in } D_{\partial U}(U, Y; L, T). \tag{3.2} \]

Theorem 3.1. Let \( X, Y, U, L \) and \( T \) be as above and assume that (3.1) and (3.2) hold. Suppose \( F \in D_{\partial U}(U, Y; L, T) \). Then the following conditions are equivalent:

(i) \( F \) is inessential in \( D_{\partial U}(U, Y; L, T) \);

(ii) there exists a map \( G \in D_{\partial U}(U, Y; L, T) \) with \( Lx \notin G(x) \) for \( x \in \overline{U} \cap \text{dom} \ L \) and \( F \cong G \) in \( D_{\partial U}(U, Y; L, T) \).

Proof. To show that (i) implies (ii) let \( G \in D_{\partial U}(U, Y; L, T) \) with \( G|_{\partial U} = F|_{\partial U} \) and \( Lx \notin G(x) \) for \( x \in \overline{U} \cap \text{dom} \ L \). Also let \( r \) be as in Theorem 2.1 and \( F^*(x) = F(r(x)) = G(r(x)) \) for \( x \in \overline{U} \). Let

\[ H(x, \lambda) = G(2\lambda r(x) + (1 - 2\lambda)x) = G \circ j(x, \lambda) \quad \text{for } (x, \lambda) \in \overline{U} \times \left[ 0, \frac{1}{2} \right], \]

where \( j : \overline{U} \times [0, \frac{1}{2}] \to \overline{U} \) is given by \( j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x \). Clearly \( H \) is an \((L, T)\) upper semicontinuous, \((L, T)\) compact map. In addition, assumption (3.1) guarantees that \((L + T)^{-1}H \in \mathcal{A}(\overline{U} \times [0, \frac{1}{2}], X) \). Now if there exists \( x \in \partial U \cap \text{dom} \ L \) and \( \lambda \in \left[ 0, \frac{1}{2} \right] \) with \( Lx \in H_x(x) \) then since \( r(x) = x \) we have \( Lx \in G_x \), a contradiction. Consequently

\[ G \cong F^* \text{ in } D_{\partial U}(U, Y; L, T). \tag{3.3} \]

Similarly with

\[ Q(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x) \quad \text{for } (x, \lambda) \in \overline{U} \times \left[ \frac{1}{2}, 1 \right], \]

we have

\[ F^* \cong F \text{ in } D_{\partial U}(U, Y; L, T). \tag{3.4} \]

Now (3.2)–(3.4) imply \( G \cong F \) in \( D_{\partial U}(U, Y; L, T) \).

We next show that (ii) implies (i). Let \( N : \overline{U} \times [0, 1] \to 2^Y \) denote the \((L, T)\) upper semicontinuous, \((L, T)\) compact map with \( N_t \in D_{\partial U}(U, Y; L, T) \) for each \( t \in [0, 1] \) and with \( N_1 = F \) and \( N_0 = G \).
[in particular \(Lx \notin N_t(x)\) for \(x \in \partial U \cap \text{dom} L\) and for \(t \in [0,1]\)]). Let

\[
B = \{ x \in \overline{U} \cap \text{dom} L : Lx \in N(x,t) \text{ for some } t \in [0,1] \}.
\]

Of course, it is immediate that

\[
B = \{ x \in \overline{U} : x \in (L + T)^{-1}(N_t + T)(x) \text{ for some } t \in [0,1] \}.
\]

If \(B = \emptyset\) then \(F\) is inessential in \(D_{\partial U}(\overline{U}, Y; L, T)\). So it remains to consider the case when \(B \neq \emptyset\). Now \(B\) is closed and \(\partial U \cap B = \emptyset\) so there exists a continuous function \(\mu : \overline{U} \to [0,1]\) with \(\mu(\partial U) = 1\) and \(\mu(B) = 0\). Define a map \(J\) by \(J(x) = N(x, \mu(x)) = N \circ j(x)\) where \(j : \overline{U} \to \overline{U} \times [0,1]\) is given by \(j(x) = (x, \mu(x))\). Clearly \(J\) is an \((L, T)\) upper semicontinuous, \((L, T)\) compact map. Also from (3.1) we know that \((L + T)^{-1}J \in \mathcal{A}(\overline{U}, X)\). Note also that \(J_{|\partial U} = F_{|\partial U}\). Finally \(Lx \notin J(x)\) for \(x \in \overline{U} \cap \text{dom} L\) since if \(Lx \in J(x)\) for \(x \in \overline{U} \cap \text{dom} L\) then \(x \in B\) and so \(\mu(x) = 0\) (i.e. \(Lx \in G(x)\)), a contradiction. Thus \(J \in D_{\partial U}(\overline{U}, Y; L, T)\) with \(J_{|\partial U} = F_{|\partial U}\) and \(Lx \notin J(x)\) for \(x \in \overline{U} \cap \text{dom} L\). As a result \(F\) is inessential in \(D_{\partial U}(\overline{U}, Y; L, T)\) and we are finished.

Now (3.2) together with Theorem 3.1 yields the following continuation theorem.

**Theorem 3.2.** Let \(X, Y, U, L\) and \(T\) be as above and assume that (3.1) and (3.2) hold. Suppose \(F\) and \(G\) are two maps in \(D_{\partial U}(\overline{U}, Y; L, T)\) with \(F \cong G\) in \(D_{\partial U}(\overline{U}, Y; L, T)\). Then \(F\) is essential in \(D_{\partial U}(\overline{U}, Y; L, T)\) if and only if \(G\) is essential in \(D_{\partial U}(\overline{U}, Y; L, T)\).

**Remark 3.1.** If \(L = I\) and \(T = 0\) the results in this section improve on those in Section 2. In this case also we could discuss maps in \(\mathcal{A}(\overline{U}, C)\) where \(\overline{B_2}\) and \(C\) are as defined in Section 2.

**References**


