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Discrete Mathematics 308 (2008) 1102–1123
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A few more Kirkman squares and doubly near resolvable BIBDs with block size 3

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Received 12 September 2005; received in revised form 8 May 2006; accepted 3 April 2007
Available online 6 April 2007

Abstract

A Kirkman square with index λ, latinicity μ, block size k, and v points, $\mathrm{KS}_k(v;\mu,\lambda)$, is a $t\times t$ array ($t=\lambda(v-1)/\mu(k-1)$) defined on a v-set V such that (1) every point of V is contained in precisely μ cells of each row and column, (2) each cell of the array is either empty or contains a k-subset of V, and (3) the collection of blocks obtained from the non-empty cells of the array is a (v,k,λ)-BIBD. In a series of papers, Lamken established the existence of the following designs: $\mathrm{KS}_3(v;1,2)$ with at most six possible exceptions [E.R. Lamken, The existence of doubly resolvable $(v,3,2)$-BIBDs, J. Combin. Theory Ser. A 72 (1995) 50–76], $\mathrm{KS}_3(v;2,4)$ with two possible exceptions [E.R. Lamken, The existence of $\mathrm{KS}_3(v;2,4)s$, Discrete Math. 186 (1998) 195–216], and doubly near resolvable $(v,3,2)$-BIBDs with at most eight possible exceptions [E.R. Lamken, The existence of doubly near resolvable $(v,3,2)$-BIBDs, J. Combin. Designs 2 (1994) 427–440]. In this paper, we construct designs for all of the open cases and complete the spectrum for these three types of designs. In addition, Colbourn, Lamken, Ling, and Mills established the spectrum of $\mathrm{KS}_3(v;1,1)$ in 2002 with 23 possible exceptions. We construct designs for 11 of the 23 open cases.

MSC: 05B07

Keywords: Kirkman square; Doubly resolvable; Doubly near resolvable; Starter; Adder; Frame

1. Introduction

A balanced incomplete block design (BIBD) D is a collection B of subsets (blocks) taken from a finite set V of v elements with the properties:

(1) Every pair of distinct elements of V is contained in precisely λ blocks of B.
(2) Every block contains exactly k elements.

We denote such a design as a (v,k,λ)-BIBD. The necessary conditions for the existence of a (v,k,λ)-BIBD are

$$\lambda(v-1)\equiv 0\ (\mathrm{mod}\ k-1)\quad\text{and}\quad\lambda v(v-1)\equiv 0\ (\mathrm{mod}\ k(k-1)).\qquad(*)$$

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.04.001

0,1,2	0,3,4	∞,3,1		∞,2,4
∞,3,0	1,2,3	1,4,0	∞,4,2	
	∞,4,1	2,3,4	2,0,1	∞,0,3
∞,1,4		∞,0,2	3,4,0	3,1,2
4,2,3	∞,2,0		∞,1,3	4,0,1

Fig. 1. A $KS_3(6; 2, 4)$, [11].

A (v, k, λ)-BIBD D is said to be μ-resolvable if the blocks of D can be partitioned into classes $R_1, R_2, \ldots, R_t$ (resolution classes) where $t = \lambda(v-1)/\mu(k-1)$ such that each element of D is contained in precisely μ blocks of each class. The classes $R_1, R_2, \ldots, R_t$ form a resolution of D. If $\mu=1$, the design is said to be resolvable and is usually denoted by (v, k, λ)-RBIBD. The necessary conditions for the existence of a (v, k, λ)-RBIBD are $(*)$ and $v \equiv 0 \pmod{k}$.

A (v, k, λ)-BIBD is said to be near resolvable if the blocks of D can be partitioned into classes (resolution classes) $R_1, R_2, \ldots, R_v$ such that for each element x of D there is precisely one class which does not contain x in any of its blocks and each class contains precisely $v-1$ distinct elements of the design. The classes $R_1, R_2, \ldots, R_v$ form a resolution of D and D is denoted by NR(v, k, λ)-BIBD. Two necessary conditions for the existence of an NR(v, k, λ)-BIBD are $v \equiv 1 \pmod{k}$ and $\lambda = k-1$.

Let R and R' be two resolutions of the blocks of a (v, k, λ)-BIBD D. R and R' are said to be orthogonal if $|R_i \cap R'_j| \leqslant 1$ for all $R_i \in R$ and $R'_j \in R'$. (It should be noted that the blocks of the design are considered as labeled so that if a subset of the elements occurs as a block more than once, the blocks are treated as distinct.) If D is a (v, k, λ)-RBIBD with a pair of orthogonal resolutions, it is called doubly resolvable and is denoted by DR(v, k, λ)-BIBD. If D is an NR(v, k, λ)-BIBD with a pair of orthogonal near resolutions, it is called doubly near resolvable and is denoted by DNR(v, k, λ)-BIBD.

The existence of a μ-resolvable (v, k, λ)-BIBD with a pair of orthogonal μ-resolutions is equivalent to the existence of a Kirkman square, $KS_k(v; \mu, \lambda)$ [8,14,15]. In particular, the existence of a DR(v, k, λ)-BIBD is equivalent to the existence of a $KS_k(v; 1, \lambda)$.

A Kirkman square with index λ, latinicity μ, block size k, and v points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a v-set V such that

(1) every point of V is contained in precisely μ cells of each row and column,
(2) each cell of the array is either empty or contains a k-subset of V, and
(3) the collection of blocks obtained from the non-empty cells of the array is a (v, k, λ)-BIBD.

We can use a pair of orthogonal μ-resolutions of a (v, k, λ)-BIBD to construct a $KS_k(v; \mu, \lambda)$. We index the rows and columns of a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) with a pair of orthogonal resolutions, R and R'. In the cell labeled (R_i, R'_j), we place the block from $R_i \cap R'_j$ for all $R_i \in R$ and $R'_j \in R'$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. It is easy to verify that this array is a $KS_k(v; \mu, \lambda)$. Similarly, it is easy to see that a $KS_k(v; \mu, \lambda)$ displays a pair of orthogonal μ-resolutions of a (v, k, λ)-BIBD. To illustrate these definitions, a $KS_3(6; 2, 4)$ is displayed in Fig. 1.

Similarly, a pair of orthogonal resolutions of a DNR(v, k, λ)-BIBD can be used to construct a $v \times v$ array. (For convenience, we often refer to this array as a DNR(v, k, λ)-BIBD.) We index the rows and columns of the array with the pair of orthogonal resolutions R and R'. In the cell labeled (R_i, R'_j), we place the block from $R_i \cap R'_j$ for all $R_i \in R$ and $R'_j \in R'$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. The rows of the array will contain the resolution classes of the resolution R and the columns will contain the resolution classes of the orthogonal resolution R'. If the DNR(v, k, λ)-BIBD has the additional property that under an appropriate ordering of the resolution classes R and R', $R_i \cup R'_i$ contains precisely $v-1$ distinct elements of the design and $R_i \cap R'_i = \emptyset$ for all i, then the array is called a $(1, \lambda; k, v, 1)$-frame. The diagonal of a $(1, \lambda; k, v, 1)$-frame is empty and a unique element of the design can be associated with each cell (i, i). The DNR$(10, 3, 2)$-BIBD in Fig. 2 is a $(1, 2; 3, 10, 1)$-frame. The element associated with cell (i, i) is i for $i = 0, 1, \ldots, 9$. We note that it is not always possible to permute the rows and columns of a DNR(v, k, λ)-BIBD to form a $(1, \lambda; k, v, 1)$-frame; we refer to [13] for examples off DNR(v, k, λ)-BIBDs which are not $(1, \lambda; k, v, 1)$-frames. (The distinction between DNR(v, k, λ)-BIBDs and $(1, \lambda; k, v, 1)$-frames is important in recursive constructions.)

		8,3,4	6,7,1				9,5,2		
			9,4,0	7,8,2				5,6,3	
8,9,3				5,0,1					6,7,4
6,1,2	9,5,4				7,8,0				
	7,2,3	5,6,0				8,9,1			
		7,9,1					6,8,4		3,2,0
			8,5,2		4,3,1			7,9,0	
				9,6,3		0,4,2			8,5,1
5,7,4					9,6,2		1,0,3		
	6,8,0					5,7,3		2,1,4	

Fig. 2. A (1, 2; 3, 10, 1)-frame, [3].

The existence of $KS_2(v; \mu, \lambda)$ has been completely settled, [8,14,15]. The first case for $\mu = \lambda = 1$ is a well-known design; a $KS_2(v; 1, 1)$ (or a DR(v, 2, 1)-BIBD) is also called a Room square of side $v - 1$. Although the first example of a Room square, an RS(7), was constructed by Kirkman in 1850 [6], the spectrum of Room squares was not completed until 1975 [18]. (An extensive bibliography is available on Room squares, see [18] and [4].)

Theorem 1.1 (*Mullin and Wallis [18]*). *Let v be a positive integer*, $v \equiv 0 \pmod 2$, $v \neq 4, 6$. *There exists a* $KS_2(v; 1, 1)$ *or a Room square of side* $v - 1$. *Furthermore*, *Room squares of side* $v - 1$ *do not exist for* $v = 4$ *and* 6.

The existence of $(1, 1; 2, v, 1)$-frames follows immediately from the existence of Room squares.

Theorem 1.2. *There exists a* $(1, 1; 2, v, 1)$-*frame* (*a* DNR(v, 2, 1)-*BIBD*) *if and only if* $v \equiv 1 \pmod 2$ *and* $v \geqslant 7$.

Quite a lot of work was done after the Room square problem was completed to try to determine the spectrum for a second class of doubly resolvable balanced incomplete block designs; see [10,1], or the survey [12] for further information. The natural analogue of the Room square for block size $k = 3$ is the DR(v, 3, 2)-BIBD or a $KS_3(v; 1, 2)$. The spectrum of DR(v, 3, 2)-BIBDs was established (with six possible exceptions) in 1995 by using the connection between partitioned balanced tournament designs and Kirkman squares [10].

Theorem 1.3 (*Lamken [10]*). *Let v be a positive integer*, $v \equiv 0 \pmod 3$, $v \neq 6, 9$. *There exists a* $KS_3(v; 1, 2)$ *except possibly for* $v \in \{72, 78, 90, 114, 117, 126\}$. *Furthermore*, *there do not exist* $KS_3(v; 1, 2)$ *for* $v = 6$ *and* 9.

Two other related results were established using similar ideas and techniques. The first provides a second class of Kirkman squares for $k = 3$.

Theorem 1.4 (*Lamken [11]*). *Let v be a positive integer*, $v \equiv 0 \pmod 3$. *Then a* $KS_3(v; 2, 4)$ *exists except possibly for* $v \in \{60, 69\}$.

The existence of doubly near resolvable designs for block size $k = 3$ has also been established with eight possible exceptions.

Theorem 1.5 (*Lamken [9]*). *Let v be a positive integer*, $v \equiv 1 \pmod 3$, $v \geqslant 10$, *and let* $N = \{34, 70, 85, 88, 115, 124, 133, 142\}$.

(i) *There exists a* DNR(v, 3, 2)-*BIBD except possibly for* $v \in N$.
(ii) *There exists a* $(1, 2; 3, v, 1)$-*frame except possibly for* $v \in N \cup \{13, 58\}$.

Finally, the existence of $KS_3(v; 1, 1)$ with at most 23 possible exceptions was established in [1].

Theorem 1.6 (*Colbourn et al. [1]*). *Let v be a positive integer*, $v \equiv 3 \pmod 6$, $v \neq 9, 15$. *There exists a* $KS_3(v; 1, 1)$ *except possibly for* $v \in \{21, 57, 69, 93, 99, 105, 117, 141, 147, 153, 165, 177, 183, 189, 201, 231, 237, 249, 261, 267, 285, 351, 357\}$. *Furthermore*, *there do not exist* $KS_3(v; 1, 1)$ *for* $v = 9$ *and* 15.

In this paper, we use direct and recursive constructions to complete all of the remaining cases for $\mathrm{KS}_3(v; 1, 2)$, $\mathrm{KS}_3(v; 2, 4)$, and $\mathrm{DNR}(v, 3, 2)$-BIBDs. In addition, we construct several more $\mathrm{KS}_3(v; 1, 1)$ or $\mathrm{DR}(v, 3, 1)$-BIBDs. In the next section, we describe constructions for frames which will be used to construct Kirkman squares and doubly near resolvable designs. In Section 3, we construct the last six open cases for $\mathrm{KS}_3(v; 1, 2)$, and we complete the spectrum for $\mathrm{KS}_3(v; 2, 4)$ by constructing the last two cases for $\mathrm{KS}_3(v; 2, 4)$, $v = 60, 69$. The remaining eight cases for $\mathrm{DNR}(v, 3, 2)$-BIBDs are constructed in Section 4. Finally, in Section 5 we construct several more $\mathrm{DR}(v, 3, 1)$-BIBDs and then we summarize our results in the last section.

2. Frames

Frames can be used to construct Kirkman squares and doubly near resolvable balanced incomplete block designs. We will use frames to construct several of the remaining cases for $\mathrm{KS}_3(v; 1, 2)$ and $\mathrm{DNR}(v, 3, 2)$-BIBDs and for one of the remaining $\mathrm{KS}_3(v; 2, 4)$. In order to describe the constructions, we need several definitions. A group divisible design (GDD) is a triple $(X, \mathscr{G}, \mathscr{B})$ which satisfies the following properties.

(1) $\mathscr{G}$ is a partition of X into subsets called groups; $\mathscr{G} = \{G_1, G_2, \ldots, G_m\}$.
(2) $\mathscr{B}$ is a collection of subsets of X, called blocks, such that a group and a block contain at most one element in common.
(3) Every pair of elements from distinct groups occurs in precisely λ blocks. (λ is then called the index of the GDD.)

Let K be a set of positive integers. A $\mathrm{GDD}(v; K; G_1, G_2, \ldots, G_m; 0, \lambda)$ G is a GDD of index λ with $|X| = v$, $|b| \in K$ for every $b \in \mathscr{B}$, and $\mathscr{G} = \{G_1, G_2, \ldots, G_m\}$. The type of G is the multiset $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We usually use exponential notation to describe the type; G has type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \leqslant i \leqslant l$. A $\mathrm{GDD}(v; K; G_1, \ldots, G_m; 0, 1)$ is often denoted as a $K - \mathrm{GDD}$ of type $t_1^{u_1} t_2^{u_2} \cdots t_\ell^{u_\ell}$.

Let V be a set of v elements. Let $G_1, G_2, \ldots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \ldots, G_m\}$-frame F with block size k, index λ, and latinicity μ is a square array of side $t = (\lambda v)/(\mu(k-1))$ which satisfies the properties listed below. Let $t_i = (\lambda |G_i|)/(\mu(k-1))$, and let $g_k = \sum_{i=1}^{k} t_i$ and define $g_0 = 0$. We index the rows and columns of F with the elements $0, 1, \ldots, (\lambda v)/(\mu(k-1)) - 1$.

(1) Each cell is either empty or contains a k-subset of V.
(2) Let F_i be the subsquare of F indexed by $g_{i-1}, g_{i-1} + 1, \ldots, g_i - 1$. F_i is empty for $i = 1, 2, \ldots, m$. (i.e. The main diagonal of F consists of empty subsquares of sides $t_i \times t_i$ for $i = 1, 2, \ldots, m$.)
(3) Let $x \in \{g_{i-1}, g_{i-1} + 1, \ldots, g_i - 1\}$. Row x of F contains each element of $V - G_i\,\mu$ times and column x of F contains each element of $V - G_i\ \mu$ times.
(4) The blocks obtained from the nonempty cells of F form a $\mathrm{GDD}(v; k; G_1, G_2, \ldots, G_m; 0, \lambda)$.

The type of a $\{G_1, G_2, \ldots, G_m\}$-frame is the multiset $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We usually use exponential notation to describe the type; a frame is said to have type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \leqslant i \leqslant \ell$. A $\{G_1, G_2, \ldots, G_m\}$-frame with block size k, index λ, and latinicity μ is denoted as a $(\mu, \lambda; k, \{G_1, \ldots, G_m\})$-frame or a $(\mu, \lambda; k)$-frame of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \leqslant i \leqslant \ell$. If $|G_i| = h$ for all i, the frame F is often called a $(\mu, \lambda; k, m, h)$-frame.

Starters and adders can be used to construct frames. We use the standard frame starter and adder construction [2,3,21] to construct the first frame.

Lemma 2.1. *There exists a* $(1, 2; 3)$*-frame of type* 10^7.

Proof. Let $V = \mathbb{Z}_{70}$ and let $G_i = \{i, i+7, \ldots, i+63\}$ for $i = 0, 1, \ldots, 6$. A frame starter and adder for a $(1, 2; 3, \{G_0, \ldots, G_6\})$-frame over $\mathbb{Z}_{70}$ are listed in Table 1. □

Intransitive starters and adders can also be used to construct frames (see [1]). An intransitive starter over $\mathbb{Z}_{3hu}$ for a $(1, 2; 3)$-frame of type $(3h)^u (3l)^1$ written on the symbol set $\mathbb{Z}_{3hu} \cup \{\infty_1, \infty_2, \ldots, \infty_{3l}\}$ with groups $G_i = \{i, i + u, \ldots, i + (3h-1)u\}$ for $i = 0, 1, \ldots, u-1$, and $G_u = \{\infty_1, \infty_2, \ldots, \infty_{3l}\}$ is defined to be a triple (S, C, R),

Table 1
Starter blocks and adders for a (1, 2; 3)-frame of type 10^7

Starter	Adder	Starter	Adder	Starter	Adder
{11, 12, 34}	11	{53, 54, 58}	43	{13, 15, 33}	25
{30, 38, 67}	51	{47, 52, 6}	27	{8, 20, 40}	26
{59, 65, 4}	6	{3, 19, 43}	17	{2, 5, 36}	52
{46, 61, 10}	41	{1, 9, 26}	38	{45, 51, 62}	10
{16, 29, 48}	66	{41, 44, 66}	34	{23, 39, 50}	44
{60, 69, 17}	69	{25, 27, 37}	16	{18, 31, 57}	19
{64, 68, 24}	5	{22, 32, 55}	30		

Table 2
(S, C, R) and A for a (1, 2; 3)-frame of type $12^8 18^1$

C	R
20, 22, 27	62, 67, 81
5, 9, 19	18, 19, 20
55, 66, 83	9, 29, 52
1, 18, 38	41, 66, 91
17, 43, 69	74, 78, 85
34, 62, 93	31, 53, 75

Starter	Adder	Starter + Adder
2, 11, 14	3	5, 14, 17
3, 6, 12	27	30, 33, 39
4, 10, 23	45	49, 55, 68
7, 25, 58	21	28, 46, 79
13, 46, 67	23	36, 69, 90
15, 30, 77	12	27, 42, 89
21, 42, 57	1	22, 43, 58
28, 59, 94	6	34, 65, 4
29, 68, 78	9	38, 77, 87
31, 60, 89	13	44, 73, 6
∞_1, 26, 76	19	∞_1, 45, 95
∞_2, 35, 65	28	∞_2, 63, 93
∞_3, 39, 73	83	∞_3, 26, 60
∞_4, 49, 85	34	∞_4, 83, 23
∞_5, 54, 81	52	∞_5, 10, 37
∞_6, 61, 79	42	∞_6, 7, 25
∞_7, 70, 82	85	∞_7, 59, 71
∞_8, 53, 92	39	∞_8, 92, 35
∞_9, 45, 86	25	∞_9, 70, 15
∞_{10}, 41, 95	58	∞_{10}, 3, 57
∞_{11}, 52, 87	95	∞_{11}, 51, 86
∞_{12}, 36, 91	81	∞_{12}, 21, 76
∞_{13}, 47, 90	7	∞_{13}, 54, 1
∞_{14}, 51, 74	10	∞_{14}, 61, 84
∞_{15}, 33, 84	14	∞_{15}, 47, 2
∞_{16}, 37, 75	71	∞_{16}, 12, 50
∞_{17}, 44, 71	38	∞_{17}, 82, 13
∞_{18}, 50, 63	44	∞_{18}, 94, 11

where

(1) $S = \{\{x_i, y_i, z_i\} | i = 1, 2, \ldots, t\} \cup \{\{\infty_i, u_i, v_i\} | i = 1, 2, \ldots, 3l\}$.
(2) $C = \{\{c_{i1}, c_{i2}, c_{i3}\} | i = 1, 2, \ldots, l\}$.
(3) $R = \{\{d_{i1}, d_{i2}, d_{i3}\} | i = 1, 2, \ldots, l\}$,

Table 3
(S, C, R) and A for a $(1, 2; 3)$-frame of type $12^9 15^1$

C		R
1, 11, 24		31, 38, 48
21, 43, 68		42, 46, 50
16, 48, 83		11, 22, 33
3, 37, 71		6, 23, 43
6, 20, 34		60, 79, 98
Starter	Adder	Starter + Adder
14, 38, 62	2	16, 40, 64
15, 46, 67	6	21, 52, 73
23, 44, 87	5	28, 49, 92
29, 80, 96	8	37, 88, 104
30, 79, 85	4	34, 83, 89
31, 59, 101	10	41, 69, 3
32, 78, 93	7	39, 85, 100
33, 66, 91	11	44, 77, 102
47, 73, 104	14	61, 87, 10
40, 70, 100	16	56, 86, 8
49, 64, 107	31	80, 95, 30
41, 74, 103	17	58, 91, 12
42, 77, 106	26	68, 103, 24
2, 4, 5	15	17, 19, 20
7, 8, 10	105	4, 5, 7
12, 17, 25	53	65, 70, 78
13, 39, 98	84	97, 15, 74
∞_1, 19, 35	87	∞_1, 106, 14
∞_2, 22, 75	62	∞_2, 84, 29
∞_3, 26, 65	40	∞_3, 66, 105
∞_4, 28, 94	65	∞_4, 93, 51
∞_5, 50, 89	21	∞_5, 71, 2
∞_6, 51, 58	82	∞_6, 25, 32
∞_7, 52, 84	23	∞_7, 75, 107
∞_8, 53, 105	56	∞_8, 1 , 53
∞_9, 55, 95	39	∞_9, 94, 26
∞_{10}, 56, 61	1	∞_{10}, 57, 62
∞_{11}, 57, 69	106	∞_{11}, 55, 67
∞_{12}, 60, 97	107	∞_{12}, 59, 96
∞_{13}, 76, 88	67	∞_{13},35, 47
∞_{14}, 82, 102	19	∞_{14}, 101, 13
∞_{15}, 86, 92	98	∞_{15}, 76 , 82

where $t = h(u-1) - 3l$, $x_i, y_i, z_i, u_j, v_j \in \mathbb{Z}^*_{3hu}$ for $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, 3l$, $c_{ij}, d_{ij} \in \mathbb{Z}^*_{3hu}$ for $j = 1, 2, 3$ and $i = 1, 2, \ldots, l$, and $\mathbb{Z}^*_{3hu} = \mathbb{Z}_{3hu} \backslash \{0, u, \ldots, (3h-1)u\}$. (S, C, R) satisfies the following properties:

(1) $(\bigcup_{i=1}^{t}\{x_i, y_i, z_i\}) \cup (\bigcup_{i=1}^{3l}\{u_i, v_i\}) \cup (\bigcup_{i=1}^{l}\{c_{i1}, c_{i2}, c_{i3}\}) = \mathbb{Z}^*_{3hu}$.
(2) Every element of $\mathbb{Z}^*_{3hu}$ occurs precisely twice as a difference in (S, C, R), i.e. $(\bigcup_{i=1}^{t}\{\pm(x_i - y_i), \pm(y_i - z_i), \pm(z_i - x_i)\}) \cup (\bigcup_{i=1}^{3l}\{\pm(u_i - v_i)\}) \cup (\bigcup_{i=1}^{l}\{\pm(c_{i1} - c_{i2}), \pm(c_{i2} - c_{i3}), \pm(c_{i3} - c_{i1})\}) \cup (\bigcup_{i=1}^{l}\{\pm(d_{i1} - d_{i2}), \pm(d_{i2} - d_{i3}), \pm(d_{i3} - d_{i1})\}) = 2 \cdot \mathbb{Z}^*_{3hu}$.
(3) Every triple in C contains three distinct residues modulo 3.
(4) Every triple in R contains three distinct residues modulo 3.

An adder for (S, C, R) is an injection A: $S \to \mathbb{Z}^*_{3hu}$, such that

$$\left(\bigcup_{i=1}^{t}\{x_i + a_i, y_i + a_i, z_i + a_i\}\right) \cup \left(\bigcup_{i=1}^{3l}\{u_i + a_{t+i}, v_i + a_{t+i}\}\right) \cup \left(\bigcup_{i=1}^{l}\{d_{i1}, d_{i2}, d_{i3}\}\right) = \mathbb{Z}^*_{3hu},$$

where $a_i = A(\{x_i, y_i, z_i\})$ for $i = 1, 2, \ldots, t$, and $a_{t+i} = A(\{\infty_i, u_i, v_i\})$ for $i = 1, 2, \ldots, 3l$.

Table 4
(S, C, R) and A for a $(1, 2; 3)$-frame of type $9^{11}12^1$

C	R
1, 26, 51	7, 12, 20
3, 23, 46	21, 35, 49
12, 29, 49	1, 30, 59
8, 18, 31	5, 36, 70

Starter	Adder	Starter + Adder
2, 6, 97	78	80, 84, 76
4, 7, 10	65	69, 72, 75
9, 14, 78	4	13, 18, 82
13, 94, 96	1	14, 95, 97
15, 21, 58	2	17, 23, 60
16, 42, 54	3	19, 45, 57
20, 35, 67	6	26, 41, 73
24, 63, 82	5	29, 68, 87
28, 52, 80	10	38, 62, 90
30, 62, 93	9	39, 71, 3
34, 60, 87	14	48, 74, 2
36, 72, 89	7	43, 79, 96
38, 56, 86	71	10, 28, 58
41, 76, 83	50	91, 27, 34
45, 69, 90	62	8, 32, 53
∞_1, 5, 65	84	∞_1, 89, 50
∞_2, 17, 19	23	∞_2, 40, 42
∞_3, 25, 40	38	∞_3, 63, 78
∞_4, 27, 37	19	∞_4, 46, 56
∞_5, 32, 81	49	∞_5, 81, 31
∞_6, 48, 57	37	∞_6, 85, 94
∞_7, 50, 59	64	∞_7, 15, 24
∞_8, 53, 95	8	∞_8, 61, 4
∞_9, 61, 73	25	∞_9, 86, 98
∞_{10}, 68, 84	98	∞_{10}, 67, 83
∞_{11}, 70, 71	80	∞_{11}, 51, 52
∞_{12}, 74, 75	18	∞_{12}, 92, 93
39, 79, 98	26	65, 6, 25
43, 64, 91	72	16, 37, 64
47, 85, 92	61	9, 47, 54

Theorem 2.2. *Suppose that there exists an intransitive starter* (S, R, C) *for a* $(1, 2; 3)$*-frame of type* $(3h)^u(3l)^1$ *over* $\mathbb{Z}_{3hu}$ *and a corresponding adder A for* (S, C, R)*, then there is a* $(1, 2; 3)$*-frame of type* $(3h)^u(3l)^1$.

Proof. This proof is similar to the proof of [1, Theorem 4.11]. We include it for completeness. First we construct a $3hu \times 3hu$ array M using starter S and a corresponding adder A. As before, we label the rows $0, 1, \ldots, 3hu-1$ and the columns $0, 3hu-1, \ldots, 2, 1$. Let $T_i = \{x_i, y_i, z_i\}$ for $i = 1, 2, \ldots, h(u-1) - 3l$ and $T_{h(u-1)-3l+i} = \{\infty_i, u_i, v_i\}$ for $i = 1, 2, \ldots, 3l$. In row j and column $a_i - j$, place the triple $T_i + j$ for all $i = 1, 2, \ldots, h(u-1)$ and all $j \in \mathbb{Z}_{3hu}$, where $a_i = A(T_i)$ for $i = 1, 2, \ldots, h(u-1)$. Then we add $3l$ new rows and $3l$ new columns to this array.

Let B_i be a $3hu \times 3$ array; label the rows $0, 1, \ldots, 3hu-1$ and the columns 0, 1, 2. Let $C_i = \{c_{i1}, c_{i2}, c_{i3}\}$. In row j, column j modulo 3 of B_i, place the triple $C_i + j$ for $j = 0, 1, \ldots, 3hu-1$. Let $B = [B_1, B_2, \ldots, B_l]$. B is a $3hu \times 3l$ array. Similarly, we construct a $3l \times 3hu$ array D from R. Let D_i be a $3 \times 3hu$ array; label the columns $0, 1, \ldots, 3hu-1$ and the rows 0, 1, 2. Let $R_i = \{d_{i1}, d_{i2}, d_{i3}\}$. In column j, row j modulo 3 of D_i, place the triple $R_i + j$ for $j = 0, 1, \ldots, 3hu-1$. Let $D = [D_1, D_2, \ldots, D_l]^T$.

Table 5
(S, C, R) and A for a $(1, 2; 3)$-frame of type $9^{11}15^1$

C		R
1, 6, 14		16, 18, 23
3, 13, 26		21, 28, 38
15, 34, 53		36, 56, 79
31, 59, 90		1, 48, 74
29, 60, 94		51, 67, 83
Starter	Adder	Starter + Adder
20, 49, 79	5	25, 54, 84
23, 47, 74	6	29, 53, 80
27, 64, 85	4	31, 68, 89
30, 54, 91	7	37, 61, 98
36, 72, 84	3	39, 75, 87
37, 67, 95	9	46, 76, 5
42, 71, 96	1	43, 72, 97
38, 70, 87	20	58, 90, 8
41, 68, 93	18	59, 86, 12
2, 4, 5	2	4, 6, 7
7, 8, 16	19	26, 27, 35
9, 12, 18	51	60, 63, 69
10, 24, 28	54	64, 78, 82
17, 21, 35	74	91, 95, 10
19, 58, 73	73	92, 32, 47
∞_1, 25, 61	32	∞_1, 57, 93
∞_2, 32, 75	69	∞_2, 2, 45
∞_3, 39, 78	63	∞_3, 3, 42
∞_4, 40, 82	41	∞_4, 81, 24
∞_5, 43, 89	30	∞_5, 73, 20
∞_6, 45, 57	94	∞_6, 40, 52
∞_7, 46, 81	83	∞_7, 30, 65
∞_8, 48, 97	64	∞_8, 13, 62
∞_9, 50, 92	21	∞_9, 71, 14
∞_{10}, 51, 86	98	∞_{10}, 50, 85
∞_{11}, 52, 98	42	∞_{11}, 94, 41
∞_{12}, 56, 76	40	∞_{12}, 96, 17
∞_{13}, 62, 83	86	∞_{13}, 49, 70
∞_{14}, 63, 69	45	∞_{14}, 9, 15
∞_{15}, 65, 80	53	∞_{15}, 19, 34

It is straightforward to verify that the following array constructed from M, B and D gives us a $(1, 2; 3)$-frame of type $(3h)^u(3l)^1$.

M	B
D	

□

Lemma 2.3. *There exist* $(1, 2; 3)$*-frames of types* $12^8 18^1$, $12^9 15^1$, $9^{11}12^1$, $9^{11}15^1$, $9^{12}15^1$, *and* $9^{13}15^1$.

Proof. Intransitive starters (S, C, R) with their corresponding adders A are listed in Tables 2–7. In each case, an intransitive starter (S, C, R) and a corresponding adder A for a $(1, 2; 3)$-frame of type $t^u w^1$ are defined on a set $V = \mathbb{Z}_{tu} \cup \{\infty_1, ..., \infty_w\}$ with groups $\{i, i+u, \dots, i+u(t-1)\}$ for $i = 0, 1, \dots, u-1$ and $\{\infty_1, \dots, \infty_w\}$. □

Intransitive starters and adders can also be used to construct $(2, 4; 3)$-frames. The definitions are similar to those for $(1, 2; 3)$-frames. An intransitive starter over $\mathbb{Z}_{3hu}$ for a $(2, 4; 3)$-frame of type $(3h)^u(3l)^1$ written on the symbol set $\mathbb{Z}_{3hu} \cup \{\infty_1, \infty_2, \dots, \infty_{3l}\}$ with groups $G_i = \{i, i+u, \dots, i+(3h-1)u\}$ for $i = 0, 1, \dots, u-1$, and $G_u = \{\infty_1, \infty_2, \dots,$

Table 6
(S, C, R) and A for a (1, 2; 3)-frame of type $9^{12}15^1$

C		R
1, 6, 14		16, 18, 23
3, 13, 26		21, 28, 38
15, 34, 53		37, 57, 80
31, 65, 99		3, 59, 85
30, 61, 92		51, 67, 83
Starter	Adder	Starter + Adder
18, 33, 73	1	19, 34, 74
19, 54, 63	7	26, 61, 70
20, 37, 57	5	25, 42, 62
21, 50, 87	6	27, 56, 93
27, 70, 88	3	30, 73, 91
29, 59, 80	10	39, 69, 90
32, 71, 93	8	40, 79, 101
38, 77, 105	9	47, 86, 6
46, 79, 104	18	64, 97, 14
40, 62, 90	4	44, 66, 94
41, 68, 97	13	54, 81, 2
44, 69, 107	2	46, 71, 1
42, 74, 101	56	98, 22, 49
43, 76, 106	52	95, 20, 50
2, 4, 5	27	29, 31, 32
7, 8, 11	81	88, 89, 92
9, 17, 23	95	104, 4, 10
10, 16, 25	90	100, 106, 7
∞_1, 22, 89	94	∞_1, 8, 75
∞_2, 28, 81	85	∞_2, 5, 58
∞_3, 35, 56	47	∞_3, 82, 103
∞_4, 39, 83	16	∞_4, 55, 99
∞_5, 45, 94	23	∞_5, 68, 9
∞_6, 47, 82	104	∞_6, 43, 78
∞_7, 49, 103	38	∞_7, 87, 33
∞_8, 51, 102	51	∞_8, 102, 45
∞_9, 52, 98	25	∞_9, 77, 15
∞_{10}, 55, 66	105	∞_{10}, 52, 63
∞_{11}, 58, 100	49	∞_{11}, 107, 41
∞_{12}, 64, 78	41	∞_{12}, 105, 11
∞_{13}, 67, 85	76	∞_{13}, 35 , 53
∞_{14}, 75, 86	98	∞_{14}, 65 , 76
∞_{15}, 91, 95	30	∞_{15}, 13 , 17

$\infty_{3l}\}$ is defined to be a triple (S, C, R), where

(1) $S = \{\{x_i, y_i, z_i\} | i = 1, 2, \ldots, 2h(u-1) - 6l\} \cup \{\{\infty_i, u_i, v_i\}, \{\infty_i, u'_i, v'_i\} | i = 1, 2, \ldots, 3l\}$.
(2) $C = \{\{c_{i1}, c_{i2}, c_{i3}\}, \{c'_{i1}, c'_{i2}, c'_{i3}\} | i = 1, 2, \ldots, l\}$.
(3) $R = \{\{d_{i1}, d_{i2}, d_{i3}\}, \{d'_{i1}, d'_{i2}, d'_{i3}\} | i = 1, 2, \ldots, l\}$,

where $x_i, y_i, z_i, u_j, v_j, u'_j, v'_j \in \mathbb{Z}^*_{3hu}$ for $i = 1, 2, \ldots, h(u-1) - 3l$, $j = 1, 2, \ldots, 3l$, $c_{ij}, d_{ij}, c'_{ij}, d'_{ij} \in \mathbb{Z}^*_{3hu}$ for $j = 1, 2, 3$ and $i = 1, 2, \ldots, l$, and $\mathbb{Z}^*_{3hu} = \mathbb{Z}_{3hu} - \{0, u, \ldots, (3h-1)u\}$. (S, C, R) satisfies the following properties:

(1) $(\bigcup_{i=1}^{h(u-1)-3l} \{x_i, y_i, z_i\}) \cup (\bigcup_{i=1}^{3l} \{u_i, v_i\}) \cup (\bigcup_{i=1}^{l} \{c_{i1}, c_{i2}, c_{i3}\}) = 2 \cdot \mathbb{Z}^*_{3hu}$.
(2) Every element of $\mathbb{Z}^*_{3hu}$ occurs precisely four times as a difference in (S, C, R).
(3) Every triple in C contains three distinct residues modulo 3.
(4) Every triple in R contains three distinct residues modulo 3.

Table 7
(S, C, R) and A for a $(1, 2; 3)$-frame of type $9^{13}15^1$

C		R
1, 6, 17		16, 18, 23
4, 14, 33		21, 28, 38
28, 44, 66		36, 56, 79
25, 59, 93		57, 82, 110
19, 56, 96		2, 49, 84
Starter	Adder	Starter + Adder
24, 84, 99	1	25, 85, 100
30, 45, 63	3	33, 48, 66
31, 64, 94	4	35, 68, 98
32, 50, 98	5	37, 55, 103
34, 76, 105	7	41, 83, 112
35, 60, 107	9	44, 69, 116
38, 62, 106	2	40, 64, 108
40, 71, 112	6	46, 77, 1
49, 92, 115	10	59, 102, 8
48, 86, 110	19	67, 105, 12
41, 68, 95	12	53, 80, 107
43, 80, 116	8	51, 88, 7
46, 77, 113	15	61, 92, 11
47, 79, 111	11	58, 90, 5
2, 3, 5	17	19, 20, 22
7, 8, 11	63	70, 71, 74
9, 15, 21	111	3, 9, 15
10, 18, 22	77	87, 95, 99
16, 27, 57	16	32, 43, 73
12, 20, 29	94	106, 114, 6
23, 73, 82	98	4, 54, 63
∞_1, 36, 97	95	∞_1, 14, 75
∞_2, 37, 85	59	∞_2, 96, 27
∞_3, 42, 100	67	∞_3, 109, 50
∞_4, 51, 108	43	∞_4, 94, 34
∞_5, 53, 114	33	∞_5, 86, 30
∞_6, 54, 109	80	∞_6, 17, 72
∞_7, 55, 101	109	∞_7, 47, 93
∞_8, 58, 72	90	∞_8, 31, 45
∞_9, 61, 83	28	∞_9, 89, 111
∞_{10}, 67, 87	92	∞_{10}, 42, 62
∞_{11}, 69, 90	108	∞_{11}, 60, 81
∞_{12}, 70, 89	57	∞_{12}, 10,29
∞_{13}, 74, 88	27	∞_{13}, 101, 115
∞_{14}, 75, 103	38	∞_{14}, 113, 24
∞_{15}, 81, 102	112	∞_{15}, 76, 97

An adder for (S, C, R) is an injection A: $S \to \mathbb{Z}^*_{3hu}$, such that $(\bigcup_{i=1}^{t} \{x_i + a_i, y_i + a_i, z_i + a_i\}) \cup (\bigcup_{i=1}^{3l} \{u_i + a_{t+i}, v_i + a_{t+i}\}) \cup (\bigcup_{i=1}^{l} \{d_{i1}, d_{i2}, d_{i3}\}) \cup (\bigcup_{i=1}^{3l} \{u'_i + a_{t+3l+i}, v'_i + a_{t+3l+i}\}) \cup (\bigcup_{i=1}^{l} \{d'_{i1}, d'_{i2}, d'_{i3}\}) = 2 \cdot \mathbb{Z}^*_{3hu}$, where $t = 2h(u-1) - 6l$, $a_i = A(\{x_i, y_i, z_i\})$ for $i = 1, 2, \ldots, t$, and $a_{t+i} = A(\{\infty_i, u_i, v_i\})$, $a_{t+3l+i} = A(\{\infty_i, u'_i, v'_i\})$ for $i = 1, 2, \ldots, 3l$.

The proof of the next result is similar to that of Theorem 2.2 and omitted.

Theorem 2.4. *Suppose that there exists an intransitive starter (S, R, C) for a $(2, 4; 3)$-frame of type $(3h)^u(3l)^1$ over $\mathbb{Z}_{3hu}$ and a corresponding adder A for (S, C, R), then there is a $(2, 4; 3)$-frame of type $(3h)^u(3l)^1$.*

Table 8
(S, C, R) and A for a (2, 4; 3)-frame of type $12^5 6^1$

C		R
4, 6, 17		51, 52, 53
14, 18, 22		17, 36, 58
19, 26, 33		2, 16, 33
8, 16, 27		14, 27, 43
Starter	Adder	Starter + Adder
21, 32, 44	12	33, 44, 56
24, 36, 47	52	16, 28, 39
23, 41, 57	56	19, 37, 53
28, 37, 49	29	57, 6, 18
1, 2, 3	46	47, 48, 49
28, 42, 56	16	44, 58, 12
26, 42, 59	47	13, 29, 46
8, 11, 39	13	21, 24, 52
9, 41, 47	42	51, 23, 29
4, 7, 13	34	38, 41, 47
2, 6, 9	2	4, 8, 11
1, 3, 7	11	12, 14, 18
12, 53, 59	49	1, 42, 48
12, 21, 38	1	13, 22, 39
13, 44, 52	19	32, 3, 11
24, 37, 58	59	23, 36, 57
∞_1, 14, 38	3	∞_1, 17, 41
∞_1, 16, 43	6	∞_1, 22, 49
∞_2, 22, 48	21	∞_2, 43, 9
∞_2, 27, 39	7	∞_2, 34, 46
∞_3, 29, 51	33	∞_3, 2, 24
∞_3, 29, 51	37	∞_3, 6, 28
∞_4, 31, 48	38	∞_4, 9, 26
∞_4, 31, 58	23	∞_4, 54, 21
∞_5, 34, 52	4	∞_5, 38, 56
∞_5, 34, 57	57	∞_5, 31, 54
∞_6, 46, 49	18	∞_6, 4, 7
∞_6, 46, 53	41	∞_6, 27, 34
11, 19, 43	48	59, 7, 31
17, 33, 54	9	26, 42, 3
23, 32, 56	36	59, 8, 32
18, 36, 54	43	1, 19, 37

Lemma 2.5. *There exists a* (2, 4; 3)*-frame of type* $12^5 6^1$.

Proof. Let $V = \mathbb{Z}_{60} \cup \{\infty_i \mid i = 1, 2, \ldots, 6\}$. The groups are $\{i, i+5, \ldots, i+55\}$ for $i = 0, 1, \ldots, 4$, and $\{\infty_1, \infty_2, \ldots, \infty_6\}$. An intransitive starter (S, C, R) and adder A are displayed in Table 8. $\square$

3. $\mathrm{KS}_3(v; 1, 2)$ and $\mathrm{KS}_3(v; 2, 4)$

In this section, we complete the spectrum of $\mathrm{KS}_3(v; 1, 2)$ or DR$(v, 3, 2)$-BIBDs by constructing $\mathrm{KS}_3(v; 1, 2)$ for $v \in \{72, 78, 90, 114, 117, 126\}$. We first use starters and adders to construct three $\mathrm{KS}_3(v; 1, 2)$.

A starter S for a $\mathrm{KS}_3(3n; 1, 2)$ defined on $\mathbb{Z}_{3n-1} \cup \{\infty\}$ is a set of n triples

$$S = \{\{x_i, y_i, z_i\} | i = 1, 2, \ldots, n-1\} \cup \{\{\infty, u, v\}\},$$

Table 9
Starters and adders for $\mathrm{KS}_3(v; 1, 2)$, $v = 72, 78, 90$

v	Starter	Adder	Starter	Adder	Starter	Adder
72	{19, 27, 69}	0	{5, 16, 36}	1	{9, 11, 18}	2
	{29, 44, 50}	3	{14, 34, 52}	4	{17, 40, 57}	5
	{8, 33, 42}	6	{3, 6, 61}	9	{15, 22, 67}	11
	{31, 32, 60}	19	{1, 45, 55}	20	{2, 20, 26}	23
	{7, 41, 46}	27	{21, 43, 47}	33	{10, 24, 51}	36
	{35, 58, 70}	37	{23, 28, 56}	38	{0, 25, 39}	42
	{4, 12, 53}	59	{13, 65, 68}	61	{37, 38, 48}	63
	{49, 62, 64}	66	{30, 54, 66}	69	{∞, 59, 63}	43
78	{34, 38, 41}	0	{31, 36, 44}	1	{24, 37, 70}	2
	{19, 30, 39}	3	{23, 32, 72}	4	{45, 73, 74}	5
	{17, 52, 69}	6	{22, 27, 49}	8	{1, 47, 55}	9
	{5, 6, 71}	14	{10, 42, 68}	21	{48, 64, 66}	26
	{12, 33, 54}	28	{13, 29, 51}	30	{15, 21, 60}	34
	{8, 35, 59}	36	{11, 28, 40}	40	{0, 25, 58}	48
	{46, 50, 53}	52	{7, 9, 43}	58	{4, 61, 75}	62
	{3, 18, 65}	66	{16, 26, 63}	67	{2, 20, 56}	68
	{14, 57, 67}	72	{∞, 62, 76}	15		
90	{14, 55, 66}	1	{4, 18, 36}	10	{12, 20, 82}	12
	{29, 39, 65}	14	{3, 51, 71}	20	{5, 41, 60}	22
	{37, 83, 86}	23	{32, 40, 74}	26	{17, 52, 76}	31
	{25, 48, 79}	32	{2, 8, 81}	33	{34, 43, 59}	34
	{23, 45, 47}	39	{10, 78, 87}	41	{1, 6, 53}	44
	{42, 46, 64}	46	{26, 27, 30}	48	{21, 44, 50}	52
	{19, 62, 69}	53	{0, 15, 77}	61	{57, 70, 85}	63
	{31, 63, 80}	64	{13, 33, 58}	68	{56, 67, 84}	69
	{28, 35, 61}	70	{11, 16, 49}	74	{24, 54, 68}	75
	{7, 9, 38}	80	{22, 72, 73}	86	{∞, 75, 88}	66

where $x_i, y_i, z_i \in \mathbb{Z}_{3n-1}$ for $i = 1, 2, \ldots, n-1$, and $u, v \in \mathbb{Z}_{3n-1}$, satisfying

(1) $(\bigcup_{i=1}^{n-1} \{x_i, y_i, z_i\}) \cup \{u, v\} = \mathbb{Z}_{3n-1}$, and
(2) $(\bigcup_{i=1}^{n-1} \{\pm(x_i - y_i), \pm(y_i - z_i), \pm(z_i - x_i)\}) \cup \{\pm(u - v)\} = 2 \cdot (\mathbb{Z}_{3n-1} \backslash \{0\})$.

An adder for S is an injection A: $S \to \mathbb{Z}_{3n-1}$, such that

$$\left(\bigcup_{i=1}^{n-1} \{x_i + a_i, y_i + a_i, z_i + a_i\}\right) \cup \{u + a_n, v + a_n\} = \mathbb{Z}_{3n-1},$$

where $a_i = A(\{x_i, y_i, z_i\})$ for $i = 1, 2, \ldots, n-1$ and $a_n = A(\{\infty, u, v\})$.

Theorem 3.1 (*Colbourn [3] and Lamken and Vanstone [17]*). *If there exists a starter S for a* $\mathrm{KS}_3(3n; 1, 2)$ *and a corresponding adder A*, *then there is a* $\mathrm{KS}_3(3n, 1, 2)$.

We use Theorem 3.1 to construct three new Kirkman squares.

Lemma 3.2. *There exists a* $\mathrm{KS}_3(v; 1, 2)$ *for* $v \in \{72, 78, 90\}$.

Proof. Let $V = \mathbb{Z}_{v-1} \cup \{\infty\}$. Starters and adders for $\mathrm{KS}_3(v; 1, 2)$ for $v \in \{72, 78, 90\}$ are listed in Table 9. □

The remaining three cases for $\mathrm{KS}_3(v; 1, 2)$ are done using the Basic Frame Construction from [17] .

Table 10
Starter and adder for a complementary $KS_3(60; 1, 2)$

Starter	Adder	Starter	Adder	Starter	Adder
{42, 46, 47}	6	{2, 25, 36}	44	{28, 44, 49}	39
{3, 16, 53}	1	{7, 33, 48}	2	{19, 54, 57}	7
{30, 31, 37}	40	{6, 14, 26}	13	{1, 21, 32}	41
{4, 23, 56}	37	{5, 13, 27}	31	{8, 52, 55}	29
{12, 29, 58}	16	{22, 24, 38}	9	{18, 20, 45}	12
{11, 34, 40}	32	{0, 10, 50}	49	{9, 41, 51}	14
{15, 39, 43}	36	{∞, 17, 35}	21		

Theorem 3.3 (*Lamken and Vanstone [17], Basic Frame Construction*). *Suppose that there exists a* $(1, 2; 3)$*-frame of type* $t_1 t_2 \dots t_m$. *If there exists a* $KS_3(t_i + u; 1, 2)$ *containing as a subarray a* $KS_3(u; 1, 2)$ *for all* i, $1 \leqslant i \leqslant m - 1$ *and a* $KS_3(t_m + u; 1, 2)$, *then there is a* $KS_3(v; 1, 2)$. *Furthermore, if there is also a* $KS_3(t_m + u; 1, 2)$ *containing as a subarray a* $KS_3(u; 1, 2)$, *then there exists a* $KS_3(v; 1, 2)$ *containing as a subarray a* $KS_3(u; 1, 2)$, *where* $v = (\sum_{i=1}^{m} t_i) + u$.

Lemma 3.4. *There exist* $KS_3(v; 1, 2)$ *for* $v \in \{114, 117, 126\}$.

Proof. We use the Basic Frame Construction, Theorem 3.3, with $u = 3$ and the $(1, 2; 3)$ frames of types $9^{11}12^1$, $12^8 18^1$, and $12^9 15^1$ from Lemma 2.3. The Kirkman squares required by the construction, a $KS_3(12; 1, 2)$ which contains as a subarray a $KS_3(3; 1, 2)$, a $KS_3(15; 1, 2)$ which contains as a subarray a $KS_3(3; 1, 2)$, a $KS_3(18; 1, 2)$ and a $KS_3(21; 1, 2)$ are provided in [10]. □

We note that both the $KS_3(114, 1, 2)$ and the $KS_3(117; 1, 2)$ constructed by Lemma 3.4 contain as subarrays a $KS_3(3; 1, 2)$.

Next, we construct the remaining two cases for $KS_3(v; 2, 4)$, namely, $v = 60$ and 69.

A $KS_3(v; 1, 2)$, A, is called complementary if there exists a $KS_3(v; 1, 2)$, B, which can be written in the empty cells of A. The next result follows immediately from this definition.

Lemma 3.5 (*Lamken and Vanstone [16] and Lamken [11]*). *If there exists a complementary* $KS_3(v; 1, 2)$, *then there exists a* $KS_3(v; 2, 4)$.

We construct a $KS_3(60; 2, 4)$ by using starters and adders to construct a complementary $KS_3(60; 1, 2)$.

Lemma 3.6. *There exists a* $KS_3(60; 2, 4)$.

Proof. A starter–adder pair for a $KS_3(60; 1, 2)$ is listed in Table 10. Multiplying this starter and adder by -1 gives a starter–adder pair for a complementary $KS_3(60; 1, 2)$. By applying Lemma 3.5, we obtain a $KS_3(60; 2, 4)$. □

The last case for $KS_3(v; 2, 4)$ is done using the Basic Frame Construction from [16,11].

Theorem 3.7 (*Lamken and Vanstone [16] and Lamken [11]*). *Suppose a* $(2, 4; 3)$*-frame of type* $t_1 t_2 \dots t_m$ *exists. If there exists a* $KS_3(t_i + u; 2, 4)$ *containing as a subarray a* $KS_3(u; 2, 4)$ *for* $1 \leqslant i \leqslant m - 1$ *and a* $KS_3(t_m + u; 2, 4)$, *then there is a* $KS_3(v; 2, 4)$ *where* $v = (\sum_{i=1}^{m} t_i) + u$.

Lemma 3.8. *There exists a* $KS_3(69; 2, 4)$.

Proof. By Lemma 2.5, there exists a $(2, 4; 3)$-frame of type $12^5 6^1$. We apply Theorem 3.7 using a $KS_3(15; 2, 4)$ which contains as a subarray a $KS_3(3; 2, 4)$ and a $KS_3(9; 2, 4)$ (see [11,16]). □

We summarize the results of this section as follows.

Lemma 3.9. (1) *There exist* $\mathrm{KS}_3(v; 1, 2)$ *for* $v = 72, 78, 90, 114, 117, 126$.
(2) *There exist* $\mathrm{KS}_3(v; 2, 4)$ *for* $v = 60$ *and* $v = 69$.

4. DNR$(v, 3, 2)$-BIBDs

We construct $(1, 2; 3, v, 1)$-frames for $v = 34, 58, 70, 85, 88, 115, 124, 133, 142$ in this section. This will complete the spectrum of DNR$(v, 3, 2)$-BIBDs.

We start by using starters and adders for bicyclic frames to construct $(1, 2; 3, v, 1)$-frames for $v = 34$ and $v = 58$. Starters and adders for bicyclic frames defined on $\mathbb{Z}_{3n+2} \times \{0, 1\}$ were first described in [3,13]. In this case, the frames are defined on $\mathbb{Z}_v$ for $v = 6n + 4$, and the blocks are developed by adding +2 modulo v. So we modify the definitions for starters and adders for this case. A starter S is a set of $4n + 2$ triples, $S = \{\{x_i, y_i, z_i\} \mid i = 1, 2, \dots, 4n + 2\}$, which are partitioned into four sets S_{11}, S_{12}, S_{21}, and S_{22} with the following properties:

(1) $S_{11} \cup S_{12} = V - \{v/2\}$.
(2) $S_{21} \cup S_{22} = V - \{0\}$.
(3) Let $\Delta = \{\pm(x_i - y_i), \pm(x_i - z_i), \pm(y_i - z_i) \mid i = 1, 2, \dots, 4n + 2\}$. Every element of $V - \{0, v/2\}$ occurs four times in Δ and the element $v/2$ occurs twice in Δ.
(4) Let $d \in V - \{0, v/2\}$. Then d occurs as a difference in four pairs in the triples of S. Suppose these pairs are $\{u_i, v_i\}$ for $i = 1, \dots, 4$. The pairs are related as follows: (i) $\{u_1, v_1\} + j = \{u_2, v_2\}$ for some j odd, (ii) $\{u_3, v_3\} + k = \{u_4, v_4\}$ for some k odd, and (iii) $\{u_1, v_1\} + \ell = \{u_3, v_3\}$ for some ℓ even.

Since the blocks are developed by adding $+2$ modulo v, a corresponding adder $A(S)$ for S is a set of $4n + 2$ elements of V such that $a \equiv 0 \pmod 2$ for $a \in A(S)$. $A(S)$ is partitioned into four sets A_{11}, A_{12}, A_{21}, and A_{22} with the following properties:

(1) For any pair (i, j) with $i, j \in \{1, 2\}$, the elements of A_{ij} are distinct.
(2) $|A_{ij}| = |S_{ij}|$ for $i, j = 1, 2$.
(3) $(S_{11} + A_{11}) \cup (S_{21} + A_{21}) = V - \{v/2\}$.
(4) $(S_{12} + A_{12}) \cup (S_{22} + A_{22}) = V - \{0\}$.

The proof of the starter and adder construction for bicyclic frames over $\mathbb{Z}_{6n+4}$ is similar to that of the construction for bicyclic frames in [3,13,9].

Lemma 4.1. *If there is a starter for a bicyclic* $(1, 2; 3, 6n + 4, 1)$*-frame over* $\mathbb{Z}_{6n+4}$ *with a corresponding adder* A, *then there is a* $(1, 2; 3, 6n + 4, 1)$*-frame*.

Proof. Let $v = 6n + 4$. We use S_{ij} and the corresponding A_{ij} to construct a $(3n + 2) \times (3n + 2)$ array T_{ij}. The blocks are developed by adding $+2$ modulo v. Then the array $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ is a $(1, 2; 3, v, 1)$-frame. □

Lemma 4.2. *There exist* $(1, 2; 3, v, 1)$*-frames for* $v = 34$ *and* 58.

Proof. Starters and adders for the bicyclic frames are listed in Table 11. □

The next two direct constructions use starters and adders. Let G be an additive abelian group of order $3t + 1$. Let $G^* = G - \{0\}$. A starter of order 2 is a partition of G^* into t triples, $S_1, S_2, \dots, S_t$, such that every element in G^* occurs precisely twice in the multiset $\cup_{i=1}^{t} \Delta_i$ where $\Delta_i = \{u - v \mid u, v \in S_i\}$. An adder $A(S)$ for the starter S is a set of t distinct elements $(a_1, a_2, \dots, a_t)$ of G such that $\cup_{i=1}^{t}(S_i + a_i)$ contains $3t$ distinct elements of G. If $a_i \neq 0$ for $i = 1, 2, \dots, t$ and $\cup_{i=1}^{t}(S_i + a_i) = G^*$, then S and $A(S)$ are a starter and adder pair for a $(1, 2; 3, 3t + 1, 1)$-frame.

Theorem 4.3 (*Colbourn et al. [3,2] and Lamken and Vanstone [13]*). *Let G be an abelian group of order* $3t + 1$. *If there is a frame starter S of order* 2 *and an adder* $A(S)$, *then there exists a* $(1, 2; 3, 3t + 1, 1)$*-frame*.

Table 11
Starters and adders for bicyclic (1, 2; 3, v, 1)-frames for $v = 34, 58$

v	S_{11}	A_{11}	S_{12}	A_{12}	S_{21}	A_{21}	S_{22}	A_{22}
34	{14, 2, 13}	24	{0, 6, 27}	20	{32, 20, 1}	24	{22, 6, 29}	2
	{18, 22, 11}	10	{4, 21, 1}	8	{26, 11, 17}	22	{12, 2, 7}	20
	{20, 10, 29}	20	{28, 31, 19}	2	{14, 19, 27}	4	{18, 4, 15}	10
	{8, 16, 3}	8	{24, 25, 23}	26	{30, 13, 31}	16	{24, 28, 3}	16
	{26, 33, 7}	28	{12, 30, 32}	6	{16, 10, 8}	26	{9, 25, 5}	32
			{5, 9, 15}	30	{23, 21, 33}	20		
58	{2, 34, 53}	2	{10, 20, 7}	22	{10, 44, 51}	40	{2, 32, 3}	2
	{0, 4, 39}	8	{46, 26, 37}	52	{14, 54, 29}	14	{4, 40, 13}	4
	{8, 14, 31}	10	{16, 30, 15}	6	{42, 26, 21}	46	{6, 8, 33}	44
	{12, 17, 1}	48	{44, 47, 25}	10	{24, 35, 9}	10	{12, 27, 7}	18
	{38, 41, 51}	18	{22, 35, 49}	16	{48, 57, 23}	52	{38, 45, 17}	10
	{36, 55, 57}	38	{28, 5, 45}	36	{56, 49, 55}	34	{22, 43, 47}	24
	{50, 21, 33}	52	{32, 19, 23}	24	{46, 11, 19}	2	{52, 53, 41}	8
	{24, 56, 6}	40	{40, 42, 52}	32	{36, 50, 16}	4	{18, 30, 34}	52
	{3, 9, 43}	54	{11, 13, 27}	26	{1, 31, 39}	22	{5, 37, 15}	54
	{18, 48, 54}	4					{20, 28, 25}	48

Table 12
Starters and adders for (1, 2; 3, v, 1)-frames for $v = 85, 88$

v	Starter	Adder	Starter	Adder	Starter	Adder
85	{51, 67, 41}	67	{24, 4, 16}	20	{57, 20, 80}	44
	{12, 76, 62}	83	{53, 64, 49}	39	{60, 2, 30}	40
	{31, 48, 70}	17	{40, 35, 73}	26	{68, 26, 28}	54
	{58, 22, 56}	56	{14, 25, 38}	3	{17, 71, 78}	45
	{65, 81, 29}	55	{47, 21, 50}	60	{45, 46, 55}	8
	{32, 52, 74}	15	{63, 66, 6}	28	{9, 13, 44}	84
	{69, 75, 5}	68	{3, 10, 33}	73	{79, 11, 23}	75
	{54, 59, 72}	81	{84, 7, 39}	33	{8, 27, 61}	18
	{77, 83, 36}	79	{1, 15, 42}	4	{18, 19, 37}	38
	{34, 43, 82}	62				
88	{2, 5, 10}	37	{40, 46, 21}	60	{44, 53, 31}	73
	{36, 48, 38}	46	{69, 84, 29}	1	{9, 27, 81}	8
	{80, 13, 24}	67	{61, 85, 33}	76	{77, 16, 30}	16
	{4, 34, 3}	6	{15, 54, 50}	53	{6, 51, 71}	39
	{76, 39, 32}	26	{47, 1, 18}	74	{82, 20, 58}	14
	{73, 74, 11}	68	{35, 37, 66}	29	{23, 26, 59}	2
	{60, 64, 14}	48	{62, 67, 78}	9	{22, 28, 45}	22
	{68, 75, 87}	61	{55, 63, 83}	56	{8, 17, 49}	66
	{42, 52, 79}	72	{57, 70, 12}	87	{72, 86, 19}	59
	{25, 43, 65}	12	{41, 56, 7}	45		

Lemma 4.4. *There exist* (1, 2; 3, v, 1)*-frames for* $v = 85$ *and* 88.

Proof. Starter and adder pairs are listed in Table 12. □

Four cases ($v = 70, 115, 124, 133$) can be done using the Basic Frame Construction in [9].

Theorem 4.5 (*Lamken [9]*).

(1) *Suppose there exists a* (1, 2; 3)*-frame of type* $t_1 t_2 \ldots t_m$. *If there exist* (1, 2; 3, t_i, 1)*-frames for* $i = 1, 2, \ldots, m$, *then there is a* (1, 2; 3, $\sum_{i=1}^{m} t_i$, 1)*-frame*.

(2) *Suppose there exists a* $(1, 2; 3)$-*frame of type* $t_1 t_2 \cdots t_m$. *If there exist* $(1, 2; 3, t_i + 1, 1)$-*frames for* $i = 1, 2, \ldots, m$, *then a* $(1, 2; 3, (\sum_{i=1}^{m} t_i) + 1, 1)$-*frame exists.*

Lemma 4.6. *There exist* $(1, 2; 3, v, 1)$-*frames for* $v \in \{70, 115, 124, 133\}$.

Proof. Since there exist both a $(1, 2; 3)$-frame of type 10^7 (by Lemma 2.1) and a $(1, 2; 3, 10, 1)$-frame (Fig. 2), we can use Theorem 4.5(1) to construct a $(1, 2; 3, 70, 1)$-frame.

By Lemma 2.3, there exist $(1, 2; 3)$-frames of type $9^n 15^1$ for $n = 11, 12$ and 13. Since there exist $(1, 2; 3, t, 1)$-frames for $t = 10$ and $t = 16$ (Theorem 1.5), we can apply Theorem 4.5(2) to construct $(1, 2; 3, v, 1)$-frames for $v = 115, 124, 133$. □

The last case, $v = 142$, is done using the GDD construction from [9].

Theorem 4.7 (*Lamken [9]*). *Let G be a* $\mathrm{GDD}(v; K; G_1, G_2, \ldots, G_m; 0, 1)$. *Suppose there exists a* $(1, 2; 3, k, t)$-*frame for each* $k \in K$ *and a* $(1, 2; 3, t|G_i| + 1, 1)$-*frame for* $i = 1, 2, \ldots, m$. *Then there exists a* $(1, 2; 3, tv + 1, 1)$-*frame.*

Lemma 4.8. *There exists a* $(1, 2; 3, 142, 1)$-*frame.*

Proof. We construct a 5-GDD of type $3^{12}11^1$ by adding a new element to each resolution class of a 4-RGDD of type 3^{12} (which exists [5]). Since a $(1, 2; 3, 5, 3)$-frame exists [7], we can apply Theorem 4.7 to obtain a $(1, 2; 3)$-frame of type $9^{12}33^1$. Now, since we have $(1, 2; 3)$-frames of types 1^{10} and 1^{34} (Theorem 1.5 and Lemma 4.2), we can apply Theorem 4.5(2) to construct the required $(1, 2; 3, 142, 1)$-frame. □

We summarize the results of this section as follows:

Lemma 4.9. *There exist* $(1, 2; 3, v, 1)$-*frames for* $v \in \{34, 58, 70, 85, 88, 115, 124, 133, 142\}$.

5. DR(v, 3, 1)-BIBDs or KS$_3$(v; 1, 1)

In this section, we construct 11 of the 23 open cases for $\mathrm{KS}_3(v; 1, 1)$. Four of these are done using a 'tripling' construction. A 'tripling' construction is described in [1]. (A similar tripling construction is also indicated in [19].) This construction uses cyclically generated $(p, 3, 1)$-BIBDs and a set of three mutually orthogonal Latin squares of order p to construct a $\mathrm{KS}_3(3p; 1, 1)$ where $p = 6t + 1$. We describe an algebraic construction for 'tripling' when $p = 6t + 1$ is a prime and t odd.

Construction 5.1. *Suppose* $p = 6t + 1$ *is a prime and t is odd. Let x be a primitive element in* $\mathrm{GF}(p)$, *and let* C_t *be the multiplicative coset* $\{x^i \mid i \equiv t \pmod 6\}$. *Suppose there exists a block* $B = \{b_1, b_2, b_3\}$ *and an adder a with the following properties*:

(1) *Multiplying B by the elements of* C_0 *gives a* $(p, 3, 1)$ *difference family, D.*
(2) *B and* $B + a$ *contain between them exactly one element from each* C_t, $0 \leqslant t \leqslant 5$.

Then there exists a $\mathrm{KS}_3(3p; 1, 1)$ *or a* $\mathrm{DR}(3p, 3, 1)$-*BIBD.*

Proof. Let $V = \mathrm{GF}(p) \times \{0, 1, 2\}$. D is the set of base blocks in the $(p, 3, 1)$ difference family. Let E be the set of $(p - 1)/2$ elements in these base blocks. We describe four types of base blocks for a $(3p, 3, 1)$-*BIBD*. The blocks are developed modulo $(p, -)$.

Type 1: $\{(czb_1, i), (czb_2, i), (czb_3, i)\}$ for $i = 0, 1, 2$, $c \in C_0$, and $z = x^i$.
Type 2: $\{(z, 0), (xz, 1), (x^2 z, 2)\}$ for $z \in E$.
Type 3: $\{(z, 0), (xz, 1), (x^2 z, 2)\}$ for $z = 0$.
Type 4: $\{(w, 0), (xw, 1), (x^2 w, 2)\}$ for $w \in \mathrm{GF}(p) - \{E \cup \{0\}\}$.

Table 13
Blocks B and corresponding adders a for Construction 5.1

$v=3p$	p	B	a	$B+a$
57	19	{3, 4, 7}	9	{12, 13, 16}
93	31	{5, 12, 16}	14	{19, 26, 30}
129	43	{4, 5, 17}	23	{27, 28, 40}
201	67	{1, 2, 31}	4	{5, 6, 35}
237	79	{7, 17, 30}	14	{21, 31, 44}

The number of resolution classes in a resolvable $(3p, 3, 1)$-BIBD is $p+3t$ where $p=6t+1$. We construct a square array, S, of side $p+3t$ of the form

where A is a $p\times p$ square, X is $p\times 3t$, Y is $3t\times p$, and $\mathscr{E}$ is a $3t\times 3t$ empty square. S will contain a resolution in the rows and an orthogonal resolution in the columns. X is constructed as follows. In the top row, place the blocks of Type 4. The rest of X is constructed by developing the top row mod$(p, -)$, i.e. for $j=0, 1, 2, \ldots, p-1$, adding $(j, 0)$ (mod$(p, -)$) to any block B^* in the top row of X gives the block in row j of X and the same column as B^*. The array Y is constructed similarly by placing the blocks of Type 2 in its first column, and developing through the columns mod$(p, -)$.

The square array A is constructed using adders for the blocks of Types 1 and 3. Each base block of Type 1 is of the form $(mb_1, i), (mb_2, i), (mb_3, i)$ where $m \in C_i$. The adder for this block will be ma. The adder for the block of Type 3 is 0. Note that these adders are all distinct; if s is the integer (mod 6) such that $a \in C_s$, then these adders span $\{0\} \cup C_s \cup C_{s+1} \cup C_{s+2}$. For any $e \in \mathrm{GF}(p)$, the block with adder e is placed in the $(0, -e)$ cell of A; for other rows, the block in the (i, j) cell is obtained by adding $(j, 0)$ to the points in the block in the $(0, j-i)$ cell. (Note that the first column of A contains the blocks of Types 1 and 3 plus their corresponding adders.)

First, we note that the base blocks of Types 1, 3, and 4 form a resolution class (the top row of the array S); we obtain p resolution classes by developing mod$(p, -)$. Each Type 2 block developed mod$(p, -)$ gives a resolution class. So the rows of the array Y give $3t$ more resolution classes, and the rows of the array S provide a resolution of the $(3p, 3, 1)$-BIBD.

The columns of the array form an orthogonal resolution. It is straightforward to verify that the left column of the array S forms a resolution class; once again, we get p resolution classes by developing mod$(p, -)$. Each block of Type 4 developed mod$(p, -)$ also gives a resolution class, so the columns of the array X are also resolution classes. □

Lemma 5.2. *There exist* $\mathrm{KS}_3(v; 1, 1)$ *for* $v = 57, 93, 129, 201$ *and* 237.

Proof. We list blocks B and their corresponding adder a for $v = 57, 93, 129, 201$ and 237 in Table 13. □

We also construct six more designs using starters and adders, see [20,1]. Three of these ($v=69, 105, 117$) are obtained using Theorem 3.4 of [1]. Here, we work over $V = \mathbb{Z}_{6n} \cup \{x, y, z\}$ instead of $W = (\mathbb{Z}_{2n} \times \{0, 1, 2\}) \cup \{\infty, \alpha, \beta\}$, as in [1], and the starter blocks are developed $+3 (\mathrm{mod}\ 6n)$. We describe this construction briefly below; for more details, see [1]. We note the designs constructed by Theorem 3.4 of [1] for $v = 6n+3$ have automorphism groups which partition the rows and columns into orbits of lengths $2n$, n, and 1.

Construction 5.3. *Let* $v = 6n+3$ *and* $V = \mathbb{Z}_{6n} \cup \{x, y, z\}$. *For a* $\mathrm{DR}(v, 3, 1)$-*BIBD we give* $3n+4$ *starter blocks (three of which are short) which are developed* $+3(\mathrm{mod}\ 6n)$. *There is also one extra block*, $\{x, y, z\}$. *The starter blocks are divided into six sets* $S_1, S_2, \ldots, S_6$ *and for* $i = 1, 3, 5, 6$, *let* A_i *denote the adder corresponding to* S_i. S_i *and* A_i *possess the following properties*:

(1) S_1 *contains n blocks*; S_2 *and* S_4 *contain one each*; S_3, S_5 *contain* $n-1$ *each, and* S_6 *contains three blocks*.
(2) *The three blocks in* S_6 *remain invariant when* $3n$ *is added to them*; *thus these three blocks generate just n blocks while all other starter blocks generate* $2n$ *blocks (when developed* $+3\ (\mathrm{mod}\ 6n)$).

Table 14
Starter and adder for a $KS_3(69; 1, 1)$

	Starter	Adder		Starter	Adder
S_1:	{48, 24, 27}	30	S_2:	{15, 1, 2}	
	{46, 64, 37}	63			
	{38, 62, 59}	9	S_3:	{0, 31, 14}	0
	{57, 58, 17}	57		{30, 40, 11}	6
	{54, 34, 23}	42		{21, 10, 50}	27
	{18, 43, 35}	24		{60, 28, 5}	24
	{45, 49, 65}	39		{12, 25, 44}	30
	{3, 52, 47}	6		{36, 7, 29}	3
	{33, 61, 8}	12		{39, 55, 41}	18
	{x, 6, 13}	27		{9, 4, 32}	21
	{y, 22, 20}	15		{42, 16, 26}	12
	{z, 51, 56}	51		{63, 19, 53}	15
S_4:	{60, 52, 56}		S_5(cont.):	{32, 47, 38}	51
				{23, 11, 41}	3
S_5:	{0, 15, 6}	0		{17, 18, 45}	45
	{3, 57, 21}	48		{27, 4, 35}	60
	{30, 49, 28}	63			
	{19, 22, 7}	9	S_6:	{x, 29, 62}	27
	{1, 25, 31}	54		{y, 9, 42}	18
	{13, 59, 20}	57		{z, 10, 43}	6

(3) *For* $i \in \{1, 3, 5, 6\}$, *all elements of* A_i *are congruent to* $0 \pmod 3$.
(4) *No two points in a block from* $S_2 \cup S_4$ *are congruent* (mod 3).
(5) *Each of the following sets of blocks forms a parallel class*: $S_1 \cup S_2 \cup S_3$, $S_5 \cup (S_5 + 3n) \cup S_6$, $(S_1 + A_1) \cup S_4 \cup (S_5 + A_5)$ *and* $(S_3 + A_3) \cup (S_3 + A_3 + 3n) \cup (S_6 + A_6)$.
(6) *Each resolution contains* $3n + 1$ *parallel classes. For the first resolution, developing the blocks in* $S_1 \cup S_2 \cup S_3$ $+3 \pmod{6n}$ *gives* $2n$ *parallel classes and developing those in* $S_5 \cup (S_5 + 3n) \cup S_6$ *produces n more parallel classes. Since no two points in a block from* S_4 *are congruent* (mod 3), S_4 (*developed* $+3 \pmod{6n}$) $\cup \{x, y, z\}$ *provides the final resolution class. For the orthogonal resolution similarly*, $2n$ *parallel classes come from* $(S_1 + A_1) \cup S_4 \cup (S_5 + A_5)$; *n parallel classes come from* $(S_3 + A_3) \cup (S_3 + A_3 + 3n) \cup (S_6 + A_6)$ (*developing these blocks* $+3 \pmod{6n}$) and a last class comes from S_2 (developed ($+3 \pmod{6n}$)) $\cup \{x, y, z\}$.
(7) *For* $i \in \{3, 5, 6\}$, *no two elements of the adder* A_i *are congruent* $\pmod{3n}$. *Also, no two elements of* A_1 *are congruent* $\pmod{6n}$.

Lemma 5.4. *There exists a* $KS_3(v; 1, 1)$ *for* $v \in \{69, 105, 117\}$.

Proof. Apply Construction 5.3 with $n = 11$, 17 and 19. For each v, the starters and corresponding adders are listed in Tables 14–16. □

We construct three more designs using starters and adders, see [20,1]. As in Section 4, we work over $\mathbb{Z}_v$ where $v = 2m$ and $m \equiv 1 \pmod 2$, and we develop the blocks by adding $+2$ modulo v. So the definitions of starters and adders are modified. A starter S for a $KS_3(6n + 3; 1, 1)$ defined on $V = \mathbb{Z}_{6n+2} \cup \{\infty\}$ is a set of $2n + 1$ triples, $T_0, T_1, \ldots, T_{2n}$, where $T_0 = \{\infty, 0, 3n + 1\}$ and $T_i = \{x_i, y_i, z_i\}$ for $i = 1, 2, \ldots, 2n$, which satisfy the following properties:

(1) $\cup_{i=0}^{2n} T_i = V$.
(2) Let $\Delta = \{\pm(x_i - y_i), \pm(x_i - z_i), \pm(y_i - z_i) \mid i = 1, \ldots, 2n\}$. Every element of $\mathbb{Z}_{6n+2} - \{0, 3n + 1\}$ occurs precisely twice in Δ.
(3) Let $d \in \mathbb{Z}_{6n+2} - \{0, 3n + 1\}$. Then d occurs as a difference in two pairs in the triples of S. Suppose these pairs are $\{u_1, v_1\}$ and $\{u_2, v_2\}$. Then the relationship between the pairs is $\{u_1, v_1\} + j = \{u_2, v_2\}$ for some j odd.

Table 15
Starter and adder for a $\mathrm{KS}_3(105; 1, 1)$

	Starter	Adder		Starter	Adder
S_1:	{48, 76, 46}	93	S_2:	{93, 55, 23}	
	{85, 26, 41}	60			
	{90, 84, 95}	21	S_3:	{0, 70, 43}	0
	{22, 31, 100}	33		{52, 32, 74}	66
	{71, 8, 80}	66		{12, 9, 47}	99
	{6, 42, 51}	87		{3, 37, 16}	93
	{67, 79, 82}	48		{10, 29, 5}	96
	{77, 44, 89}	45		{66, 24, 17}	3
	{60, 33, 45}	24		{54, 13, 49}	36
	{15, 1, 92}	18		{19, 83, 86}	21
	{87, 40, 68}	36		{63, 96, 53}	81
	{36, 58, 2}	78		{88, 94, 34}	90
	{18, 7, 20}	72		{14, 50, 98}	12
	{69, 25, 11}	27		{39, 78, 57}	9
	{72, 64, 101}	96		{91, 28, 73}	75
	{x, 30, 61}	90		{59, 65, 38}	27
	{y, 97, 56}	3		{27, 99, 75}	57
	{z, 21, 62}	51		{81, 4, 35}	33
S_4:	{42, 7, 5}		S_5(cont.):	{24, 70, 95}	36
				{36, 16, 56}	96
S_5:	{0, 1, 17}	0		{78, 25, 77}	21
	{54, 91, 101}	27		{66, 82, 89}	60
	{6, 10, 2}	81		{30, 13, 14}	48
	{12, 85, 62}	3		{72, 22, 98}	75
	{48, 88, 35}	6		{60, 79, 32}	93
	{42, 49, 20}	33			
	{84, 58, 92}	66	S_6:	{x, 23, 74}	6
	{96, 4, 8}	69		{y, 39, 90}	33
	{18, 97, 80}	90		{z, 43, 94}	18

The blocks are generated by adding +2 modulo $(6n+2)$. Let $A(S)=(a_0, a_1, \ldots, a_{2n})$ where $a_i \in \mathbb{Z}_{6n+2}$, $a_i \equiv 0 \pmod 2$, and no element of $\mathbb{Z}_{6n+2}$ occurs more than once in $A(S)$. $A(S)$ is an adder for S if $S + A(S) = \{T_i + a_i \mid i = 0, 1, \ldots, 2n\} = V$.

The proof of the next construction is similar to that of the starter-adder constructions in [1,20].

Lemma 5.5. *If there exists a starter S for a* $\mathrm{KS}_3(6n+3; 1, 1)$ *on* $\mathbb{Z}_{6n+2} \cup \{\infty\}$ *and a corresponding adder* $A(S)$, *then there exists a* $\mathrm{KS}_3(6n+3; 1, 1)$.

The next construction is similar to that of [1, Lemma 3.2] except we use 32, 16 or 20 initial blocks instead of 4 or 8 and we use Lemma 5.5.

Lemma 5.6. *There exists a* $\mathrm{KS}_3(v; 1, 1)$ *for* $v = 99$, 147 *and* 183.

Proof. Let $V = \mathbb{Z}_{v-1} \cup \{\infty\}$. These designs all contain the starter block $\{\infty, 0, (v-1)/2\}$ with corresponding adder 0. For $v = 99$, an additional 32 starter blocks and their adders are listed in Table 17. Develop these blocks $+2 \pmod{98}$. For $v = 147$ and 183, an additional 16 or 20 starter blocks are listed in Tables 18 and 19. Multiply these starter blocks and their adders by 1, 81 and 137 (mod 146) for $v = 147$, and by 1, 9 and 81 (mod 182) for $v = 183$. This gives us a total of 49 blocks to be developed $+2 \pmod{146}$ for $v = 147$, and 61 blocks to be developed $+2 \pmod{182}$ for $v = 183$. □

The method in the last few examples can also be used to obtain certain (1, 1; 3)-frames. In [1], (1, 1; 3)-frames of types 6^7, 6^9, 6^{13}, and 6^{19} were obtained this way, and types 6^8, 6^{10}, and $6^{13}12^1$ were obtained by a similar method using intransitive starters and adders.

Table 16
Starter and adder for a $KS_3(117; 1, 1)$

	Starter	Adder		Starter	Adder
S_1:	{90, 97, 85}	84	S_2:	{36, 10, 110}	
	{58, 86, 83}	39			
	{65, 30, 72}	48	S_3:	{0, 103, 5}	0
	{51, 61, 76}	78		{96, 73, 68}	102
	{55, 92, 101}	81		{39, 37, 35}	39
	{113, 66, 42}	75		{111, 108, 63}	48
	{12, 52, 112}	72		{87, 3, 9}	87
	{49, 41, 62}	69		{69, 6, 60}	12
	{20, 102, 21}	60		{93, 105, 78}	72
	{27, 43, 104}	3		{15, 54, 33}	33
	{24, 46, 80}	66		{79, 40, 70}	99
	{57, 91, 8}	36		{109, 31, 64}	84
	{81, 4, 59}	57		{25, 22, 88}	18
	{48, 94, 74}	9		{34, 16, 106}	36
	{45, 100, 56}	63		{28, 7, 1}	9
	{75, 19, 23}	12		{50, 89, 17}	3
	{18, 82, 47}	24		{71, 44, 95}	108
	{x, 99, 67}	33		{32, 26, 14}	54
	{y, 13, 2}	45		{38, 53, 98}	6
	{z, 11, 84}	18		{29, 107, 77}	21
S_4:	{33, 109, 53}		S_5(cont.):	{51, 22, 65}	99
				{27, 7, 29}	36
S_5:	{0, 1, 98}	0		{48, 31, 107}	21
	{24, 55, 14}	75		{45, 37, 89}	51
	{18, 61, 68}	87		{6, 10, 101}	69
	{42, 91, 59}	3		{3, 16, 104}	48
	{30, 82, 83}	9		{9, 28, 92}	45
	{15, 76, 95}	6		{12, 40, 80}	111
	{39, 106, 77}	27			
	{33, 103, 56}	39	S_6:	{x, 17, 74}	9
	{54, 13, 62}	72		{y, 36, 93}	33
	{21, 100, 110}	90		{z, 52, 109}	36

Table 17
Starter blocks and adders for a $KS_3(99; 1, 1)$

Starter	Adder	Starter	Adder	Starter	Adder
{63, 13, 95}	8	{2, 52, 64}	12	{20, 72, 88}	50
{82, 38, 6}	66	{46, 54, 56}	72	{8, 12, 53}	80
{22, 28, 55}	56	{4, 60, 81}	34	{10, 68, 27}	38
{14, 74, 69}	28	{18, 32, 91}	92	{16, 80, 77}	74
{24, 42, 9}	10	{76, 96, 21}	4	{58, 30, 7}	84
{70, 44, 73}	64	{62, 86, 51}	58	{92, 75, 79}	60
{50, 3, 57}	24	{36, 61, 67}	32	{40, 33, 41}	22
{26, 39, 97}	90	{34, 15, 25}	52	{66, 71, 35}	16
{90, 45, 11}	68	{78, 19, 87}	86	{94, 93, 65}	62
{84, 1, 23}	46	{48, 59, 83}	48	{29, 31, 43}	30
{89, 37, 17}	6	{5, 47, 85}	96		

Lemma 5.7. *There exists a* $(1, 1; 3)$-*frame of type* 6^{11}.

Proof. Starter blocks and their corresponding adders (generated over $\mathbb{Z}_{66}$) are listed in Table 20. Develop these blocks by adding even residues (mod 66) to them. □

Table 18
Initial starter blocks and adders for a $KS_3(147; 1, 1)$

Starter	Adder	Starter	Adder	Starter	Adder
{76, 88, 4}	36	{8, 86, 140}	48	{26, 78, 109}	62
{6, 82, 23}	10	{10, 90, 43}	12	{50, 52, 131}	18
{38, 142, 19}	94	{14, 128, 89}	128	{100, 77, 45}	110
{60, 29, 15}	124	{118, 139, 97}	20	{126, 49, 137}	46
{144, 103, 7}	78	{30, 127, 57}	100	{113, 115, 123}	82
{107, 39, 9}	126				

Table 19
Initial starter blocks and adders for a $KS_3(183; 1, 1)$

Starter	Adder	Starter	Adder	Starter	Adder
{46, 54, 138}	64	{96, 126, 48}	28	{16, 20, 31}	100
{62, 64, 51}	82	{8, 24, 69}	76	{28, 60, 77}	80
{103, 109, 23}	132	{26, 146, 79}	16	{32, 90, 1}	128
{140, 154, 117}	74	{92, 142, 101}	98	{170, 131, 39}	6
{160, 3, 7}	178	{118, 11, 177}	18	{148, 33, 67}	102
{110, 15, 83}	78	{159, 161, 5}	44	{114, 107, 121}	94
{133, 59, 89}	160	{18, 78, 151}	12		

Table 20
Starter blocks and adders for a (1, 1; 3)-frame of type 6^{11}

Starter	Adder	Starter	Adder	Starter	Adder
{6, 41, 45}	18	{50, 5, 51}	32	{2, 59, 29}	46
{14, 7, 21}	58	{10, 27, 39}	54	{46, 26, 28}	28
{8, 23, 31}	26	{20, 24, 34}	6	{32, 38, 13}	48
{40, 12, 52}	64	{48, 56, 53}	14	{4, 47, 57}	38
{60, 36, 19}	24	{16, 25, 1}	20	{18, 54, 49}	40
{64, 63, 35}	34	{62, 30, 9}	16	{17, 65, 15}	36
{42, 58, 61}	10	{3, 37, 43}	4		

Our final design is obtained recursively using the following lemma:

Lemma 5.8 (*Colbourn et al.* [1]). *If a* $(1, 1; 3)$*-frame of type* $t_1 t_2 \ldots t_m$ *and a* $\mathrm{TD}(5, g)$ *both exist, then there exists a* $(1, 1; 3)$*-frame of type* $(gt_1)^1 (gt_2)^1 \ldots (gt_m)^1$. *If further,* $u = 1$ *or* 3 *and a* $KS_3(gt_i + u; 1, 1)$ *exists for* $i = 1, 2, \ldots, m$, *then there exists a* $KS_3(v, 1, 1)$ *for* $v = (\sum_{i=1}^{m} gt_i) + u$.

Corollary 5.9. *There exists a* $KS_3(267; 1, 1)$.

Proof. We start with a $(1, 1; 3)$-frame of type 6^{11}, and then we apply Lemma 5.8 with $g = 4$, $u = 3$. This gives a $(1, 1; 3)$-frame of type 24^{11}; since a $KS_3(27; 1, 1)$ exists, we can also construct a $KS_3(267; 1, 1)$. □

We summarize the results of this section as follows.

Lemma 5.10. *A* $KS_3(v; 1, 1)$ *exists for* $v \in \{57, 69, 93, 99, 105, 117, 147, 183, 201, 237, 267\}$.

6. Summary

In Section 3, we constructed designs for all of the possible exceptions listed for $KS_3(v; 1, 2)$ and $KS_3(v; 2, 4)$. Combining Lemma 3.9 and Theorems 1.3 and 1.4, we have the following existence results for these Kirkman squares.

Theorem 6.1. *Let* v *be a positive integer*, $v \equiv 0 \pmod 3$, $v \neq 6, 9$. *There exists a* $\mathrm{KS}_3(v; 1, 2)$. *Furthermore*, *no* $\mathrm{KS}_3(v; 1, 2)$ *exists for* $v = 6$ *or* 9.

Theorem 6.2. *Let* v *be a positive integer*, $v \equiv 0 \pmod 3$. *There exists a* $\mathrm{KS}_3(v; 2, 4)$.

We constructed nine $(1, 2; 3, v, 1)$-frames in Section 4. These include all eight of the open cases listed in Theorem 1.5(i) and one of the missing frames listed in Theorem 1.5(ii). So this completes the spectrum for $\mathrm{DNR}(v, 3, 2)$-BIBDs. We note that the only case where a $\mathrm{DNR}(v, 3, 2)$-BIBD is known to exist and a $(1, 2; 3, v, 1)$-frame remains unknown is for the case $v = 13$. Combining Theorem 1.5 and Lemma 4.9 provides the next existence result.

Theorem 6.3. *Let* v *be a positive integer*, $v \equiv 1 \pmod 3$, $v \geqslant 10$.

(i) *There exists a* $\mathrm{DNR}(v, 3, 2)$*-BIBD*.
(ii) *There exists a* $(1, 2; 3, v, 1)$*-frame except possibly for* $v = 13$.

We also constructed 11 open cases for $\mathrm{KS}_3(v; 1, 1)$. So there are now 12 possible exceptions for the existence of these designs. Combining Theorem 1.6 and Lemma 5.10 gives the following existence result for $\mathrm{KS}_3(v; 1, 1)$.

Theorem 6.4. *Let* v *be a positive integer*, $v \equiv 3 \pmod 6$, $v \neq 9, 15$. *Then there exists a* $\mathrm{KS}_3(v; 1, 1)$ *except possibly for* $v \in N$ *where* $N = \{21, 141, 153, 165, 177, 189, 231, 249, 261, 285, 351, 357\}$. *Furthermore*, *there do not exist* $\mathrm{KS}_3(v; 1, 1)$ *for* $v = 9$ *and* $v = 15$.

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