Error estimates for Hermite interpolation
on spheres

J. Levesley * and Z. Luo

Department of Mathematics, University of Leicester, University Rd, Leicester LE1 7RH, UK
Received 16 January 2001
Submitted by C. Rogers

Abstract

In this paper, we prove convergence rates for spherical spline Hermite interpolation on the sphere
$S^{d-1}$ via an error estimate given in a technical report by Luo and Levesley. The functionals in
the Hermite interpolation are either point evaluations of pseudodifferential operators or rotational
differential operators, the desirable feature of these operators being that they map polynomials
to polynomials. Convergence rates for certain derivatives are given in terms of maximum point
separation.

© 2003 Elsevier Science (USA). All rights reserved.

Keywords: Hermite interpolation; Pseudodifferential operator; Rotational differential operator; Convergence rate

1. Introduction

Let $S^{d-1}$ denote the unit sphere in the Euclidean space $\mathbb{R}^d$, and $C(S^{d-1})$ (respectively
$C^r(S^{d-1})$) be the set of continuous (respectively $r$-times differentiable) functions on $S^{d-1}$.
Let $xy$ denote the usual inner product of $x, y \in \mathbb{R}^d$, and $d(x, y) = \cos^{-1} xy$ be the geodesic
distance between $x, y \in S^{d-1}$. Let $\mu$ be the normalised rotation invariant surface measure
on $S^{d-1}$ and for $f, g \in L_2(S^{d-1})$, the square integrable functions on $S^{d-1}$, set

$$[f, g] = \int_{S^{d-1}} fg \, d\mu.$$
Let $P_m$ be the subspace of spherical polynomials of degree no greater than $m$, $H_m = P_m \cap (P_{m-1})^\bot$, be the set of spherical harmonics of degree $m$, and $d_m = \dim(H_m)$. Choose an orthonormal basis $\{Y_{m,1}, \ldots, Y_{m,d_m}\}$ for $H_m$, $m = 0, 1, \ldots$. Then, every $f \in L^1(S^{d-1})$ (the integrable functions on the sphere) has a Fourier series expansion

\[
\hat{f}_{m,j} = \int_{S^{d-1}} f Y_{m,j} \, d\mu.
\]

Let $h : [-\pi, \pi] \to \mathbb{R}$ be a continuous function, and $X = \{x_1, \ldots, x_N\}$ be a set of distinct interpolation points. The usual interpolation problem is to find a function

\[
s(x) = \sum_{j=1}^N c_j h(d(x, x_j)) + p(x), \quad p \in P_{m-1},
\]

such that

\[
s(x_i) = f(x_i), \quad x_i \in X,
\]

where $c_j$, $j = 1, \ldots, N$, satisfy

\[
\sum_{j=1}^N c_j q(x_j) = 0, \quad \forall q \in P_{m-1}.
\]

The node set $X$ is assumed to be $P_{m-1}$-unisolvent, which means that if $p \in P_{m-1}$ and $p(x) = 0$ for all $x \in X$, then $p = 0$. Furthermore $N \geq \dim(P_{m-1})$.

In this paper we consider the problem of Hermite interpolation. Fix $r \in \mathbb{N}$. Given a set of $N$ functionals $\Lambda = \{L_1, \ldots, L_N\} \subset (C^r(S^{d-1}))'$ (the dual of $C^r(S^{d-1})$), the Hermite interpolation problem is to find a function

\[
s(x) = \sum_{j=1}^N c_j L_j^y(h(d(x, y))) + p(x), \quad p \in P_{m-1},
\]

such that

\[
L_i(s) = L_i(f), \quad i = 1, \ldots, N,
\]

where $c_j$, $j = 1, \ldots, N$, satisfy

\[
\sum_{j=1}^N c_j L_j(q) = 0, \quad \forall q \in P_{m-1}.
\]

We use the notation $L^y$ to denote action on the $y$ variable. Of course, to make the above problem make sense the function $h$ must be $2r$ times continuously differentiable. We also assume that the functional set $\Lambda$ is $P_{m-1}$-unisolvent, i.e., if for some $q \in P_{m-1}$, $L(q) = 0$ for all $L \in \Lambda$, then $q = 0$. 
To ensure that the above interpolation problem is solvable we use a strictly conditionally positive definite function of order $m$ (see [11] for a definition) for $h$. To make the presentation here more straightforward we let $h$ be a strictly positive definite function, the extension to the conditionally positive definite case being elementary. In [19], Schoenberg shows that if $h : [-\pi, \pi] \to \mathbb{R}$ has the absolutely convergent expansion

$$h(t) = \sum_{k=0}^{\infty} h_k P_{(d-3)/2}(\cos t),$$

then $h$ is positive definite if and only if $h_k \geq 0$ for every $k \geq 0$. We will make the stronger assumption: $h_k > 0$ for all $k \geq 0$. The polynomials $P_{(d-3)/2}$ are Gegenbauer polynomials; see Szegő [20].

In this case, we wish to find a function

$$s(x) = \sum_{j=1}^{N} c_j L_j^2(h(d(x, y))),$$

such that

$$L_i(s) = L_i(f), \quad i = 1, \ldots, N.$$
investigated. There it is assumed that information about a fixed derivative is known at a set of points which are becoming dense. The convergence rates for derivatives of this fixed derivative are given. The same approach is adopted here.

The question of exactly what a derivative on the sphere is does not have a simple answer. In this article we restrict ourselves to rotational derivatives and pseudoderivatives (see Section 2.2.2) which have the property that they map polynomials to polynomials. It is natural to consider rotational derivatives since many natural phenomena, from hurricanes and ocean currents to the motion of the tectonic plates, have rotational symmetry. These derivatives are described in greater detail in Section 2.2. Also, as is shown in [17] the results given here can be used to prove convergence rates for the solution of partial differential equations on spheres. The main result of the paper is given in Section 3. We end this introduction by defining the space of functions which we can approximate, and a statement of the error estimate result proved in [11].

As is shown in [13], the variational approach leads us to consider the approximation of the Hilbert space of functions

$$S_h = \left\{ f: \sum_{k=0}^{\infty} \frac{d_k}{h_k} \sum_{j=1}^{k} \left( \hat{f}_{k,j} \right)^2 =: \| f \|_h^2 < \infty \right\}.$$ 

In the terminology of Schaback [18] this is referred to as the native space.

Using the variational approach (see [11] for details) we can prove the following:

**Theorem 1.1.** Let $h \in C^2[-\pi, \pi]$ be strictly positive definite. Further, let $L_1, \ldots, L_N \in (C^r(S^{d-1}))'$, $V = \text{span}[L_1, \ldots, L_N]$, and $s_f$ be the $h$-spline Hermite interpolant to the function $f \in S_h$. Let $L_0 \in (C^r(S^{d-1}))'$. Then the following error bound holds:

$$\| L_0 (f - s_f) \| \leq C(L_0) \| f \|_h,$$

where

$$C(L_0) = \inf_{\Gamma \in L_0 + V} \left\{ \Gamma^x \circ \Gamma^y \left(h(d(x, y))\right) \right\}^{1/2}.$$

**Remark 1.** Convergence rates will be obtained in Section 3 by choosing the functional $L \in V$ so as to provide as small an upper bound on $C(L_0)$ as possible.

2. The coordinate system, differential operators, and point scaling

2.1. Coordinates

We will use the nonstandard spherical polar coordinates defined by

$$x_2 = \cos \theta_1 \sin \theta_2,$$

$$x_3 = \cos \theta_1 \cos \theta_2 \sin \theta_3,$$

$$\vdots$$
\[ x_{d-1} = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{d-2} \sin \theta_{d-1}, \]
\[ x_1 = \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{d-2} \cos \theta_{d-1}, \]
\[ x_d = \sin \theta_1. \] (1)

The coordinates lie in the set \( M^d := [-\pi, \pi]^{d-2} \times [-\pi/2, \pi/2] \), and we denote by \( x(\theta) \) the point on the sphere corresponding to the coordinates \( \theta \in M^d \). For \( \lambda \in \mathbb{R} \), let \( \lambda \theta \) be the point with spherical coordinates \( \lambda \theta_1, \ldots, \lambda \theta_{d-1} \).

**Lemma 1.** For \( \theta = (\theta_1, \ldots, \theta_{d-1}) \), \( \phi = (\phi_1, \ldots, \phi_{d-1}) \in M^d \),
\[ x(\theta) x(\phi) = 1 - \sum_{i=1}^{d-1} (1 - \cos(\theta_i - \phi_i)) \prod_{j=1}^{d} \cos \theta_j \cos \phi_j, \]
where the empty product is interpreted as 1.

**Proof.** The proof follows by induction on dimension. The result is trivial for \( d = 2 \). For the purposes of this proof let us denote the inner product of \( x, y \in S^{d-1} \) by \( x^d y^d \). Then, for \( \theta' = (\theta_1, \ldots, \theta_{d-1}, \theta_d) \), \( \phi' = (\phi_1, \ldots, \phi_{d-1}, \phi_d) \in M^{d+1} \),
\[ x^{d+1}(\theta') x^{d+1}(\phi') = x^d(\theta) x^d(\phi) - \cos \theta_1 \ldots \cos \theta_{d-1} \cos \phi_1 \ldots \cos \phi_{d-1} \]
\[ + \cos \theta_1 \ldots \cos \theta_d \cos \phi_1 \ldots \cos \phi_d + \cos \theta_1 \ldots \sin \theta_d \cos \phi_1 \ldots \sin \phi_d \]
\[ = x^d(\theta) x^d(\phi) - \cos \theta_1 \ldots \cos \theta_{d-1} \cos \phi_1 \ldots \cos \phi_{d-1} \]
\[ + \theta_1 \ldots \cos \theta_{d-1} \cos \phi_1 \ldots \cos \phi_{d-1} \cos(\theta_d - \phi_d) \]
\[ = x^d(\theta) x^d(\phi) - \left( 1 - \cos(\theta_d - \phi_d) \right) \prod_{j=1}^{d-1} \cos \theta_j \cos \phi_j, \]
and the result follows using the inductive hypothesis. \( \square \)

For future reference we require scale factors and base vectors for this coordinate system. Let \( e_i \) be a unit vector in the direction of increasing \( x_i \), and let \( r = \sum_{i=1}^{d} x_i e_i \) be the position vector in Cartesian coordinates. Then, using (1), the scale factor
\[ s_i = \left| \frac{\partial r}{\partial \theta_i} \right| = \prod_{j=1}^{i-1} |\cos \theta_j|, \] (2)
and the base unit vector
\[ \hat{\theta}_i = \frac{1}{s_i} \frac{\partial r}{\partial \theta_i}. \] (3)

### 2.2. Differential operators

In this subsection we discuss the types of differential operator we will allow in the interpolation process. One property these operators will enjoy is that they map polynomials to polynomials, and this will prove important for our convergence analysis.
2.2.1. Rotational differentiation

In $\mathbb{R}^d$ it is easy to make sense of the idea of directional differentiation. The corresponding idea on the sphere constructs derivatives by limiting a rotation rather than a translation, since polynomials are invariant under rotation. To formally define this process requires the introduction of a one parameter subgroup of the appropriate rotation group, and a limiting process using the orbit of this subgroup on the sphere. To this end let $A$ be an element of the Lie algebra of $O(d)$, the rotation group of $\mathbb{R}^d$, and $\sigma = \exp A$. Then the differential operator $A_\sigma$ is defined by

$$A_\sigma f(x) = \lim_{t \to 0} \frac{f(\exp(tA)x) - f(x)}{t}, \quad x \in S^{d-1}.$$

We observe that $A_\sigma : P_k \to P_k$ for all $k \geq 0$. For $\sigma = (\sigma_1, \ldots, \sigma_j) \in (O(d))^j$ write $A_\sigma = A_{\sigma_j} \circ \cdots \circ A_{\sigma_1}$ to denote a $j$th order rotational derivative.

For the analysis that follows we need only bound such derivatives. To do this we note that, at $x \in S^{d-1}$, the rotation $\sigma$ is in a direction $n$, tangent to $S^{d-1}$. Then, the rotational derivative at $x$ is just a multiple (depending on $\sigma$) of the directional derivative $D_n$, in the direction $n$, multiplied by the sine of $\alpha$, the angle between $x$ and the axis of rotation of $\sigma$. Thus, to bound the rotational derivative at $x$ we need only bound directional derivatives at $x$.

Let $r$ be the radial coordinate in $\mathbb{R}^d$, with scale factor $s_r$ and base unit vector $\hat{r}$. Then $(r, \theta_1, \ldots, \theta_{d-1})$ is an orthogonal curvilinear coordinate system for $\mathbb{R}^d$. In this coordinate system the gradient operator is

$$\nabla = \frac{\hat{r}}{s_r} \frac{\partial}{\partial r} + \sum_{i=1}^{d-1} \frac{\theta_i}{s_i} \frac{\partial}{\partial \theta_i}.$$

Since $n$ is tangent to the sphere we can write $n = n_1 \hat{\theta}_1 + \cdots + n_{d-1} \hat{\theta}_{d-1}$. Thus

$$D_n = n \cdot \nabla = \sum_{i=1}^{d-1} \frac{n_i}{s_i} \frac{\partial}{\partial \theta_i}.$$

Thus, to bound the rotational derivative we need only bound the derivatives with respect to the spherical coordinates, as the scale factors $s_i$ are all bounded below in the region in which we will be interested. We will bound these derivatives in the next section.

We remark here that it is trivial to write any directional derivative as a rotational derivative.

2.2.2. Pseudodifferential operators

A pseudodifferential operator $\Gamma$ not only maps $P_k$ to $P_k$, but maps the space $H_k$ into itself. In this paper we will consider the operators such that, if $Y \in H_k$ then $\Gamma Y = \gamma_k Y$. Thus, the action of $\Gamma$ on $L^2(S^{d-1})$ is prescribed by the sequence of eigenvalues $\gamma_0, \gamma_1, \ldots$. The set $\{\gamma_0, \gamma_1, \ldots\}$ is called the symbol of $\Gamma$. Formally, if

$$f = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} a_{kj}(f) Y_{kj}$$
then
\[ \Gamma f = \sum_{k=0}^{\infty} \gamma_k \sum_{j=1}^{\infty} a_{kj}(f)Y_{kj}. \]

Of course, we require summability conditions above, but in our setting the above series will always converge in the appropriate sense. If \(|\gamma_k| \leq Ck^r\) for some fixed constant \(C\) then \(\Gamma\) is a pseudodifferential operator of order \(r\).

If the symbol of \(\Gamma\) has coefficients which decay with enough rapidity then the action of \(\Gamma\) can be realised as convolution (see, e.g., [4, p. 63] for a definition) with a function
\[ k_\Gamma = \sum_{k=0}^{\infty} \frac{d_k \gamma_k}{P(d-k)/2} P(d-k)/2. \]

The Laplace–Beltrami operator, \(\Delta\), for the sphere is an example of such a pseudodifferential operator. In this case the eigenvalues \(\gamma_k = -k(k+d-2)\). Thus, \(\Delta\) is a pseudodifferential operator of order 2.

We note here that the image of a zonal function under \(\Gamma\) is another zonal function. Thus, the Hermite interpolation problem with \(\Gamma\) as the differential operator and \(h\) as the basis function reduces to the usual interpolation problem with \(\Gamma x \circ \Gamma y(h(d(x,y)))\) as the basis function; see Section 3.2.

For more information on pseudodifferential operators see Freeden [4].

### 2.3. Point scaling

We see, from (2), that as \(\theta_i \to 0\) the scale factor \(s_i \to 1\), so that the angular coordinate behaves more and more like a distance coordinate as it approaches 0. Thus, scaling in \(M^d\) leads to a similar scaling on \(S^{d-1}\). Let \(\Lambda : S^{d-1} \to S^{d-1}\) denote this scaling operator.

**Lemma 2.** Fix \(\theta, \phi \in M^d\), with \(-\pi/4 < \theta_i, \phi_i < \pi/4\). Then, if \(\lambda \leq 1\),
\[ C_1 \lambda \leq d(x(\lambda \theta), x(\lambda \phi)) \leq C_2 \lambda, \]
where the constants \(C_1\) and \(C_2\) are independent of \(\lambda\).

**Proof.** In the following we will use the simple inequality \(t^2/4 \leq 1 - \cos t \leq t^2/2\), which is valid for sufficiently small \(t\). Recall also that \(\cos(d(z, y)) = z y\).

We prove the upper bound in (4) first. Since all the subtracted terms in the right-hand side of (1) are positive, we have
\[
x(\lambda \theta) x(\lambda \phi) \geq \cos(\lambda(\theta_1 - \phi_1)) - \sum_{i=2}^{d-1} (1 - \cos(\lambda(\theta_i - \phi_i)))
\]
\[ \geq 1 - \frac{\lambda^2}{2} \left( \sum_{i=1}^{d-1} (\theta_i - \phi_i)^2 \right) = 1 - A \frac{\lambda^2}{4}, \]
with the obvious definition of the positive constant \(A\). Hence,
\[ x(\lambda \theta) x(\lambda \phi) \geq \cos(\sqrt{A\lambda}), \]

and the upper bound follows as the cosine is a decreasing function on \([0, \pi]\).

For the lower bound in (4) we first consider the case \(\theta_1 \neq \phi_1\). For then

\[ x(\lambda \theta) x(\lambda \phi) \leq \cos(\lambda(\theta_1 - \phi_1)), \]

and the result follows because cosine is decreasing on \([0, \pi]\). If \(\theta_1 = \phi_1\), then \(\theta_i \neq \phi_i\) for some \(i = 2, \ldots, d - 1\). Thus,

\[ x(\lambda \theta) x(\lambda \phi) \leq 1 - \frac{\lambda^2}{4} (\theta_i - \phi_i)^2 \prod_{j=1}^{i-1} \cos \lambda \theta_j \cos \lambda \phi_j \leq 1 - B \lambda^2, \]

for some positive constant \(B\). Hence,

\[ x(\lambda \theta) x(\lambda \phi) \leq \cos(\sqrt{B}/2), \]

and the result follows as cosine decreases on \([0, \pi]\). \(\Box\)

Let \(\Theta := \{\theta^1, \ldots, \theta^{\mu_n}\} \subset M^d\), where \(\mu_n := \dim(P_n)\), be a set of interpolation points, and denote by \(\Theta_k\) the set \(\lambda \theta^1, \ldots, \lambda \theta^{\mu_n}\). Let \(p_i^k[\Theta]\), \(i = 1, \ldots, \mu_n\), be Lagrange polynomials for interpolation on \(x(\theta^i)\), i.e., \(p_i^k[\Theta](x(\lambda \theta^i)) = \delta_{ij}\). Write \(p_i[\Theta] = p_i^1[\Theta]\).

The following lemma tells us that we can select, from our interpolation points, a subset of points which may be obtained by \(\lambda\) scaling a set of points for which the derivatives, with respect to the spherical coordinates, of the Lagrange polynomials are bounded. To prove the lemma it is useful to define

\[ B_\epsilon(y) = \{x: d(x, y) \leq \epsilon\}. \]

**Lemma 3.** Let \(V\) be a fixed neighbourhood of \(e_1\) and \(Z \subset V\) be a set of points. Let

\[ \rho := \max_{y \in V} \min_{x \in Z} d(x, y). \]

Then, for sufficiently small \(\rho\), there exists a set \(\Phi = \{\phi^1, \ldots, \phi^{\mu_n}\} \subset M^d\), and \(\lambda > C\rho\), for some fixed \(C > 0\), such that the set \(\{x(\lambda \psi^i): i = 1, \ldots, \mu_n\} \subset Z\) is \(P_n\)-unisolvent. Furthermore, all derivatives of \(p_i[\Phi]\), with respect to spherical coordinates, up to order \(k \geq 0\) are bounded by a constant.

**Proof.** First of all we choose a set of points \(\Psi = \{\psi^1, \ldots, \psi^{\mu_n}\}\) such that \(x(\Psi)\) is a unisolvent set for Lagrange interpolation on the sphere. Then, all derivatives of \(p_i[\Psi]\), up to order \(k\), are bounded on the sphere. By continuity there exists \(\epsilon > 0\) such that all derivatives of \(p_i[\Theta]\), up to order \(k\), are bounded, for \(\Theta = \{\theta^1, \ldots, \theta^{\mu_n}\}\) with \(x(\theta^i) \in B_\epsilon(x(\psi^i)), i = 1, \ldots, \mu_n\).

By the previous lemma, if \(d(x(\lambda \theta^i), x(\lambda \psi^i)) = \epsilon\) then, for \(\lambda < 1\),

\[ d(x(\lambda \theta^i), x(\lambda \psi^i)) \geq \epsilon C_1 / C_2. \]
So, if we choose $\lambda$ such that $\epsilon \lambda C_1 / C_2 > \rho$, we see that $B_\rho(x(\lambda\psi^i)) \subset \Lambda(B_\epsilon(x(\psi^i)))$, for $i = 1, \ldots, \mu_n$.

By the definition of the point separation $\rho$, we know that there exists $x_i \in Z$, $i = 1, \ldots, \mu_n$, such that $d(x_i, x(\lambda\psi^i)) < \rho$. Thus, $x_i = x(\lambda\phi^i) \in B_\epsilon(x(\psi^i))$, for some $\phi^i \in M^d$, $i = 1, \ldots, \mu_n$. The result follows due to the argument in the first paragraph of the proof.

Bos and de Marchi [1] show that as $\lambda \to 0$ the problem of interpolating from $P_n$ has, as a limiting case, the problem of interpolation on $M^d$ using algebraic polynomials of the form $q_1(\theta_2, \ldots, \theta_d) + \theta_1 q_2(\theta_2, \ldots, \theta_d)$, where $\deg(q_1) \leq n$, $\deg(q_2) \leq n - 1$, and $\theta_d = \theta_2 + \cdots + \theta_{d-1}$. This is a fixed problem, independent of $\lambda$, and so, for $\Psi = \{\psi^1, \ldots, \psi^{\mu_n}\} \subset M^d$ we are able to infer the boundedness of the numbers $p^i_\lambda[\Psi](x(\lambda\phi))$, for fixed $\phi \in M^d$, as $\lambda \to 0$. Let $\bar{p}_1[\Psi] := \lim_{\lambda \to 0} p^i_\lambda[\Psi]$. We generalise this result in the next lemma, where, for $\alpha \in \mathbb{N}_0^{d-1}$,

\[ D^\alpha = \frac{\partial}{\partial \theta_1^{d-1}} \cdots \frac{\partial}{\partial \theta_d^{d-1}}. \]

We note that $p_1^i[\Psi]$ is an infinitely differentiable function of all its variables and $\lambda$.

**Lemma 4.** Let $\Psi = \{\psi^1, \ldots, \psi^{\mu_n}\} \subset M^d$, and $x(\psi^1), \ldots, x(\psi^{\mu_n})$ be a unisolvent set on $S^{d-1}$. Then, for all $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| = k$,

\[ \lim_{\lambda \to 0} \lambda^k D^\alpha \left( p_1^i[\Psi](x(\lambda\theta)) \right) = D^\alpha \bar{p}_1[\Psi](\theta), \quad \theta \in M^d. \]

**Proof.** We need only prove the result for $\partial / \partial \theta_1$, the full result following by induction. Since $p_1^i[\Psi]$ is an infinitely differentiable function of all its variables and $\lambda$,

\[ \lim_{\lambda \to 0} \frac{\partial}{\partial \theta_1} \left[ p_1^i[\Psi](x(\lambda\theta)) \right] = \frac{\partial}{\partial \theta_1} \bar{p}_1[\Psi](\theta). \]

The result follows since

\[ \frac{\partial}{\partial \theta_1} \left[ p_1^i[\Psi](x(\lambda\theta)) \right] = \lambda \frac{\partial p_1^i[\Psi]}{\partial \theta_1}(x(\lambda\theta)). \]

3. Convergence rate

In this section we find a convergence rate for Hermite interpolation using rotational differential operators and then use that result to prove a convergence rate for pseudodifferential operators. The letter $C$ will be used to denote a constant which need not have the same value each time it appears.

3.1. Rotational differential operators

**Lemma 5.** Let $\phi \in \mathcal{C}^{2r}[-\epsilon, \epsilon]$ be even. Then, $\phi$ has the following expansion
\[ \phi(\theta) = \sum_{k=0}^{r-1} a_k (1 - \cos \theta) + R_{2r}(\phi, \theta), \]

where

\[ \left| \frac{d^j}{d\theta^j} R_{2r}(\phi, \theta) \right| \leq C \theta^{2r-j}, \]

the constant C depending on \( \phi \), but not on \( \theta \).

**Proof.** We first define the function \( g \) by

\[ g(\sqrt{2} \sin(\theta/2)) = \phi(\theta). \]

Then, since \( g(t) = \phi(2 \sin(t/\sqrt{2})) \), \( g \) is even and \( 2r \)-times continuously differentiable on \( [-\delta, \delta] \), where \( \delta = 2 \sin^{-1}(\epsilon/\sqrt{2}) \),

\[ g(t) = \sum_{k=0}^{r-1} a_k t^{2k} + \overline{R}_{2r}(g, t), \]

(5)

for some coefficients \( a_k, k = 0, \ldots, r - 1 \). Here \( \overline{R}_{2r}(g, t) \) is the Taylor series remainder term.

Making the change of variable \( t = \sqrt{2} \sin(\theta/2) \) in (5) gives

\[ \phi(\theta) = \sum_{k=0}^{r-1} a_k (2 \sin^2(\theta/2))^{k} + \overline{R}_{2r}(g, \sqrt{2} \sin(\theta/2)), \]

which gives the required representation for \( \phi \), since \( 2 \sin^2(\theta/2) = (1 - \cos \theta) \), where \( \overline{R}_{2r}(g, \sqrt{2} \sin(\theta/2)) \).

If we differentiate (5) \( j \) times with respect to \( t \) we see that

\[ \frac{d^j g}{dt^j}(t) = \sum_{k=0}^{r-1} \frac{(2k)!}{(2k-j)!} a_k t^{2k-j} + \frac{d^j}{dt^j} \overline{R}_{2r}(g, t), \]

and by the uniqueness of Taylor series expansions,

\[ \overline{R}_{2r-j}(g^{(j)}, t) = \frac{d^j}{dt^j} \overline{R}_{2r}(g, t). \]

Thus, for \( t \in [-\delta, \delta] \),

\[ \left| \frac{d^j}{dt^j} \overline{R}_{2r}(g, t) \right| \leq C \| g^{(2r)} \|_{\infty, [-\delta, \delta]} t^{2r-j}, \]

(6)

for some constant C independent of \( g \) and \( t \).

Now,

\[ \frac{d^j}{d\theta^j} R_{2r}(\phi, \theta) = \sum_{l=0}^{j} \eta_l(\theta) \frac{d^l}{d\theta^l} \overline{R}_{2r}(g, \sqrt{2} \sin(\theta/2)), \]
for some trigonometric polynomials \( p_l, l = 0, \ldots, j \). Using (6) we see that

\[
\left| \frac{d^l}{d\theta^l} R_{2r}(\phi, \theta) \right| \leq C \| g^{(2r)} \|_{\infty, [-\delta, \delta]} \sum_{l=0}^{j} \sin(\theta/2)^{2r-l} \\
\leq C \| g^{(2r)} \|_{\infty, [-\delta, \delta]} \theta^{2r-j}.
\]

\[
\blacksquare
\]

**Corollary 3.1.** For any \( \sigma \in (O^d)^k \) and \( \gamma \in (O^d)^l \)

\[
| A_\sigma \circ A_\gamma (R_{2r}(\phi, d(x, y))) | \leq C d(x, y)^{2r-k-l}.
\]

**Proof.** As remarked in Section 2.2 the size of any rotational derivative is bounded by the directional derivative in the direction of the rotation. The maximum directional derivative, at \( x \), of a radial function centred at \( y \), is in the direction of \( x \) to \( y \) (or the opposite direction). Thus, the largest directional derivative of \( R_{2r}(\phi, d(x, y)) \) is differentiation with respect to \( d(x, y) \). Therefore,

\[
| A_\sigma \circ A_\gamma (R_{2r}(\phi, d(x, y))) | \leq C \left| \frac{d^{k+l}}{d\theta^{k+l}} R_{2r}(\phi, \theta) \right|_{d(x, y)};
\]

the constant \( C \) depends on \( \sigma \) and \( \gamma \). The result follows from the previous lemma. \( \blacksquare \)

**Lemma 6.** Let \( x_1, \ldots, x_{\mu_n} \in S^{d-1} \), and \( p_1, \ldots, p_{\mu_n} \) be the unique polynomials such that

\[ p_i(x_j) = \delta_{i,j}, \quad i, j = 1, \ldots, \mu_n. \]

Then

(a) for every \( x \in S^{d-1} \) and \( \sigma \in (O^d)^k \),

\[
T_\sigma := A_\sigma - \sum_{i=1}^{n} p_i \delta_{x_i} \circ A_\sigma
\]

annihilates \( P_n \).

(b) If, furthermore, \( \gamma \in (O^d)^l \), then

\[
T_{\sigma, \gamma} := A_\gamma \circ T_\sigma
\]

also annihilates \( P_n \).

**Proof.** For \( i = 1, \ldots, \mu_n \), we have

\[
T_\sigma p_i = A_\sigma p_i - \sum_{j=1}^{n} p_j A_\sigma p_i(x_j).
\]

If we now evaluate at \( x_i, l = 1, \ldots, \mu_n \), in the last equation we see that
\[ T_\sigma p_i(x_l) = A_\sigma p_i(x_l) - \sum_{j=1}^{n} p_j(x_l) A_\sigma p_i(x_j) = A_\sigma p_i(x_l) - A_\sigma p_i(x_l) = 0, \]

since \( p_j(x_l) = \delta_{jl} \). Hence the polynomial \( T_\sigma p_i \) is zero on the unisolvent set \( x_1, \ldots, x_{\mu_n} \) and is thus zero. Since the polynomial set \( \{ p_i, i = 1, \ldots, \mu_n \} \) spans the set \( P_n \), it follows that \( T_\sigma \) annihilates \( P_n \).

The second result is trivial since, for any \( p \in P_n \),

\[ T_\sigma p = A_\gamma (T_\sigma p) = 0, \]

by statement (a).

\[ \square \]

We now state and prove our main theorem:

**Theorem 3.2.** Let \( h \in C^{2r}[-\epsilon, \epsilon] \), for some \( \epsilon > 0 \), with \( r \in \mathbb{N} \), be even, and positive definite on \( S^{d-1} \). Fix \( x \in S^{d-1} \) and let \( \Omega = \{ y \in S^{d-1} : d(x, y) \leq \epsilon/2 \} \). Assume that the interpolation distribution set \( \Lambda = \{ L_1, \ldots, L_N \} \) contains a subset \( \Lambda_1 = \{ L_i = \delta_x \circ A_\sigma : i = 1, \ldots, Q \} \), for some fixed \( \sigma \in (O^d)^k \), with \( k \leq r \), such that the supporting set \( X = \{ x_1, \ldots, x_Q \} \subseteq \Omega \) satisfies the following inequality:

\[ \max_{y \in \Omega} \min_{x_i \in X} \{ d(y, x_i) \} \leq \rho. \]

Let \( f \) be from the native space \( S_h \), and let

\[ s_f(x) := \sum_{j=1}^{N} \alpha_j L_j^\prime \{ h(d(x, y)) \}, \]

subject to the interpolation conditions

\[ L_i(s) = L_i(f), \quad i = 1, \ldots, N. \]

Then for \( \gamma \in (O^d)^l \) with \( k + l \leq r \), we have

\[ \left| [A_\gamma \circ A_\sigma (f - s_f)](x) \right| \leq C \rho r^{-k-l}. \quad (7) \]

**Proof.** If \( \rho \) is sufficiently small, using Lemma 3, we can select a subset \( \Phi = \{ \phi_1, \ldots, \phi_{\mu-1} \} \subseteq M^d \) and \( \lambda > C \rho \) such that \( \{ x(\lambda \phi^l), i = 1, \ldots, \mu r-1 \} \subset X \) is \( P_{r-1} \)-unisolvent.

Without loss of generality we may assume that \( x = e_1 = (1, 0, \ldots, 0) \). Let

\[ \Gamma = L_0 + \sum_{i=1}^{\mu r-1} c_i L_i, \]

where we have set \( L_0 = \delta_x \circ A_\gamma \circ A_\sigma \). Let \( c_i = -\delta_x \circ A_\gamma (p_i^\prime \{ \Phi \} \cdot e_1) \). Then, since rotational derivatives are bounded in size by directional derivatives, using Lemma 4 we have
Now, by Lemma 6, \( \Gamma \) annihilates \( P_{r-1} \). Employing the expansion from Lemma 5, and using the fact that \( \cos d(x, y) = xy \), we see that

\[
\left\{ \Gamma^{x} \circ \Gamma^{y} \left( h(d(x, y)) \right) \right\}^{1/2} \\
= \left\{ \Gamma^{x} \circ \Gamma^{y} \left( \sum_{k=0}^{r-1} a_k (1 - xy)^k + R_{2r} (h, d(x, y)) \right) \right\}^{1/2}
\]

\[
= \left\{ \mu_{r-1}^{x} \circ \mu_{r-1}^{y} (R_{2r} (h, d(x, y))) \right\}^{1/2}
\]

\[
= \left\{ \begin{array}{c}
\sum_{i=1}^{\mu_{r-1}} c_i L^x_i \circ L^y_i (R_{2r} (h, d(x, y))) \\
\sum_{i, j=1}^{\mu_{r-1}} c_i c_j L^x_i \circ L^y_j (R_{2r} (h, d(x, y)))
\end{array} \right\}^{1/2}.
\]

We now use bound (8) and Corollary 3.1 to show that

\[
\left\{ \Gamma^{x} \circ \Gamma^{y} \left( h(d(x, y)) \right) \right\}^{1/2} \\
\leq \left\{ \begin{array}{c}
C \rho^{2r-2(k+l)} + \sum_{i=1}^{\mu_{r-1}} \rho^{-l} \rho^{2r-(k+l)-k} + \sum_{i, j=1}^{\mu_{r-1}} \rho^{-2l} \rho^{2r-2k} 
\end{array} \right\}^{1/2}
\]

\[
\leq C \rho^{r-k-l},
\]

noting that \( d(x_i, x_j), d(x_i, x_j) \leq C \rho \) for some constant, \( C \), independent of \( 1 \leq i, j \leq \mu_{r-1} \).

The result follows from application of Theorem 1.1. \( \square \)

**Remark 2.** We remark here that if we set \( k = l = 0 \) in the above theorem we are in the case of Lagrange interpolation using a positive definite function, and we recover the result of Light and von Golitschek [8].

### 3.2. Pseudodifferential operators

In this section we shall assume that the strictly positive definite basis function \( h \in C^{2r}[-\pi, \pi] \) has the Gegenbauer expansion

\[
h(t) = \sum_{k=0}^{\infty} h_k P_k^{(d-3)/2}(\cos t),
\]

where
\[
\sum_{k=0}^{\infty} k^{2r} h_k P_{k}^{(d-3)/2}(1) < \infty.
\]  

Let \( \Gamma \) be a pseudodifferential operator of order \( p \) with symbol \([\gamma_0, \gamma_1, \ldots] \). Assume that the functional set \( \Lambda \) contains a subset \( X = \{\delta_{x_1} \circ \Gamma, \ldots, \delta_{x_Q} \circ \Gamma\} \). Then, for \( L_1 = \delta_{x_i} \circ \Gamma \) and \( L_2 = \delta_{x_j} \circ \Gamma, 1 \leq i, j \leq N \), the quantity

\[
L_{x_1} \circ L_{x_2} \{ h(d(x, y)) \} = L_{x_1} \circ L_{x_2} \left\{ \sum_{k=0}^{\infty} h_k P_{k}^{(d-3)/2}(xy) \right\} = \delta_{x_i} \circ \delta_{x_j} \left\{ \sum_{k=0}^{\infty} \gamma_k^2 h_k P_{k}^{(d-3)/2}(xy) \right\} = \delta_{x_i} \circ \delta_{x_j} \left( \tilde{h}(d(x, y)) \right),
\]

where

\[
\tilde{h}(t) = \sum_{k=0}^{\infty} \gamma_k^2 h_k P_{k}^{(d-3)/2}(\cos t),
\]

which is also strictly positive definite.

Because \( \Gamma \) is a pseudodifferential operator of order \( p \), \( \gamma_k \leq C k^p \) for some fixed constant \( C \). It is thus easy to show, via (9), that \( \tilde{h} \in C^{2r-2p}[-\pi, \pi] \). Also, if \( f \in S_h \), then \( \Gamma f \in S_h \). We may now use Theorem 3.2, with \( k = l = 0 \), and consider the \( h \)-spline interpolation problem. This gives us

**Theorem 3.3.** Let

\[
h(t) = \sum_{k=0}^{\infty} h_k P_{k}^{(d-3)/2}(\cos t), \quad h_k > 0,
\]

be the Gegenbauer expansion for \( h \in C^{2r}[-\pi, \pi] \). Assume further that

\[
\sum_{k=0}^{\infty} k^{2r} h_k P_{k}^{(d-3)/2}(1) < \infty
\]

holds. Fix \( x \in S^{d-1} \) and let \( \Omega = \{ y \in S^{d-1}: d(x, y) \leq \epsilon/2 \} \). Assume that the interpolation distribution set \( \Lambda = \{L_1, \ldots, L_N\} \) contains a subset

\[
\Lambda_1 := \{L_i = \delta_{x_i} \circ \Gamma: i = 1, \ldots, Q\},
\]

for some fixed pseudodifferential operator \( \Gamma \), of order \( p \). Further suppose that the supporting set \( X := \{x_1, \ldots, x_Q\} \subset \Omega \) satisfies the following inequality:

\[
\max_{y \in \Omega} \min_{x_i \in X} \{d(y, x_i)\} \leq \rho.
\]

Let \( f \) be from the native space \( S_h \), and let

\[
s_f(x) := \sum_{j=1}^{N} \alpha_j L_j(h(d(x, y))).
\]
subject to the interpolation conditions

\[ L_i(s) = L_i(f), \quad i = 1, \ldots, N. \]

Then,

\[ \left| \left[ \Gamma(f - sf) \right](x) \right| \leq C \rho^{r-p}. \quad (10) \]

In [17] Morton and Neamtu give error estimates for the solution of pseudodifferential equations using collocation. They show how to invert a pseudodifferential operator on appropriate Sobolev spaces, and then employ interpolation results to prove their estimate.

Acknowledgment

The authors would like to thank the referee for suggestions which have both made the paper more readable and more internally consistent.

References