Decomposition of a scalar operator into a product of unitary operators with two points in spectrum

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\begin{abstract}
We consider products of unitary operators with at most two points in their spectra, 1 and $e^{i\alpha}$. We prove that the scalar operator $e^{i\gamma}I$ is a product of $k$ such operators if $\alpha\left(1 + 1/(k - 3)\right) \leq \gamma \leq \alpha\left(k - 1 - 1/(k - 3)\right)$ for $k \geq 5$. Also we prove that for $e^{i\alpha} \neq -1$, only a countable number of scalar operators can be decomposed in a product of four operators from the mentioned class. As a corollary we show that every unitary operator on an infinite-dimensional space is a product of finitely many such operators.
\end{abstract}

1. Introduction

A well known problem on eigenvalues of sums of Hermitian matrices was solved towards the end of the last century (see e.g. [7]). A similar problem concerning the spectrum of a product of unitary matrices was also solved, see [1,2]. There exists an interesting connection between these two problems. For a collection of $k$ pairwise commuting Hermitian matrices $A_1, A_2, \ldots, A_k$, it is easy to show...
\[ e^{i(A_1+A_2+\cdots+A_k)} = e^{i\tilde{A}_1} e^{i\tilde{A}_2} \cdots e^{i\tilde{A}_k}. \] (1)

However the corresponding property for non-commuting \( A_j \),
\[ e^{i(A_1+A_2+\cdots+A_k)} = e^{i\tilde{A}_1} e^{i\tilde{A}_2} \cdots e^{i\tilde{A}_k}, \] (2)

where \( \tilde{A}_j \) is a matrix unitarily similar to \( A_j \), is not so trivial to prove. The validity of the formula (2) was noticed in [1] for sufficiently small norms of \( A_j \). The main question of the article is: how big can the norms of \( A_j \) be if we take instead of \( A_j \) multiples of orthogonal projections \( P_i, P_i^2 = P_i^* = P_i \)? The following theorem was proved in [10]:

**Theorem 1.** A scalar operator \( \lambda I \) is a sum of \( k \) orthogonal projections if and only if \( \lambda \in \Sigma_k \), where \( \Sigma_k \subset \mathbb{R} \),
\[ \Sigma_k \supseteq \left[ \frac{k-\sqrt{k^2-4k}}{2}, \frac{k+\sqrt{k^2-4k}}{2} \right] \] and contains all points of the sequences
\[
\begin{align*}
a_0 &= 0, \quad a_i = \phi(a_{i-1}), \quad i = 1, 2, \ldots, \quad \phi(x) = 1 + \frac{1}{k-1-x}, \\
b_0 &= 1, \quad b_1 = \phi(b_{i-1}), \quad i = 1, 2, \ldots, \\
k - a_j, \quad k - b_j \text{ for } j = 0, 1, 2, \ldots
\end{align*}
\]

Let us denote by \( U^\alpha \) the set of unitary operators on a separable Hilbert space \( H \) whose spectra lie in \( \{1, e^{i\alpha}\} \):
\[ U^\alpha = \{ X \in L(H) | XX^* = X^* X = I, \sigma(X) \subset \{1, e^{i\alpha}\} \} \]
and consider the equation
\[ ul = U_1 U_2 \cdots U_k. \quad U_j \in U^\alpha. \] (3)

Let \( \Omega^\alpha_k \) be the set of all unitary \( u \in \mathbb{C} \) for which a solution of Eq. (3) exists. We shall prove in Section 3 that \( \Omega^\alpha_k \supseteq \{ e^{i\alpha x} | x \in \left[ 1 + \frac{1}{k-3}, k - 1 - \frac{1}{k-3} \right] \} \) for \( k > 4 \). Whence for big enough \( k, \alpha \), the sets \( \{ e^{i\alpha x} | x \in \Sigma_k \} \) and \( \Omega^\alpha_k \) coincide with the unit circle \( \mathbb{T} \). In contrast to the equality \( \Omega^\pi_4 = \mathbb{T} \) proven in [8], we shall show that \( \Omega^\alpha_k \) is a discrete set for \( \alpha \neq \pi \). Using the described results, we conclude in the last section that every unitary operator is a product of finitely many operators from \( U^\alpha \).

Returning to the property (2), we note that the sum \( A_1 + A_2 + \cdots + A_k \) does not depend on the order of the summands. But since \( U_1 U_2 = U_2^* (U_2 U_1) U_2 \) for any unitary matrices \( U_1 \) and \( U_2 \), then for every permutation \( \omega \), there exist Hermitian matrices \( \hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k \) such that \( \hat{A}_j \) is similar to \( \tilde{A}_j \) for \( j = 1, \ldots, k \)
\[ e^{i\hat{A}_{\omega(1)}} e^{i\hat{A}_{\omega(2)}} \cdots e^{i\hat{A}_{\omega(k)}} = e^{i\tilde{A}_1} e^{i\tilde{A}_2} \cdots e^{i\tilde{A}_k}. \]

Hence the existence of (2) does not depend on the order. In Section 2 we show that (2) holds for a wide class of Hermitian matrices when
\[ \sum_{1}^{k} \|A_j\| \leq 2\pi. \] (4)

We also give examples of matrices for which both (2) and (4) do not hold.

In what follows we shall denote the trace of a matrix \( A \) by \( \text{tr} A \), the identity and zero \( n \times n \) matrices by \( I_n \) and \( 0_n \), respectively. The diagonal matrix will be denoted by \( \text{diag}(a_1, \ldots, a_n) \). Similarity we shall denote by \( \cong \).

2. Unitary reflections and dilations

We start with products of two unitary operators. Let us denote by \( R^\psi_\alpha \) the \( 2 \times 2 \) matrix
\[
\begin{pmatrix}
1 - \psi + \psi e^{i\alpha} & (e^{i\alpha} - 1) \sqrt{\psi - \psi^2} \\
(e^{i\alpha} - 1) \sqrt{\psi - \psi^2} & \psi + e^{i\alpha} - \psi e^{i\alpha}
\end{pmatrix}
\]
for $\psi \in [0, 1]$. Its eigenvalues are 1 and $e^{i\alpha}$. So $R^\alpha_\psi \subset U^\alpha$. Since for any $U \in U^\alpha$, the operator $(U - I)/(e^{i\alpha} - 1)$ is an orthoprejection, then the spectral theorem for a pair of orthoprejections [14] can be reformulated for a pair of unitary operators from $U^\alpha_1$ and $U^\alpha_2$, respectively. This means that a pair of unitary elements $u_1$ and $u_2$ of an associative algebra with the identity $e$ which satisfies the relations
\[(u_1 - e)(u_1 - e^{i\alpha_1}) = 0 \quad \text{and} \quad (u_2 - e)(u_2 - e^{i\alpha_2}) = 0\]
has only one- and two-dimensional irreducible representations in unitary operators. The one-dimensional representations are given by $u_1 \rightarrow 1, e^{i\alpha_1}, u_2 \rightarrow 1, e^{i\alpha_2}$ and all two-dimensional unitary representations up to unitary equivalence can be defined by the formulas
\[
\pi_\psi(u_1) = R^\alpha_0, \quad \pi_\psi(u_2) = R^\alpha_\psi \quad \text{for} \quad \psi \in (0, 1).
\]

The following Lemma about the spectrum of the product of two unitary operators can be derived from [3].

**Lemma 2.** Let $U_1 \in U^\alpha_1$ and $U_2 \in U^\alpha_2$. If the number $u$ does not belong to the set $\{1, e^{i\alpha_1}, e^{i\alpha_2}, e^{i(\alpha_1 + \alpha_2)}\}$, then
\[
u \in \sigma(U_1U_2) \iff e^{i\alpha_1}e^{i\alpha_2}/u \in \sigma(U_1U_2).
\]

**Proof.** Using the spectral theorem, it suffices to prove the Lemma for the case of $2 \times 2$ matrices $R^\alpha_0$ and $R^\alpha_\psi$, and this case is verified directly. □

Note that in the notations of Lemma 2, if $e^{i\phi} \in \sigma(U_1U_2)$, then for some $\psi$, $e^{i\phi}$ is an eigenvalue of $R^\alpha_0R^\alpha_\psi$. Whence $\phi$ and $\psi$ satisfy the relation
\[
\psi = \frac{(e^{i\phi} - e^{i\alpha_1}e^{i\alpha_2})(1 - e^{i\phi})}{e^{i\phi}(1 - e^{i\alpha_1})(1 - e^{i\alpha_2})}. \tag{7}
\]

**Remark 3.** For $0 \leq \alpha_1 \leq \alpha_2 < 2\pi$, $\alpha_1 + \alpha_2 < 2\pi$, we have $\psi \in [0, 1]$ if and only if $\phi \in [0, \alpha_1]$ or $\phi \in [\alpha_2, \alpha_1 + \alpha_2]$. Or for $\alpha_1 + \alpha_2 \geq 2\pi$, $\psi \in [0, 1]$ if and only if $\phi \in [\alpha_2, 2\pi]$ or $\phi \in [\alpha_1 + \alpha_2 - 2\pi, \alpha_1]$.

This gives us the first example of matrices for which the equality (2) does not hold. For example, let
\[
A_1 = \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix} \quad \text{and} \quad A_2 = B \begin{pmatrix} \psi & \sqrt{\psi - \psi^2} \\ \sqrt{\psi - \psi^2} & 1 - \psi \end{pmatrix}, \quad \pi < \beta < 2\pi.
\]

Then the similarity $A_1 + A_2 \approx\text{diag}(\beta(1 + \sqrt{\psi}), \beta(1 - \sqrt{\psi}))$ holds and so for $\psi = \pi^2 / \beta^2$,
\[
e^{i\beta} = e^{i(\beta + \pi)}I_2 = e^{i(\beta - \pi)}I_2.
\]

At the same time $e^{i\beta} \approx e^{i\beta_2} \approx R_0^\beta$. So if
\[
e^{i(\beta - \pi)}I_2 = R_0^\beta R_\psi,
\]
and hence $e^{i(\beta - \pi)} \in \sigma(R_0^\beta R_\psi)$ for some $\psi \in [0, 1]$, then by Remark 3, $\beta - \pi \in [\beta, 2\pi]$ or $\beta - \pi \in [2\beta - 2\pi, \beta]$. Therefore the equality $e^{i(\beta - \pi)}I_2 = U_1U_2$ has no solution in matrices belonging to $U^\beta$.

Another example comes from products of unitary reflections, which are matrices $U_j \in U^\psi$ with the property $\text{rank}(U_j - I) = 1$. According to Fillmore’s result [6], a Hermitian matrix $A$ is a sum of orthoprejections $P_1, P_2, \ldots$ if and only if $A \geq 0$, $\text{tr}A \in \mathbb{Z}$ and $\text{tr}A \geq \text{rank}A$.

Moreover one can choose the orthoprejections $P_1, P_2, \ldots, P_{\text{tr}A}$ so that $A = P_1 + P_2 + \cdots + P_{\text{tr}A}$ with $\text{tr}P_j = 1$ for every $j$. Note that $e^{i(\pi P_j)}$ is a unitary reflection. It was proved in [13] that any unitary $n \times n$ matrix $U$, $\det U = \pm 1$ is a product of at most $2n - 1$ reflections. And later in [5] the authors proved that if $W = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_n})$ with $\phi_j > 0, j = 1, \ldots, n$ and
\[ \phi_1 + \cdots + \phi_n = \pi, \]  
then \( W \) is not a product of \( 2n - 2 \) reflections. So for \( n \) odd, let Eq. (8) hold and \( A \) be a Hermitian matrix \( \text{diag}(\phi_1, \ldots, \phi_n + (n - 1)\pi) \). Then \( \text{tr} A = n\pi \) whence \( A \) is a multiple of a sum of \( n \) rank-one orthoprojections. As we mentioned above the matrix \( W = e^{iA} \) cannot be decomposed into a product of \( n \) unitary reflections.

Let \( \alpha_j > 0, j = 1, \ldots, n \). The following theorem reformulates Fillmore’s result for products of dilations, i.e. matrices \( U_j \) with two eigenvalues and rank \( (U_j - I) = 1 \).

**Theorem 4.** A unitary matrix \( U = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_n}) \) with \( \sum_1^n \alpha_j = k\alpha, k \geq s = \text{rank}(U - I_n) \) and \( s\alpha \leq 2\pi \) is a product of \( k \) dilations from \( U^\alpha \).

**Proof.** We may assume that \( U - I_n \) is invertible because \( U \) is a product of elements from \( U^\alpha \) if and only if \( U \oplus I \) is. The basic case is for \( k = n \) since if \( k > n \), then for some \( j \), we have \( \alpha_j > \alpha \) and putting

\[ W = \text{diag}(1, 1, \ldots, e^{i\alpha_j}, 1, \ldots, 1), \]

we have that \( \det W^*U = (k - 1)\alpha \) and all conditions of the theorem are fulfilled with the smaller \( k \).

Let \( k = n \). Then there exists \( J_1, J_2 \in \mathbb{N} \) such that \( \alpha_{J_1} \leq \alpha \leq \alpha_{J_2} \). Since for \( \psi = (e^{i\alpha_{J_1}} - e^{i(\alpha_{J_1} + \alpha_{J_2})})/(e^{i\alpha_{J_1}} (1 - e^{i\alpha}) (1 - e^{i(\alpha_{J_1} + \alpha_{J_2} - \alpha)})) \),

\[ \text{diag}(e^{i\alpha_{J_1}}, e^{i\alpha_{J_2}}) \approx R_\psi^\alpha \text{diag}(1, e^{i(\alpha_{J_1} + \alpha_{J_2} - \alpha)}) \]

then there exists \( W \approx \text{diag}(R_\psi^\alpha, 1, \ldots, 1) \) such that \( W^*U \) has the same eigenvalues as those of \( U \) except that \( e^{i\alpha_{J_1}} \), respectively \( e^{i\alpha_{J_2}} \), are replaced by 1, respectively \( e^{i(\alpha_{J_1} + \alpha_{J_2} - \alpha)} \). This reduces the size of the decomposing matrix. Repeating the process, we obtain at last a \( 2 \times 2 \) matrix with determinant \( e^{i(2\alpha)} \) which is the product of \( R_\psi^\alpha \) and \( R_\psi^\alpha \) for some \( \psi_1 \) and \( \psi_2 \). \( \square \)

For decompositions of special dilations we can weaken the inequality on the sum of \( \alpha_j \) in Theorem 4.

**Lemma 5.** Let \( \alpha_1, \phi > 0, \alpha_1 \leq \alpha \leq \pi, \gamma \geq \alpha \) and, for some integer \( m \), \( \phi + m\gamma = \alpha_1 + m\alpha \). Then the matrix \( W = \text{diag}(e^{i\gamma}I_m, e^{i\phi}) \) is the product of \( m \) dilations from \( U^\alpha \) and a dilation from \( U^{\alpha_1} \).

**Proof.** This follows from a straightforward application of formula (7), because for the notation \( L_{\psi_j} = I_{j-1} \oplus R_{\psi_j}^\gamma \oplus I_{m-j} \), the chain

\[ \begin{array}{c}
L_0^{\alpha_1} \oplus I_{m-1} \rightarrow \text{diag}(e^{i\psi_1}, e^{i(\alpha_1 + (\alpha - \gamma))}) \oplus I_{m-1} \rightarrow \cdots \\
\times L_{\psi_1}^\gamma \rightarrow \text{diag}(e^{i\psi_1}I_{m}, e^{i(\alpha_1 + s(\alpha - \gamma))}) \oplus I_{m-s} \times L_{\psi_{1+1}}^\gamma \rightarrow \cdots \times L_{\psi_m}^\gamma \rightarrow \text{diag}(e^{i\psi_1}I_{m}, e^{i\phi})
\end{array} \]

leads to transformations of \( 2 \times 2 \) matrices with eigenvalues \( 1, \alpha, \alpha_1 + j(\alpha - \gamma) \) and the existence of \( \psi_{j+1} \) comes from \( \alpha + \alpha_1 + j(\alpha - \gamma) \leq 2\alpha \leq 2\pi \) and \( \alpha_1 + (j + 1)(\alpha - \gamma) \geq 0 \). \( \square \)

**Corollary 6.** Let \( \alpha \leq \pi, 0 \leq \phi \leq \alpha \leq \gamma \) and for some integer \( m > 0, \phi + m\gamma = (m + 1)\alpha \). Then the matrix \( W = \text{diag}(e^{i\gamma}I_m, e^{i\phi}) \) is the product of \( m + 1 \) dilations from \( U^\alpha \).

### 3. Decompositions of a scalar operator

As was mentioned in the introduction our basic case is a product of operators with two points in their spectra. We are going to describe some properties of \( \Omega_k^\alpha, \alpha > 0 \). In the proofs of the following
Lemmas we shall construct various solutions of (2) for multiples of orthoprojections $\alpha P_1, \alpha P_2, \ldots, \alpha P_k$ such that $\alpha (P_1 + P_2 + \cdots + P_k) = \gamma I$, where $\gamma \in \mathbb{R}$.

**Lemma 7.** The set $\Omega_\gamma^\alpha$ has the following properties.

(1) For any $u \in \Omega_\gamma^\alpha$, $0 \leq \arg(u) \leq k\alpha$;
(2) If $u \in \Omega_\gamma^k$, then $e^{ik\alpha}/u \in \Omega_\gamma^\alpha$;
(3) $\Omega_\gamma^\alpha = \{u \in \mathbb{T} \mid u \in \Omega_\gamma^{2\pi - \alpha}\}$
(4) $\Omega_\gamma^\alpha \cap \{u \in \mathbb{C} \mid 0 < \arg(u) < \alpha\} = \emptyset$ for $k\alpha < 2\pi$;
(5) $\Omega_\gamma^1 = \{1, e^{i\alpha}, \ldots, e^{i(3\alpha/2)}\}, \Omega_\gamma^2 = \{1, e^{i\alpha}, e^{i(2\alpha)}, e^{i(3\alpha)}\}$ for $m \leq k$

and if $2\pi/3 < \alpha < 4\pi/3$, then $\Omega_\gamma^2$ contains both numbers $e^{i(3\alpha/2)}$ and $-e^{i(3\alpha/2)}$.

**Proof.** The first statement is trivial if $k\alpha \geq 2\pi$. So suppose that the equality (3) holds for some $u \in \mathbb{T}$ and $k\alpha < 2\pi$. Every unit vector $\vec{h}$ is an eigenvector of $U_1 U_2 \cdots U_k$, by definition of $U^\alpha$. Let us define $\vec{h}_0 = \vec{h}, \vec{h}_1 = U_{k-1} U_{k-2} \cdots U_h \vec{h}$ and denote by $H$ the finite dimensional Hilbert space $\{\vec{h}_0, \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_k\}$. Let $\vec{U}_i$ be a unitary pseudo-reflection (dilation) acting on $U_1 U_2 \cdots U_k$ and the eigenvector corresponding to the eigenvalue $\alpha$ be $\vec{h}_{k-1}, \vec{h}_{k-2} = 0$, otherwise let $\vec{U}_i$ be the identity matrix. By construction, $\vec{U}_i \vec{h}_i = \vec{h}_{i-1}$. Hence, $\vec{U}_1 \cdots \vec{U}_k \vec{h}_0 = \vec{h}_0$ and so $u \in \sigma(\vec{U}_1 \cdots \vec{U}_k)$. Let $u = e^{i\theta}, 0 \leq \phi < 2\pi$. To obtain that $\phi \leq k\alpha$ we apply the interlace theorem for eigenvalues of a unitary matrix perturbed by a pseudo-reflection [4]. It states that eigenvalues of the two unitary matrices $W_1$ and $W_2 = W_1 I$, where $U$ is a pseudo-reflection, are interlaced on the unit circle. Using the theorem for eigenvalues 1, $\theta_1, \theta_2 \in \sigma(\vec{U}_k \cdots \vec{U}_1)$, where $0 < \theta_1 < \theta_2 < 2\pi$, we have that $0 < \theta_1 < \alpha < \theta_2$. Moreover $\theta_1 \leq 2\pi$. Then for eigenvalues 1, $\theta_1, \theta_2, \theta_3 \in \sigma(\vec{U}_k \cdots \vec{U}_1)$, where $0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$, we have that $0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$. Hence $\theta_1 < \theta_3 < \theta_2$. By induction, we conclude that arguments of eigenvalues of $\vec{U}_1 \cdots \vec{U}_k$ are less or equal to $k\alpha$. Therefore $\phi < k\alpha$.

The second is true, since for an operator $u \in U^\alpha$, the operator $e^{i\alpha} U^\alpha$ belongs to $U^\alpha$. Thus the decomposition $u = U_k U_{k-1} \cdots U_1$ leads to the decomposition

$e^{i\alpha}/u = e^{i\alpha} U_1^* e^{i\alpha} U_2^* \cdots e^{i\alpha} U_k^*$.

Conjugating both sides of the equality (3), we obtain the third property.

To prove the fourth property we suppose that for some $\phi$, $0 < \phi < \alpha$, there exist unitary operators $U_1, \ldots, U_k \in U^\alpha$ such that $e^{i\phi} I = U_1 U_2 \cdots U_k$ and $U_1$ is not the identity. Then the operator $U_1 U_2 \cdots U_k = e^{i\phi} U_k^*$ has the eigenvalue $e^{i(2\pi + \phi - \alpha)}$. But $2\pi + \phi - \alpha > (k-1)\alpha$. Thus we obtain a contradiction to the proof of the first statement.

In the fifth statement the set $\Omega_\gamma^\alpha$ is equal to $\{1, e^{i\alpha}\}$ by definition of $U^\alpha$. It is obvious that $\Omega_\gamma^\alpha \subset \{1, e^{i(m\alpha)}\}$ for $m \leq k$. So let us show that $\Omega_\gamma^\alpha \subset \{1, e^{i\alpha}, e^{i(2\alpha)}, e^{i(3\alpha)}\}$. In view of the property (3) it suffices to consider the case $0 < \alpha < \pi$. Suppose $u = e^{i\phi} \in \{1, e^{i\alpha}, e^{i(2\alpha)}\}, 0 < \phi < 2\pi$ and $u \in \sigma(U_1 U_2)$ for some unitary operators $U_1, U_2 \in U^\alpha$. By Lemma 2, we have that $e^{i(2\pi - \phi)} \in \sigma(U_1 U_2)$. If $U_1 U_2 = u$, then $e^{i(2\pi - \phi)} = e^{i\phi}$. Whence $\phi = \alpha$ or $\phi = \alpha + \pi$. The first equality contradicts our assumption and the second denies the truth of the property (1) of Lemma 7 since $\phi \neq 2\pi$ and hence $\alpha + \pi > 2\alpha$.

Assume now that $e^{i\gamma} I = U_1 U_2 U_3$ for some non-scalar unitary operators $U_1, U_2, U_3 \in U^\alpha, 0 < \alpha < \pi, 0 < \gamma < 2\pi$. Then $e^{i\gamma} U_3^* = U_1 U_2$. So

$\sigma(U_1 U_2) = \{e^{i\gamma}, e^{i(\gamma - \alpha)}\}.$

If $\gamma \neq 0, 2\alpha$, then by Lemma 2, $e^{i(2\alpha - \gamma)} \in \sigma(U_1 U_2)$. This defines completely the points of $\Omega_\gamma^\alpha$. Really, if $e^{i(2\alpha - \gamma)} = e^{i\gamma}$, then $\gamma = \alpha$ or $\gamma = \pi + \alpha$. The number $e^{i(\pi + \alpha)}$ cannot be in the spectra $\sigma(U_1 U_2)$ by
Remark 3 because \( \alpha + \pi > 2\alpha \) and \( \gamma \neq 2\alpha \). So \( e^{i(2\alpha - \gamma)} = e^{i(\gamma - \alpha)} \), i.e. \( \gamma = 3\alpha/2 \) or \( \gamma = 3\alpha/2 \pm \pi \). If \( 3\alpha/2 < \pi \), then \( 3\alpha/2 + \pi > 3\alpha \). Whence \( 3\alpha/2 + \pi \notin \Omega_{3\alpha/2}^\alpha \) by property (1).

Let us show that \( e^{i(3\alpha/2)} \in \Omega_{3\alpha/2}^\alpha \) for every \( \alpha \leq \pi \) and \( -e^{i(3\alpha/2)} \in \Omega_{3\alpha/2}^\alpha \) for \( 2\pi/3 < \alpha \leq \pi \). By Theorem 4, there exist \( 2 \times 2 \) matrices \( U_1 \) and \( U_2 \) from \( U^\alpha \) such that

\[
U_1 U_2 = \text{diag}(e^{i(3\alpha/2)}, e^{i(\alpha/2)}).
\]

Putting \( U_3 = \text{diag}(1, e^{i\alpha}) \), we obtain \( e^{i(3\alpha/2)} I_2 = U_1 U_2 U_3 \). Using Theorem 4 again, one can find \( 2 \times 2 \) matrices \( V_1 \) and \( V_2 \) from \( U^\alpha \) such that

\[
V_1 V_2 = \text{diag}(e^{i(3\alpha/2 - \pi)}, e^{i(\pi + \alpha/2)}).
\]

for \( 2\pi/3 < \alpha \leq \pi \). Thus \( -e^{i(3\alpha/2)} I_2 = V_1 V_2 U_3 \).

The case \( \alpha > \pi \) follows from the property (3). \( \square \)

In the following two Lemmas we construct decompositions of a scalar operator on an infinite dimensional Hilbert space. These Lemmas are analogous of corresponding ones for sums of orthoprojections discussed in \([10,11]\).

**Lemma 8.** Let \( k \geq 4 \) and \( 0 < \alpha \leq \pi \). The set \( \Omega_k^\alpha \) contains every number \( u = e^{i\gamma} \) with \( 2\alpha \leq \gamma \leq (k - 2)\alpha \).

**Proof.** Obviously \( u = e^{i(2\alpha)} \in \Omega_4^\alpha \) and only \( \gamma = 2\alpha \) satisfies the conditions of the Lemma for \( k = 4 \).

It is sufficient to prove the Lemma for \( k = 5 \) because if \( e^{i\gamma} I = U_1 U_2 \cdots U_s \) then \( e^{i\gamma} e^{i\alpha} I = U_1 U_2 \cdots U_s \) is a product of \( s + 1 \) operators from \( U^\alpha \).

Thus let \( k = 5 \) and \( 2\alpha \leq \gamma \leq 2.5\alpha \). Let \( \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \) be the orthonormal basis of a Hilbert space \( H \). For \( 0 \leq \phi \leq 2\alpha \) and \( \psi \) calculated by the formula (7) with \( \alpha_1 = \alpha_2 = \alpha \), the product \( R_0^\alpha R_\psi^\alpha \) has two eigenvalues:

\[
R_0^\alpha R_\psi^\alpha \approx \text{diag}(e^{i\phi}, e^{i(2\alpha - \phi)}).
\]

So for any sequence \( 0 \leq \phi_j \leq 2\alpha, j = 1, 2, \ldots \), there exist two operators \( U_1, U_2 \in U^\alpha \) such that

\[
U_1 U_2 = \text{diag}(e^{i(2\pi - \phi_j)}, e^{i\phi_j}, e^{i(2\alpha - \phi_j)}, e^{i(2\alpha - \phi_j)}), \quad \text{where } \phi_j = 0, 1, 2, \ldots.
\]

The operator \( U_3 \) is defined by the formulas \( U_3 \vec{e}_j = e^{i(\alpha + \pi)} \vec{e}_j \), where \( \tau_{2j - 1} = 0 \),

\[
\tau_{2j} = \begin{cases} 
0, & \text{if } \theta_j < \alpha, \\
1, & \text{otherwise}.
\end{cases}
\]

Then

\[
U_1 U_2 U_3 U_4 U_5 = \text{diag}(e^{i(2\alpha + \phi_1)}, e^{i(2\alpha + \phi_1 + \phi_2)}, e^{i(2\alpha + \phi_1 + \phi_2 + \phi_3)}, \ldots).
\]

Putting \( \theta_1 = \gamma - 2\alpha, \)

\[
\theta_{j+1} = \theta_j + 2\gamma - (4 + \tau_{2j})\alpha \quad \text{and} \quad \phi_j = \theta_j + \gamma - (2 + \tau_{2j})\alpha,
\]

we have that \( U_1 U_2 U_3 U_4 U_5 = e^{i\gamma} I \). The only property we need to prove is that \( 0 \leq \theta_j \leq 2\alpha \) and \( 0 \leq \phi_j \leq 2\alpha \). Note that \( 0 \leq \theta_1 < \alpha \) and if \( 0 \leq \theta_j < \alpha \), then \( \theta_j + 1 = (2\gamma - 4\alpha) + \theta_j \leq 2\alpha \). On the other hand, if \( \theta_j > \alpha \), then \( \theta_{j+1} = (2\gamma - 5\alpha) + \theta_j \leq \theta_j \). The inequality \( 0 \leq \phi_j \leq 2\alpha \) can be checked by a similar reasoning. So for all \( \gamma, 2\alpha < \gamma < 2.5\alpha \), the decomposition \( U_1 U_2 U_3 U_4 U_5 = e^{i\gamma} I \) holds. Using property (2) of Lemma 7, we complete the proof. \( \square \)
Lemma 9. Let \( k \geq 5 \). The set \( \Omega_k^\alpha \) contains every unitary number \( u = e^{iv} \) with \( (1 + \frac{1}{k-3})\alpha \leq \gamma \leq (1 + \frac{1}{k-4})\alpha \) for \( \alpha \leq \pi \).

Proof. If \( \gamma = (1 + \frac{1}{k-3})\alpha \), then \((k-3)\gamma = (k-2)\alpha\) and by Corollary 6, the matrix \( \text{diag}(e^{iv}, e^{i(\gamma - \alpha)}) \) is a product of \( k-3 \) matrices from \( U_\alpha \). Whence, the scalar matrix

\[
e^{iv}I_{k-3} = \text{diag}(e^{iv}, e^{i(\gamma - \alpha)}) \text{diag}(I_{k-4}, e^{i\alpha})
\]

is a product of \( k-2 \) matrices from \( U_\alpha \). So we assume further that

\[
\left( 1 + \frac{1}{k-3} \right)^2 \alpha < \gamma < \left( 1 + \frac{1}{k-4} \right)\alpha.
\]

Let \( 0 \leq \phi_j \leq \alpha \) and \( 0 \leq \theta_j \leq \alpha \), \( j = 1, 2, \ldots \), be two sequences of real numbers. By Lemma 5 for all \( j \in \mathbb{N} \), there exist unitary matrices \( V_j^{(s)} \in M_{k_j+1} \), and \( W_j^{(l)} \in M_{q_j+1} \), \( V_j^{(s)}, W_j^{(l)} \in U_\alpha \) such that

\[
V_j^{(1)} \ldots V_j^{(k_j)} \text{diag}(e^{i\phi_j}, I_{k_j}) \approx \text{diag}(e^{iv}I_{k_j}, e^{i(\phi_j - k_j(\gamma - \alpha))})
\]

and

\[
\text{diag}(e^{i\theta_j}, I_{q_j})W_j^{(1)} \ldots W_j^{(q_j)} \approx \text{diag}(e^{iv}I_{q_j}, e^{i(\theta_j - q_j(\gamma - \alpha))})
\]

with

\[
0 \leq \phi_j - k_j(\gamma - \alpha) < \gamma - \alpha \quad \text{and} \quad 0 \leq \theta_j - q_j(\gamma - \alpha) < \gamma - \alpha.
\]

To simplify the formulas further we put \( V_j^{(l)} = I_{k_j+1} \) if \( i > k_j \) and \( W_j^{(i)} = I_{q_i+1} \) if \( i > q_i \) and define the direct sums of matrices:

\[
V_1 := V^{(s)}_1 \oplus V^{(s)}_2 \oplus V^{(s)}_3 \oplus \cdots \quad \text{and} \quad W_1 := W^{(l)}_1 \oplus W^{(l)}_2 \oplus W^{(l)}_3 \oplus \cdots,
\]

where \( s, l = 1, \ldots, k-4 \) and

\[
\Phi = \text{diag}(e^{i\phi_1}, I_{k_1}, e^{i\phi_2}, I_{k_2}, e^{i\phi_3}, I_{k_3}, \ldots), \quad \Psi = \text{diag}(e^{i\theta_1}, I_{q_1}, e^{i\theta_2}, I_{q_2}, e^{i\theta_3}, I_{q_3}, \ldots).
\]

We shall show below that \( k_j \) and \( q_j \) will be less than \( k-3 \). Matrices \( V_1, \ldots, V_{k-4} \) and \( \Phi \) define unitary operators on a separable Hilbert space \( G_1 \). By the relations (11) and (12), we obtain

\[
V_1V_2 \ldots V_{k-4}\Phi \approx \text{diag}(e^{iv}I_{k_1}, e^{i(\phi_1 - k_1(\gamma - \alpha))}, e^{iv}I_{k_2}, e^{i(\phi_2 - k_2(\gamma - \alpha))}, \ldots)
\]

So in the orthogonal basis of eigenvectors \( \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots\} \), the matrix associated with the operator \( V_1V_2 \ldots V_{k-4} \Phi \) is diagonal. We split this basis in two parts \( \tilde{f}_{m_1}, \tilde{f}_{m_2}, \ldots, \tilde{f}_{m_s} \) and \( \tilde{f}_{n_1}, \tilde{f}_{n_2}, \ldots, \tilde{f}_{n_l} \) such that

\[
V_1V_2 \ldots V_{k-4}\Phi\tilde{f}_{m_i} = e^{iv}\tilde{f}_{m_i}, \quad i = 1, 2, 3, \ldots,
\]

\[
V_1V_2 \ldots V_{k-4}\Phi\tilde{f}_{n_j} = e^{i(\phi_j - k_j(\gamma - \alpha))}\tilde{f}_{n_j}, \quad j = 1, 2, 3, \ldots
\]

Let

\[
H_1 = (\tilde{f}_{m_1}, \tilde{f}_{m_2}, \ldots, \tilde{f}_{m_s})
\]

be a Hilbert space, the closure of the linear span of the vectors \( \tilde{f}_{m_1}, \tilde{f}_{m_2}, \ldots, \) and \( H_2 = H_1^\perp \). Then in the basis \( \{\tilde{f}_{m_1}, \tilde{f}_{m_2}, \ldots, \tilde{f}_{n_1}, \tilde{f}_{n_2}, \ldots\} \) the matrix associated with \( V_1V_2 \ldots V_{k-4} \Phi \) will be of the form

\[
V_1 \ldots V_{k-4}\Phi = e^{iv}I_{H_1} \oplus \text{diag}(e^{i(\phi_1 - k_1(\gamma - \alpha))}, e^{i(\phi_2 - k_2(\gamma - \alpha))}, e^{i(\phi_3 - k_3(\gamma - \alpha))}, \ldots),
\]

where \( I_{H_1} \) is the identity operator on \( H_1 \).
Matrices $W_1, \ldots, W_{k-4}$ and $Ψ$ define unitary operators on a separable Hilbert space $G_2$. We use the same argument as above for the unitary operator $ΨW_1 \cdots W_{k-4}$. One can find an orthogonal basis $\langle \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_1, \vec{h}_2, \ldots \rangle$ in $G_2$ such that in this basis

$$ΨW_1 \cdots W_{k-4} = e^{iγ} I_{H_3} \oplus \text{diag}(e^{i(θ_1-φ_1)(γ-α)}, e^{i(θ_2-φ_2)(γ-α)}, e^{i(θ_3-φ_3)(γ-α)}, \ldots),$$

where $H_3 = \langle \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_1, \vec{h}_2, \ldots \rangle$. Let $H_4 = H_2^\perp$ in $G_2$. The needed $k$ operators $U_1, \ldots, U_k \in U^α$ will act on the Hilbert space $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$. We fix now the following basis in $H$:

$$\{j \vec{f}_1, j \vec{f}_2, \ldots, j \vec{f}_1, j \vec{f}_2, \ldots, j \vec{h}_1, j \vec{h}_2, \ldots \}.$$

Let $U_j = V_j \oplus W_j$ for $j = 1, \ldots, k - 4, U_{k-1}$ and $U_k$ be such that

$$U_{k-1}U_k = Φ \oplus I_{H_3} \oplus \text{diag}(e^{i(2α-φ_2)}, e^{i(2α-φ_3)}, \ldots) \quad \text{(15)}$$

for $φ_1 = α$ and $U_1$ and $U_2$ be such that

$$U_1U_2 = I_{H_1} \oplus \text{diag}(e^{i(2α-θ_1)}, e^{i(2α-θ_2)}, e^{i(2α-θ_3)}, \ldots) \oplus Ψ. \quad \text{(16)}$$

Note that in Eqs. (15) and (16) by $Φ$ and $Ψ$ we do not mean the diagonal form (14), but the act of the operators on the corresponding subspaces. The product

$$U_1U_2 \cdots U_k = e^{iγ} I_{H_3} \oplus \text{diag}(e^{i(2α+φ_1-θ_1-k_1(γ-α))}, e^{i(2α+φ_2-θ_2-k_2(γ-α))}, \ldots)$$

$$\oplus e^{iγ} I_{H_3} \oplus \text{diag}(e^{i(2α+φ_1-θ_1-k_1(γ-α))}, e^{i(2α+φ_2-θ_2-k_2(γ-α))}, \ldots)$$

is the scalar operator $e^{iγ} I_{H}$ if

$$2α + φ_j - θ_j - k_j(γ-α) = γ \quad \text{and} \quad 2α + φ_j - θ_{j+1} - q_j(γ-α) = γ. \quad \text{(17)}$$

Thus putting $φ_1 = α$,

$$θ_j = α + φ_j - (k_j + 1)(γ-α),$$

$$φ_{j+1} = α + φ_j - (q_j + 1)(γ-α),$$

where we define $k_j$ and $q_j$ to be the unique integers satisfying (13), we have (17). Beside this in view of (13), the inequality $φ_1 \leq α$ inductively yields

$$α - (γ - α) \leq θ_j < α \quad \text{and} \quad α - (γ - α) \leq φ_{j+1} < α. \quad \text{(18)}$$

Since $α/(k - 4) > γ - α > α/(k - 3)$, it follows directly from (13) and (18) that $0 \leq q_j \leq k - 4$ and $0 \leq k_j \leq k - 4$. Therefore, all operators $U_1, \ldots, U_k$ are defined correctly and this completes the proof.

**Theorem 10.** For $0 < α \leq π$ and $k > 4$, we have the set inclusion

$$Ω_k^α \supset \{u ∈ \mathbb{T} | (1 + \frac{1}{k - 3}) α ≤ \arg(u) ≤ (k - 1 - \frac{1}{k - 3}) α \}. \quad \text{(19)}$$

**Proof.** By Lemmas 8 and 9, $Ω_2^α \supset \{e^{iγ} | γ ∈ [2α, 3α]\}$ and $Ω_3^α \supset \{e^{iγ} | γ ∈ [1.5α, 2α]\}$. Using the property (2) of Lemma 7, we conclude that $Ω_2^α \supset \{e^{iγ} | γ ∈ [1.5α, 3.5α]\}$. Since $Ω_k^α \subset Ω_{k+1}^α$, in view of Lemma 9, we obtain the set inclusion (19).

**Corollary 11.** For big enough $k$ the value

$$\left(k - 1 - \frac{1}{k - 3}\right) α ≥ 2π + \left(1 + \frac{1}{k - 3}\right) α$$

Therefore $Ω_k^α = \mathbb{T}$ in this case.
We denote by $\mu(\alpha)$ the value of the biggest root of the equation

$$
(x - 1 - 1/(x - 3))\alpha = \pi,
$$

$$
\mu(\alpha) = \frac{\pi/\alpha + 4 + \sqrt{(\pi/\alpha - 2)^2 + 4}}{2}.
$$

In the following corollary the expression $[x]$ means the smallest integer $n \geq x$.

**Corollary 12.** Let $0 < \alpha \leq \pi$ and $U$ be a unitary symmetry on a Hilbert space $H$, i.e. $U^2 = 1_H$. If the dimension $\dim \ker (U + I_H) = \infty$, then $U$ is a product of 6 operators from $U^\alpha$ for $2\pi/3 < \alpha < 6\pi/7$ and of $\max(5, \lceil \mu(\alpha) \rceil)$ operators from $U^\alpha$ for other cases of $\alpha$.

**Proof.** Let $H_1 = \operatorname{Im} (U + I_H)$ and $H_2 = \operatorname{Im} (U - I_H)$. We are going to prove that $-I_{H_1}$ is a product of $k$ operators from $U^\alpha$, where

$$
\left\{ \begin{array}{ll}
k = 6, & \text{if } 2\pi/3 < \alpha < 6\pi/7, \\
\max(5, \lceil \mu(\alpha) \rceil), & \text{otherwise}.
\end{array} \right.
$$

Really if $2\pi/3 \leq \alpha \leq 6\pi/7$, then $4\alpha/3 \leq 3\pi \leq 14\alpha/3$. Hence by Theorem 10, $-1 = e^{i(3\pi)} \in \Omega_6^\alpha$. Also if $6\pi/7 \leq \alpha \leq \pi$, then $3\alpha/2 \leq 3\pi \leq 7\alpha/2$. Whence $-1 \in \Omega_7^\alpha$. Besides this, by definition of $\mu(\alpha)$, the inequalities

$$
\left( 1 + \frac{1}{k - 3} \right) \alpha \leq \pi \leq \left( k - 1 - \frac{1}{k - 3} \right) \alpha
$$

hold for $k \geq \mu(\alpha)$ and $\alpha \leq 2\pi/3$. Therefore $-I_{H_1} = U_1 U_2 \cdots U_k$, where $U_i \in U^\alpha$ and

$$
U = (U_1 \oplus I_{H_2})(U_2 \oplus I_{H_2}) \cdots (U_k \oplus I_{H_2})
$$

is a product of $k$ unitary operators on $H$ with $U_i \oplus I_{H_2} \in U^\alpha$, $i = 1, \ldots, k$. \qed

**Theorem 13.** Let $\alpha < \pi$. Then the set inclusion holds:

$$
\Omega_4^\alpha \subset \left\{ e^{i\gamma} | \gamma = 2\alpha - \frac{s\alpha + 2q\pi}{m}, s = 0, \pm 1, \pm 2, q \in \mathbb{Z}, m \in \mathbb{N} \right\}.
$$

**Remark 14.** It was proved in [8] that any unitary operator in a separable Hilbert space is a product of four operators from $U^2$. In finite dimensional spaces for any matrix $U$ with $\det U = \pm 1$, a corresponding result was obtained in [12]. So $\Omega_4^{\pi/2} = \mathbb{T}$.

**Proof.** Suppose that for some $\gamma \in \mathbb{R}$ and unitary operators $U_1, \ldots, U_4 \in U^\alpha$, one has $e^{i\gamma} I = U_1 U_2 U_3 U_4$. Then

$$
e^{i\gamma} U_4 U_3^s U_2^t = U_1 U_2.
$$

Let $\gamma \neq 2\alpha - (s\alpha + 2q\pi)/m$, where the parameters $s$, $q$ and $m$ are from the formulation of the theorem and $e^{i\phi} \in \sigma(U_1 U_2)$. Then the sequence of the numbers

$$
e^{i\phi}, e^{i(\phi + (2\alpha - \gamma))}, e^{i(\phi + 2(2\alpha - \gamma))}, \ldots, e^{i(\phi + n(2\alpha - \gamma))}
$$

or the sequence

$$
e^{i(s_1 \alpha)}, e^{i(s_1 \alpha + (2\alpha - \gamma))}, e^{i(s_1 \alpha + 2(2\alpha - \gamma))}, \ldots, e^{i(s_1 \alpha + n(2\alpha - \gamma))}
$$

are the points of the union of the spectrum $\sigma(U_1 U_2)$ and the spectrum $\sigma(U_3 U_4)$ for some integer $s_1 \in \{0, 1, 2\}$. Let us show this by two steps.

**Step 1.** By Eq. (20), we have $e^{i(\gamma - \phi)} \in \sigma(U_3 U_4)$. Suppose for a moment that

$$
e^{i(\phi + j(2\alpha - \gamma))} \neq e^{i(s_2 \alpha)}, \quad \forall j = 1, 2, 3, \ldots, s_2 = 0, 1, 2,
$$

where

$$
\gamma = 2\alpha - \frac{s\alpha + 2q\pi}{m}, \quad s = 0, \pm 1, \pm 2, q \in \mathbb{Z}, m \in \mathbb{N}.
$$
then we deduce by the equivalence stated in Lemma 2, that $e^{i(\phi+(2\alpha-\gamma))} \in \sigma(U_2U_4)$. By Eq. (20), $e^{i\gamma(\phi+(2\alpha-\gamma))} \in \sigma(U_1U_2)$ and using Lemma 2, we get

$$e^{i(\phi+2(2\alpha-\gamma))} \in \sigma(U_1U_2).$$

Repeating such a process $n$ times, we obtain that $n$ elements of the sequence (21) have to be in $\sigma(U_1U_2) \cup \sigma(U_2U_4)$.

Step 2. If (23) is not true, then on some step of the process we have that one of the number $1, e^{i\alpha}$ or $e^{i(2\alpha)}$ is in $\sigma(U_1U_2)$ or in $\sigma(U_2U_4)$. Let $s_1 = \{0, 1, 2\}$ be fixed. Without loss of generality we can assume that $e^{i(2\alpha)} \in \sigma(U_1U_2)$. Starting the described process with $\phi = s_1 \alpha$, we conclude by Step 1 that the sequence (22) belongs to $\sigma(U_1U_2) \cup \sigma(U_2U_4)$. This is really so since

$$s_1 \alpha + j(2\alpha - \gamma) = s_2 \alpha + 2l \pi \iff \gamma = 2\alpha - \frac{(s_2 - s_1)\alpha + 2l \pi}{j},$$

whence the property (23) is fulfilled.

By assumption, $e^{i(2\alpha-\gamma)}$ is an irrational rotation of a unit circle. So for every $\phi$, there exist $n_1$ and $p_1$ such that

$$\tilde{\phi} = \phi + n_1(2\alpha - \gamma) - 2p_1 \pi \in (2\alpha, 2\pi).$$

In view of Remark 3, the number $e^{i\tilde{\phi}}$ cannot belong to $\sigma(U_1U_2)$ or to $\sigma(U_2U_4)$. Therefore $\sigma(U_1U_2)$ is empty and hence such a $\gamma$ is not in $\Omega_4^{\alpha}$. 

A product of two matrices from $U^\alpha$ in some orthonormal basis $\vec{e}_1, \vec{e}_2, \ldots$, has the form (9), putting aside from the consideration the common eigen-subspaces. Hence by Eq. (20), $U_2U_4$ is also a diagonal matrix in the basis. Therefore the construction from the proof of Lemma 8 provides a general scheme for finding $U_1, \ldots, U_4$ that satisfy Eq. (20). For example, we define the operators $U_1, U_2, U_3$ and $U_4$ such that Eqs. (9) and (10) hold. Let $\gamma = (2 + \frac{1}{m}) \alpha$ and $m$ be even. Putting $\tau_1 = 0, \theta_1 = \gamma - 2\alpha = \alpha/m,$

$$\theta_{j+1} = \theta_j + 2\gamma - 4\alpha = \frac{2j + 1}{m} \alpha, \quad 1 \leq j \leq m/2,$$

$$\theta_j = \theta_j + \gamma - 2\alpha = \frac{2j}{m} \alpha, \quad 1 \leq j \leq m/2,$$

$$\theta_j = 0 \text{ for } j \geq m/2 + 1 \text{ and } l \geq (m + 1)/2,$$

we obtain that

$$U_1U_2U_3U_4 = e^{i\gamma}I_m \oplus (e^{i\alpha}) \oplus e^{i(2\alpha)}I.$$ 

The product $U_1U_2$ has the form

$$U_1U_2 = \text{diag}(e^{i(2\alpha)}, e^{i(2\alpha)/m}, \ldots, e^{i\alpha}, e^{-i\alpha}) \oplus \text{diag}(1, e^{i(2\alpha)}, 1, e^{i(2\alpha)}, 1, \ldots).$$

We choose the pair $U_1$ and $U_2$ so that $U_1\vec{e}_m = \vec{e}_m$ and $U_2\vec{e}_{m+1} = \vec{e}_{m+1}$. The space $H_1 = \langle \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m \rangle$ is invariant under the action of every operator $U_j$. Whence the restriction of the product $U_1U_2U_3U_4 |_{H_1} = e^{i\gamma}I_m$.

**Corollary 15.** Let $0 < \alpha \leq \pi$. Then $e^{i\alpha(2 \pm 1/m)} \in \Omega_4^{\alpha}$ for every even $m$.

4. **Concluding remarks**

1. We described various decompositions of a scalar operator. In all cases known to us the formula (2) remains true provided $\sum_k |A_k| \leq 2\pi$, even on a separable Hilbert space. It is interesting to find a basic explanation for such a property.

2. By Corollary 11, we have that $\Omega_4^{\alpha} = \mathbb{T}$ for big enough $k$, e.g. $(k - 3)\alpha \geq 2\pi$. The structure of the set $\Omega_4^{\alpha} \cap \{e^{i\gamma} | 0 < \gamma < \alpha \}$ for
remains unclear. This set contains points that have no connections with $\Sigma_k$. For example, $e^{i(k\alpha - 2\pi s)/s} \in \Omega_k^\alpha$ for $s = 2, \ldots, k-1$. Really for $\alpha < 2\pi/(k-1)$ and $\phi = (k\alpha - 2\pi)/s$, the matrix $e^{i\phi} I_{s-1} \oplus e^{i(s\alpha - s\phi + \phi)}$ is a product of $s$ operators from $U^\alpha$ by Theorem 4 and hence
\[
\left( \text{diag}(I_{s-1}, e^{i\alpha}) \right)^{k-s} \text{diag}(e^{i\phi} I_{s-1}, e^{i(s\alpha - s\phi + \phi)}) = e^{i\phi} I_s
\]
is a product of $k$ operators from $U^\alpha$.

3. We may conjecture by analogy with results of [8,12] that any unitary operator is a product of $k$ operators from $U^\alpha$ for $(k - c_k)\alpha > 2\pi$, where $c_k \in [0,5]$. As we mentioned above every unitary operator is a product of four symmetries on infinite-dimensional space [8]. Moreover for a unitary operator $U$, the authors constructed the decomposition
\[
U = V_1 V_2 V_3 V_4, \quad V_j \in U^\pi
\]
such that the subspace $\ker(V_j + I)$ is infinite-dimensional for every $j = 1, 2, 3, 4$. Therefore, it follows directly from Corollary 12 that every unitary operator is a product of $\max(24, 4 \lceil \mu(\alpha) \rceil)$ operators from $U^\alpha$. See also [15] for various other decompositions.

4. It is interesting to see whether the equation
\[
\Omega_k^\alpha = \{ e^{k\alpha} | x \in \Sigma_k \}
\]
holds or is violated for $k\alpha < 2\pi$. One of the methods for finding new decompositions of operators comes from representation theory. In [10] a transformation (a reflection functor) was found such that for a decomposition of a scalar operator into a sum of ortho-projections, it gives a decomposition of different from former scalar operator into a sum of ortho-projections. It will be worthwhile to construct similar transformations for products of unitary operators. For finite matrices satisfying additional conditions, the existence of such transformations was found in [9].

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